Distributed & Embedded Real-Time Systems (0640751)

Lecture: 11

Stability of Real-Time Embedded Systems

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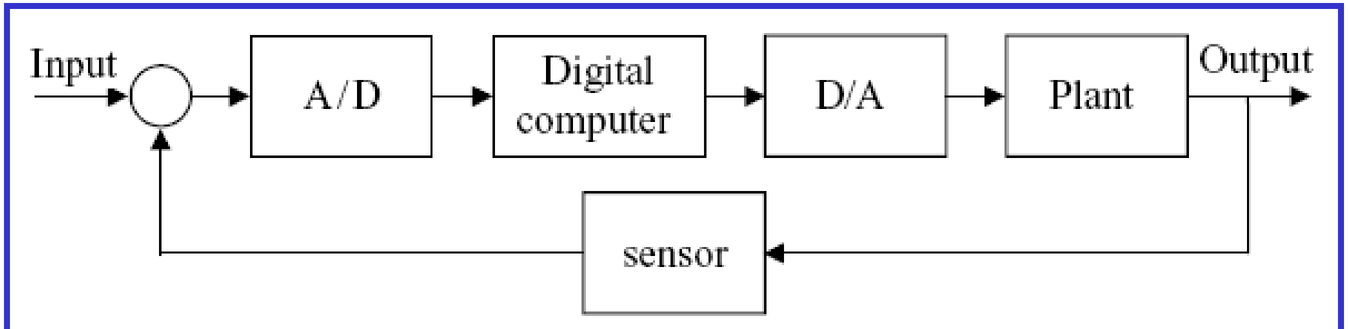
Course Objectives:

This unit is concerned with:

- Sampled-data system analysis.
- Stability analysis of embedded real-time systems.
- Various techniques available for the analysis of the stability.
- Hardware and software design issues with stability requirements.

Sampled Data Systems:

- A sampled data system operates on discrete-time rather than continuous-time signals. A digital computer is used as the controller in such a system. A D/A converter is usually connected to the output of the computer to drive the plant. We will assume that all the signals enter and leave the computer at the same fixed times, known as the sampling times.
- The digital computer performs the controller or the compensation function within the system. The A/D converter converts the error signal, which is a continuous signal, into digital form so that it can be processed by the computer.
- At the computer output the D/A converter converts the digital output of the computer into a form which can be used to drive the plant.



Sampled data control system

Stability of Sampled Data Systems:

Suppose we have a closed-loop sampled data system transfer function:

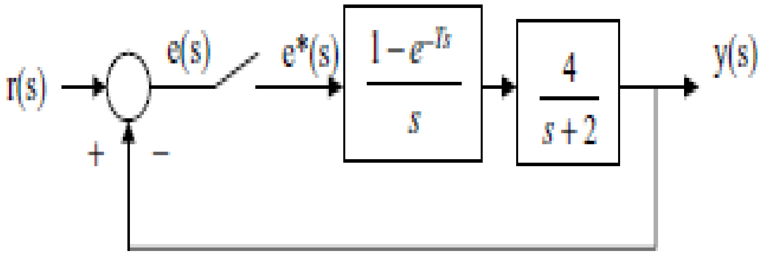
$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

where $D(z) = 1 + GH(z) = 0$ is known as the characteristic equation.

- We can say that a system in the z-plane will be stable if all the roots of the characteristic equation, $D(z) = 0$, lie inside the unit circle.
- There are several methods available to check for the stability of a discrete-time system:
  1. Factorize $D(z) = 0$ and find the positions of its roots, and hence the position of the closed loop poles.
  2. Determine the system stability without finding the poles of the closed-loop system, such as Jury’s test.
  3. Transform the problem into the s-plane and analyze the system stability using the well established s-plane techniques, such as frequency response analysis or the Routh–Hurwitz criterion.
  4. Use the root-locus graphical technique in the z-plane to determine the positions of the system poles.

Example:

For the given system, determine whether or not the system is stable. Assume *T = 1 s.*



Solution

The closed-loop system transfer function is

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1+G(z)},$$

where

$$G(z) = Z\left\{\left[\frac{1-e^{-Ts}}{s}\frac{4}{s+2}\right]\right\} = (1-z^{-1})Z\left\{\left[\frac{4}{s(s+2)}\right]\right\} = (1-z^{-1})\frac{2z(1-e^{-2T})}{(z-1)(z-e^{-2T})}$$

$$= \frac{2(1-e^{-2T})}{z-e^{-2T}}.$$

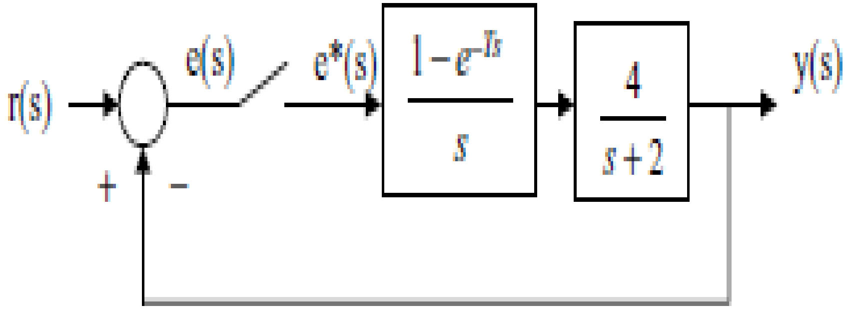
For $T = 1$ s,

$$G(z) = \frac{1.729}{z-0.135}.$$

The roots of the characteristic equ. are 1 + *G(z) = 0, or 1 + 1.729/(z − 0.135) = 0,*
Then; *z = −1.594 which is outside the unit circle, i.e. the system is not* stable.

