Philadelphia University

Lecture Notes for 650364

Probability & Random Variables

Lecture 9: Moments

Department of Communication & Electronics Engineering

Instructor

Dr. Qadri Hamarsheh

Email: qhamarsheh@philadelphia.edu.jo
Website: http://www.philadelphia.edu.jo/academics/qhamarsheh

Moments

- ✓ An application of the expected value of a function **g(.)** of random variable **X** is in calculating **moments**. Two types of moments are of interest, those **about the origin** and those **about the mean**.

Moments about the Origin:

- ✓ The $\mathbf{n_{th}}$ moment of the random variable **X** about the origin

$$m_n = E[X^n] = \int_{-\infty}^{+\infty} x^n \, f_X(x)dx$$

Clearly:

$$m_0 = E[X^0] = 1$$

$$m_1 = E[X^1] = \overline{X} \quad , \text{ the mean value of } X$$

$$m_2 = E[X^2] = \overline{X^2} \quad , \text{ the mean square value of } X$$

Central Moments:

- ✓ The $\mathbf{n_{th}}$ central moment of the random variable **X**

$$\mu_n = E[(X - \overline{X})^n] = \int_{-\infty}^{+\infty} (x - \overline{X})^n \, f_X(x)dx$$

Clearly:

$$\mu_0 = E[(X - \overline{X})^0] = 1$$

$$\mu_1 = E[(X - \overline{X})^1] = 0$$

$$\mu_2 = E[(X - \overline{X})^2] = \sigma_X^2 \quad , \text{ the variance of } X$$

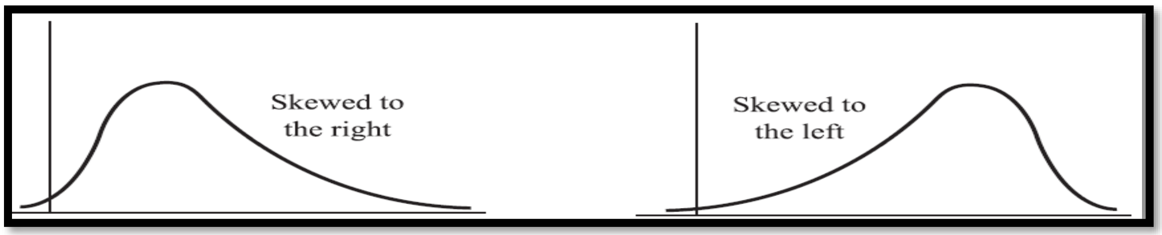
σ_X , is called the standard deviation of X

Variance and Skew:

- ✓ The **second central moment** is called the **variance** of **X**

$$\sigma_X^2 = \mu_2 = E[(X-\overline{X})^2] = E[X^2 - 2\overline{X}X + \overline{X}^2]$$
$$= E[X^2] - 2\overline{X}E[X] + \overline{X}^2$$
$$= E[X^2] - \overline{X}^2 = m_2 - m_1^2$$

- ✓ Often a distribution is **not symmetric** about any value but instead has one of its **tails longer than the other**. If the longer tail occurs to the right, as in Fig, the distribution is said to be **skewed to the right**, while if the longer tail occurs to the left, as in Fig, it is said to be **skewed to the left**.
- ✓ Measure describing this asymmetry called **coefficient of skewness**, or briefly **skewness**.
- ✓ The measure will be **positive** or **negative** according to whether the distribution is skewed to the right or left, respectively.



✓ The **third central moment** is a measure of the **asymmetry** of the density function about the mean value **m_0** and called the skew of the density function

$$\mu_3 = E[(X - \overline{X})^3]$$

✓ The **normalized third** central moment

$$\mu_3 / \sigma_X^3$$

is called the **coefficient of skewness**.

✓ If a density is **symmetric** about the mean value **m_0**, it has zero skew.

✓ **Example**: (**Exponential Density Function**)

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

$$E[X] = m_1 = \int_{-\infty}^{+\infty} x\, f_X(x) dx = \int_a^{\infty} x \frac{1}{b} e^{-(x-a)/b} dx = a + b$$

$$E[X^2] = m_2 = \int_{-\infty}^{+\infty} x^2\, f_X(x) dx = \int_a^{\infty} x^2 \frac{1}{b} e^{-(x-a)/b} dx = b^2 + (a+b)^2$$

$$\sigma_X^2 = \mu_2 = m_2 - m_1^2 = b^2$$

$$\mu_3 = \int_{-\infty}^{+\infty} (x - \overline{X})^3\, f_X(x) dx = \int_a^{\infty} (x - a - b)^3 \frac{1}{b} e^{-(x-a)/b} dx = 2b^3$$

$$\text{Coefficien t of skewness} = \frac{\mu_3}{\sigma_X^3} = \frac{2b^3}{b^3} = 2$$

Functions that Give Moments

- ✓ Although the moments of most distributions can be determined **directly** by **evaluating the necessary integrals or sums**, an **alternative procedure** sometimes provides considerable simplifications. This technique utilizes **moment-generating functions**.
- ✓ Two functions can be defined that allow moments to be calculated for a random variable **X**.
  - o **Characteristic Function:**
    - The **characteristic function** of a random variable **X** is defined by

