Real-Time Systems
(0630581)

Lecture: 7

Stability of Real-Time Systems

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Course Objectives:

This unit is concerned with:

- stability analysis of microcontroller-based real-time systems.
- the various techniques available for the analysis of the stability.
- hardware and software design issues with stability requirements.

Stability of Real-Time Systems:

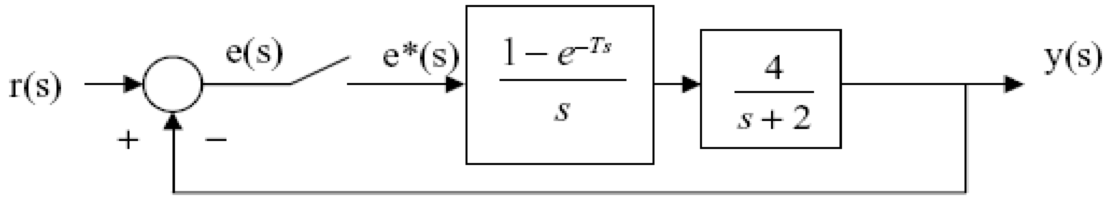
Suppose we have a closed-loop system transfer function:

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

where $D(z) = 1 + GH(z) = 0$ is known as the characteristic equation.

- The stability of the system depends on the location of the poles of the closed-loop transfer function, or the roots of the characteristic equation $D(z) = 0$.
- The left-hand side of the *s*-plane (a continuous system is stable) maps into the interior of the unit circle in the *z*-plane.
- We can say that a system in the *z*-plane will be stable if all the roots of the characteristic equation, $D(z) = 0$, lie inside the unit circle.
- There are several methods available to check for the stability of a discrete-time system:
    1. Factorize $D(z) = 0$ and find the positions of its roots, and hence the position of the closed loop poles.
    2. Determine the system stability without finding the poles of the closed-loop system, such as Jury's test.
    3. Transform the problem into the *s*-plane and analyze the system stability using the well established *s*-plane techniques, such as frequency response analysis or the Routh–Hurwitz criterion.
    4. Use the root-locus graphical technique in the *z*-plane to determine the positions of the system poles.

Example 1:



$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)}$$

where

$$G(z) = Z\left\{\left[\frac{1 - e^{-Ts}}{s}\frac{4}{s+2}\right]\right\} = (1 - z^{-1})Z\left\{\left[\frac{4}{s(s+2)}\right]\right\}$$

$$= (1 - z^{-1})\frac{2z(1 - e^{-2T})}{(z-1)(z - e^{-2T})} = \frac{2(1 - e^{-2T})}{z - e^{-2T}}.$$

For $T = 1$ s,

$$G(z) = \frac{1.729}{z - 0.135}.$$

- The roots of the characteristic equation are;

$1 + G(z) = 0, 1 + 1.729/(z - 0.135) = 0,$

- The solution of which is $z = -1.594$ which is outside the unit circle, i.e. the system is not stable.

Example 2: For the system given in Example 1, find the value of T for which the system is stable.

$$G(z) = \frac{2(1-e^{-2T})}{z-e^{-2T}}$$

$$1 + G(z) = 0, \text{ or } 1 + 2(1-e^{-2T})/(z-e^{-2T}) = 0$$

$$= z - e^{-2T} + 2(1-e^{-2T}) = 0$$

$$z = 3e^{-2T} - 2$$

$$|3e^{-2T} - 2| < 1$$

$$2T < \ln\left(\tfrac{1}{3}\right) \quad \text{or} \quad T < 0.549$$

- Thus, the system will be stable as long as the sampling time $T < 0.549$.

Jury's Stability Test:

- Jury's stability test is similar to the Routh–Hurwitz stability criterion used for continuous time systems.
- Jury's test can be applied to characteristic equations of any order, and its complexity increases for high-order systems.
- To describe Jury's test, express the characteristic equation of a discrete-time system of order *n* as

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

where *an* > 0.

- We now form the array shown in the following table. The elements of this array are defined as follows:
- The elements of each of the even-numbered rows are the elements of the preceding row, in reverse order.
- The elements of the odd-numbered rows are defined as:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ n_{n-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$

Table Array for Jury's stability tests

z^0	z^1	z^2	$\dots$	z^{n-k}	$\dots$	z^{n-1}	z^n
a_0	a_1	a_2	$\dots$	a_{n-k}	$\dots$	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	$\dots$	a_k	$\dots$	a_1	a_0
b_0	b_1	b_2	$\dots$	b_{n-k}	$\dots$	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	$\dots$	b_{k-1}	$\dots$	b_0	
c_0	c_1	c_2	$\dots$	c_{n-k}	$\dots$		
c_{n-2}	c_{n-3}	c_{n-4}	$\dots$	c_{k-2}	$\dots$		
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

Jury's Stability Test Conditions:

- The necessary and sufficient conditions for the characteristic equation to have roots inside the unit circle are given as:

1. $F(1) > 0,$
2. $(-1)^n F(-1) > 0,$
3. $|a_0| < a_n,$
4. $$|b_0| > b_{n-1}$$
$$|c_0| > c_{n-2}$$
$$|d_0| > d_{n-3}$$
$$\dots$$

- Jury's test is then applied as follows:
- Check the first three conditions and stop if any of these conditions is not satisfied.
- Construct the array given in the Table and check the fourth conditions given above. Stop if any condition is not satisfied.

Example 3:

Check stability of a system has an open loop transfer function:

$$G(z) = \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2}$$

Solution

The characteristic equation is

$$1 + G(z) = 1 + \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2} = 0$$

$$z^2 - z + 0.7 = 0.$$

Applying Jury's test,

$$F(1) = 0.7 > 0, \quad F(-1) = 2.7 > 0, \quad 0.7 < 1$$

All the conditions are satisfied and the system is stable

Example 4

The characteristic equation of a system is given by

$$1 + G(z) = 1 + \frac{K(0.2z + 0.5)}{z^2 - 1.2z + 0.2} = 0$$

Determine the value of K for which the system is stable.

Solution

The characteristic equation is

$$z^2 + z(0.2K - 1.2) + 0.5K = 0, \quad \text{where } K > 0.$$

Applying Jurys's test,

$$F(1) = 0.7K - 0.2 > 0,$$
$$F(-1) = 0.3K + 2.2 > 0, \quad 0.5K < 1$$

Thus, the system is stable for $0.285 < K < 2$.

Example 5

The characteristic equation of a system is given by

$$F(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0$$

Determine the stability of the system.

Solution

Applying Jury's test, $a_3 = 1$, $a_2 = -2$, $a_1 = 1.4$, $a_0 = -0.1$ and

$$F(1) = 0.3 > 0, \quad F(-1) = -4.5 < 0, \quad 0.1 < 1$$

The first conditions are satisfied. Applying the other condition,

$$\left|\begin{bmatrix} -0.1 & 1 \\ 1 & -0.1 \end{bmatrix}\right| = -0.99 \quad \text{and} \quad \left|\begin{bmatrix} -0.1 & 1.4 \\ 1 & -2 \end{bmatrix}\right| = -1.2$$

since $|0.99| < |-1.2|$, the system is not stable.

Hardware and Software Design and Stability Requirements:

- This will be discussed with examples during lecture.