Example:

For the given system, find the value of *T for which the system is stable.*



Solution:

$$G(z) = \frac{2(1 - e^{-2T})}{z - e^{-2T}}$$

The roots of the characteristic equation are $1 + G(z) = 0$, or $1 + 2(1 - e^{-2T})/(z - e^{-2T}) = 0$.

$$z - e^{-2T} + 2(1 - e^{-2T}) = 0$$

$$z = 3e^{-2T} - 2.$$

The system will be stable if the absolute value of the root is inside the unit circle, i.e.

$$|3e^{-2T} - 2| < 1,$$

from which we get $2T < \ln\left(\frac{1}{3}\right)$ or $T < 0.549.$

Thus, the system will be stable as long as the sampling time $T < 0.549$.

Jury's Stability Test:

- Jury's stability test is similar to the Routh–Hurwitz stability criterion used for continuous time systems.
- Jury's test can be applied to characteristic equations of any order, and its complexity increases for high-order systems.
- To describe Jury's test, express the characteristic equation of a discrete-time system of order *n* as

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

where *an* > 0.

- We now form the array shown in the following table. The elements of this array are defined as follows:
- The elements of each of the even-numbered rows are the elements of the preceding row, in reverse order.
- The elements of the odd-numbered rows are defined as:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \; c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ n_{n-1} & b_k \end{vmatrix}, \; d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$

Table Array for Jury's stability tests

z^0	z^1	z^2	$\dots$	z^{n-k}	$\dots$	z^{n-1}	z^n
a_0	a_1	a_2	$\dots$	a_{n-k}	$\dots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\dots$	a_k	$\dots$	a_1	a_0
b_0	b_1	b_2	$\dots$	b_{n-k}	$\dots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_{k-1}	$\dots$	b_0	
c_0	c_1	c_2	$\dots$	c_{n-k}	$\dots$		
c_{n-2}	c_{n-3}	c_{n-4}	$\dots$	c_{k-2}	$\dots$		
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

Jury’s Stability Test Conditions:

- The necessary and sufficient conditions for the characteristic equation to have roots inside the unit circle are given as:

1. $F(1) > 0,$
2. $(-1)^n F(-1) > 0,$
3. $|a_0| < a_n,$
4. $|b_0| > b_{n-1}$

   $|c_0| > c_{n-2}$

   $|d_0| > d_{n-3}$

   $\ldots$

- Jury’s test is then applied as follows:
- Check the first three conditions and stop if any of these conditions is not satisfied.
- Construct the array given in the Table and check the fourth conditions given above. Stop if any condition is not satisfied.

Example 3:

Check stability of a system has an open loop transfer function:

$$G(z) = \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2}$$

Solution

The characteristic equation is

$$1 + G(z) = 1 + \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2} = 0$$

$$z^2 - z + 0.7 = 0.$$

Applying Jury's test,

$$F(1) = 0.7 > 0, \quad F(-1) = 2.7 > 0, \quad 0.7 < 1$$

All the conditions are satisfied and the system is stable

Example 4

The characteristic equation of a system is given by

$$1 + G(z) = 1 + \frac{K(0.2z + 0.5)}{z^2 - 1.2z + 0.2} = 0$$

Determine the value of K for which the system is stable.

Solution

The characteristic equation is

$$z^2 + z(0.2K - 1.2) + 0.5K = 0, \quad \text{where } K > 0$$

Applying Jurys's test,

$$F(1) = 0.7K - 0.2 > 0,$$
$$F(-1) = 0.3K + 2.2 > 0, \quad 0.5K < 1$$

Thus, the system is stable for $0.285 < K < 2$.

Example 5

The characteristic equation of a system is given by

$$F(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0$$

Determine the stability of the system.

Solution

Applying Jury's test, $a_3 = 1$, $a_2 = -2$, $a_1 = 1.4$, $a_0 = -0.1$ and

$$F(1) = 0.3 > 0, \quad F(-1) = -4.5 < 0, \quad 0.1 < 1$$

The first conditions are satisfied. Applying the other condition,

$$\left|\begin{bmatrix} -0.1 & 1 \\ 1 & -0.1 \end{bmatrix}\right| = -0.99 \quad \text{and} \quad \left|\begin{bmatrix} -0.1 & 1.4 \\ 1 & -2 \end{bmatrix}\right| = -1.2$$

since $|0.99| < |-1.2|$, the system is not stable.