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x)dx$$

    - The $\mathbf{n_{th}}$ moment of **X** is given by

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

✓ **Example: (Exponential Density Function)**

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

- The **characteristic function** for the exponential random variable is

$$\begin{aligned} \Phi_X(\omega) &= E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx \\ &= \int_a^{+\infty} \frac{1}{b} e^{-(x-a)/b} e^{j\omega x} dx = \frac{e^{a/b}}{b} \int_a^{+\infty} e^{-(1/b - j\omega)x} dx \\ &= \frac{e^{a/b}}{b} \left[\frac{e^{-(1/b - j\omega)x}}{-(1/b - j\omega)} \Bigg|_a^{\infty} \right] = \frac{e^{j\omega a}}{1 - j\omega b} \end{aligned}$$

- The **first moment** for the exponential random variable is

$$\begin{aligned} m_1 &= (-j)^1 \frac{d\Phi_X(\omega)}{d\omega} \Bigg|_{\omega=0} = -je^{j\omega a} \left[\frac{ja}{1 - j\omega b} + \frac{jb}{(1 - j\omega b)^2} \right] \Bigg|_{\omega=0} \\ &= a + b \end{aligned}$$

○ Moment Generating Function:

- Another statistical average closely related to the characteristic function is the **moment generating function**, defined by

$$M_X(v) = E[e^{vX}] = \int_{-\infty}^{+\infty} e^{vx} \; f_X(x)dx$$

- The n_{th} moment of X is given by

$$m_n = \left. \frac{d^n M_X(v)}{dv^n} \right|_{v=0}$$

✓ **Example: (Exponential Density Function)**

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b} & x > a \\ 0 & x < a \end{cases}$$

- The **moment generating function** for the exponential random variable

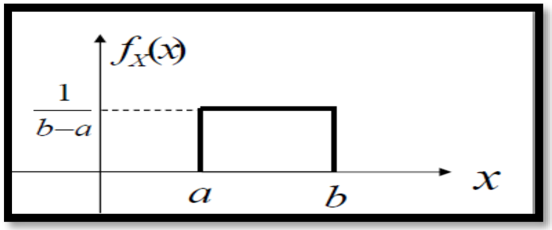
$$\begin{aligned} M_X(v) &= E[e^{vX}] = \int_{-\infty}^{+\infty} e^{vx} \; f_X(x)dx \\ &= \int_a^{+\infty} \frac{1}{b} e^{-(x-a)/b} e^{vx} \; dx = \frac{e^{a/b}}{b} \int_a^{+\infty} e^{-(1/b - v)x} dx \\ &= \frac{e^{a/b}}{b} \left[\left. \frac{e^{-(1/b - v)x}}{-(1/b - v)} \right|_a^{\infty} \right] = \frac{e^{av}}{1 - bv} \end{aligned}$$

- The **first moment** for the exponential random variable is

$$\begin{aligned} m_1 &= \left. \frac{dM_X(v)}{dv} \right|_{v=0} = \left. \left[\frac{e^{va}\left[a(1-bv)+b\right]}{(1-bv)^2} \right] \right|_{v=0} \\ &= a + b \end{aligned}$$

✓ **Example: (Uniform Density Function)**

$$f_X(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$



$$E[X] = m_1 = \int_{-\infty}^{+\infty} x\, f_X(x)dx = \int_{a}^{b} x \frac{1}{b-a} dx = \frac{a+b}{2}$$

$$E[X^2] = m_2 = \int_{-\infty}^{+\infty} x^2\, f_X(x)dx = \int_{a}^{b} x^2 \frac{1}{b-a} dx = \frac{b^3-a^3}{3(b-a)}$$

$$\sigma_X^2 = \mu_2 = m_2 - m_1^2 = \frac{b^3-a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2$$

$$\mu_3 = \int_{-\infty}^{+\infty} (x-\overline{X})^3\, f_X(x)dx = \int_{a}^{b} (x - \frac{a+b}{2})^3 \frac{1}{b-a} dx$$

$$\text{Coefficien t of skewness} = \frac{\mu_3}{\sigma_X^3}$$

Examples

Exponential Functions

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} \qquad a \text{ real or complex}$$

$$\int xe^{ax}\,dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right] \qquad a \text{ real or complex}$$

$$\int x^2e^{ax}\,dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right] \qquad a \text{ real or complex}$$

$$\int x^3e^{ax}\,dx = e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right] \qquad a \text{ real or complex}$$

$$\int e^{ax}\sin(x)\,dx = \frac{e^{ax}}{a^2+1}[a\sin(x) - \cos(x)]$$

$$\int e^{ax}\cos(x)\,dx = \frac{e^{ax}}{a^2+1}[a\cos(x) + \sin(x)]$$

Expectation of random variables

✓ **Example 1**: In a lottery, there are **200** prizes of **$5**, **20** prizes of **$25**, and **5** prizes of **$100**. Assuming that **10,000** tickets are to be issued and sold, what is a **fair price** to pay for a ticket?