Example :

Test the stability of the polynomial.

$$F(z) = z^5 + 2.6z^4 - 0.56z^3 - 2.05z^2 + 0.0775z + 0.35 = 0$$

We compute the entries of the Jury table using the coefficients of the polynomial

Jury Table

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	0.35	0.0775	−2.05	−0.56	2.6	1
2	1	2.6	−0.56	−2.05	0.0775	0.35
3	−0.8775	−2.5729	−0.1575	1.854	0.8325	
4	0.8325	1.854	−0.1575	−2.5729	−0.8775	
5	0.0770	0.7143	0.2693	0.5151		
6	0.5151	0.2693	0.7143	0.0770		
7	−0.2593	−0.0837	−0.3472			

The first two conditions require the evaluation of $F(z)$ at $z = \pm 1$.

1. $F(1) = 1 + 2.6 - 0.56 - 2.05 + 0.0775 + 0.35 = 1.4175 > 0$
2. $(-1)^5 F(-1) = (-1)(-1 + 2.6 + 0.56 - 2.05 - 0.0775 + 0.35) = -0.3825 < 0$

Conditions 3 through 6 can be checked quickly

3. $| 0.35 | < 1$
4. $| -0.8775 | > | 0.8325 |$
5. $| 0.0770 | < | 0.5151 |$
6. $| -0.2593 | < | -0.3472 |$

Conditions 2, 5, and 6 are violated, and the polynomial has roots on or outside the unit circle. In fact, the polynomial can be factored as

$$F(z) = (z - 0.7)(z - 0.5)(z + 0.5)(z + 0.8)(z + 2.5) = 0$$

and has a root at –2.5 outside the unit circle. Note that the number of conditions violated is not equal to the number of roots outside the unit circle and that condition 2 is sufficient to conclude the instability of $F(z)$.

EXAMPLE

Find the stable range of the gain K for the unity feedback digital cruise control system with the analog plant transfer function

$$G(s) = \frac{K}{s+3}$$

and with digital-to-analog converter (DAC) and analog-to-digital converter (ADC) if the sampling period is 0.02 s.

Solution

The transfer function for analog subsystem ADC and DAC is

$$G_{ZAS}(z) = (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{G(s)}{s}\right]\right\}$$

$$= (1 - z^{-1})\mathscr{Z}\left\{\mathscr{L}^{-1}\left[\frac{K}{s(s+3)}\right]\right\}$$

Using the partial fraction expansion

$$\frac{K}{s(s+3)} = \frac{K}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

we obtain the transfer function

$$G_{ZAS}(z) = \frac{1.9412 \times 10^{-2} K}{z - 0.9418}$$

For unity feedback, the closed-loop characteristic equation is

$$1+G_{ZAS}(z)=0$$

which can be simplified to

$$z-0.9418+1.9412\times10^{-2}\quad K=0$$

The stability conditions are

$$0.9418-1.9412\times10^{-2}\quad K<1$$

$$-0.9418+1.9412\times10^{-2}\quad K<1$$

Thus, the stable range of K is

$$-3<K<100.03$$

Hardware and Software Design and Stability Requirements:

- This will be discussed with examples during lecture.

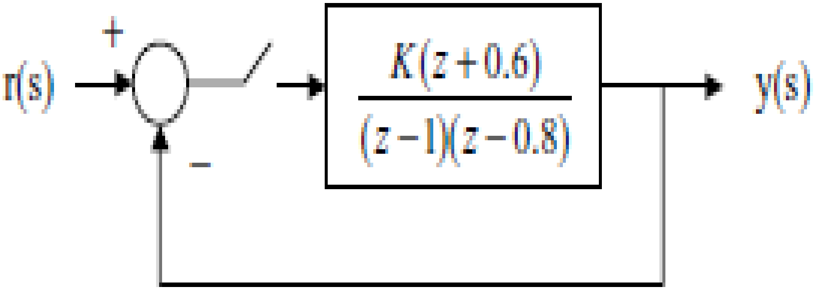
Assignment:

1. For the given system, determine the range of *K for stability by;*

   a). finding the roots of the characteristic equation.

   b). using Jury's test.

   c). using Routh–Hurwitz criterion.



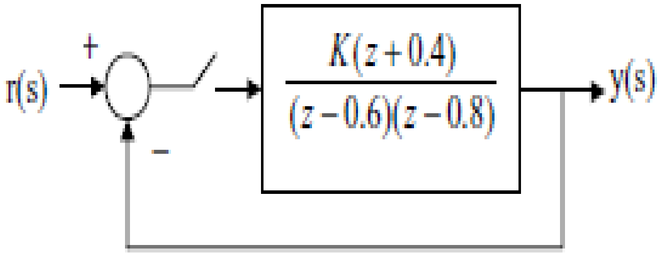
2. For the given system, determine the range of *K for stability by;*

   a). finding the roots of the characteristic equation.

   b). using Jury's test.

   c). using Routh–Hurwitz criterion.

   d). If the system is marginally stable, determine the frequency of oscillation.



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