- **Solution:** Let X be a random variable denoting the amount of money to be won on a ticket. The various values of X together with their probabilities are shown in Table

x (dollars)	5	25	100	0
$P(X = x)$	0.02	0.002	0.0005	0.9775

$E(X) = (5)(0.02) + (25)(0.002) + (100)(0.0005) + (0)(0.9775) = 0.2$ or 20 cents. Thus, the **fair price** to pay for a ticket is **20 cents**

✓ **Example 2**: A continuous random variable X has probability density given by

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Find (a) $E(X)$, (b) $E(X^2)$.

- **Solution:**

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^{\infty} x(2e^{-2x})\,dx = 2\int_0^{\infty} xe^{-2x}\,dx$$

$$= 2\left[(x)\left(\frac{e^{-2x}}{-2}\right) - (1)\left(\frac{e^{-2x}}{4}\right)\right]\Bigg|_0^{\infty} = \frac{1}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = 2\int_0^{\infty} x^2 e^{-2x}\,dx$$

$$= 2\left[(x^2)\left(\frac{e^{-2x}}{-2}\right) - (2x)\left(\frac{e^{-2x}}{4}\right) + (2)\left(\frac{e^{-2x}}{-8}\right)\right]\Bigg|_0^{\infty} = \frac{1}{2}$$

Variance and standard deviation

✓ **Example 3**: Find **(a) the variance**, **(b)** the **standard deviation** for the random variable of **example 2**.

- **Solution:**

a) Then the variance is

$$\mathrm{Var}(X) = E[(X-\mu)^2] = E\left[\left(X - \frac{1}{2}\right)^2\right] = \int_{-\infty}^{\infty}\left(x - \frac{1}{2}\right)^2 f(x)\,dx$$

$$= \int_0^{\infty}\left(x - \frac{1}{2}\right)^2 (2e^{-2x})\,dx = \frac{1}{4}$$

Another method

$$\mathrm{Var}(X) = E[(X-\mu)^2] = E(X^2) - [E(X)]^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

b)

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Moments and moment generating functions

✓ **Example 4**: A random variable ***X*** has density function given by

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the **moment generating function**

- **Solution:**

$$\begin{aligned} M(t) &= E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \\ &= \int_0^{\infty} e^{tx}(2e^{-2x})\,dx = 2\int_0^{\infty} e^{(t-2)x}\,dx \\ &= \frac{2e^{(t-2)x}}{t-2}\bigg|_0^{\infty} = \frac{2}{2-t}, \quad \text{assuming } t < 2 \end{aligned}$$

✓ **Example 5** Find the **first four moments**

- (a) about the **origin**,
- (b) about the **mean**,

For a random variable ***X*** having density Function

$$f(x) = \begin{cases} 4x(9 - x^2)/81 & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

- **Solution:**

$$\mu_1' = E(X) = \frac{4}{81}\int_0^3 x^2(9 - x^2)\,dx = \frac{8}{5} = \mu$$

$$\mu_2' = E(X^2) = \frac{4}{81}\int_0^3 x^3(9 - x^2)\,dx = 3$$

$$\mu_3' = E(X^3) = \frac{4}{81}\int_0^3 x^4(9 - x^2)\,dx = \frac{216}{35}$$

$$\mu_4' = E(X^4) = \frac{4}{81}\int_0^3 x^5(9 - x^2)\,dx = \frac{27}{2}$$

$$\mu_1 = 0$$

$$\mu_2 = 3 - \left(\frac{8}{5}\right)^2 = \frac{11}{25} = \sigma^2$$

$$\mu_3 = \frac{216}{35} - 3(3)\left(\frac{8}{5}\right) + 2\left(\frac{8}{5}\right)^3 = -\frac{32}{875}$$

$$\mu_4 = \frac{27}{2} - 4\left(\frac{216}{35}\right)\left(\frac{8}{5}\right) + 6(3)\left(\frac{8}{5}\right)^2 - 3\left(\frac{8}{5}\right)^4 = \frac{3693}{8750}$$

✓ **Example 6:** Find the **characteristic function** of the random variable **X** having density function given by

$$f(x) = \begin{cases} 1/2a & |x| < a \\ 0 & \text{otherwise} \end{cases}$$

- **Solution:**

  The characteristic function is given by

$$E(e^{i\omega X}) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx = \frac{1}{2a}\int_{-a}^{a} e^{i\omega x}\, dx$$
$$= \frac{1}{2a}\frac{e^{i\omega x}}{i\omega}\Bigg|_{-a}^{a} = \frac{e^{ia\omega} - e^{-ia\omega}}{2ia\omega} = \frac{\sin a\omega}{a\omega}$$

✓ **Example 7:** Find the coefficient of **skewness**, for the distribution of **example 5**.

- **Solution:**

From **Example 5** (b) we have

$$\sigma^2 = \frac{11}{25} \qquad \mu_3 = -\frac{32}{875}$$

$$\text{Coefficient of skewness} = \alpha_3 = \frac{\mu_3}{\sigma^3} = -0.1253$$