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Lect 12

Image Processing in Frequency domain

* Basic steps of image processing in frequency domain:

1- Transform the image to its frequency representation
2- perform image processing.
3- Compute inverse transform back to the spatial domain

* What do frequencies mean in an image processing?

- High frequencies: they correspond to pixel values that change rapidly across image (text, leaves, ...)
- Strong low frequencies: large scale features in the image (homogenous object that dominates the image).

Concepts:

- The Fourier Series:

Periodic functions can be expressed as the sum of sines and/or cosines of different frequencies each multiplied by a different coefficient.

- The Fourier Transform:

Functions that are not periodic, but with finite area under the curve can be expressed as the integral of sines and/or cosines multiplied by a weight function.

$$\boxed{F(u) = \int_{-\infty}^{\infty} f(x) \cdot e^{-j2\pi ux} \, dx}$$

"Fourier Transform"

Fourier Synthesis

Fourier Analysis

$$\boxed{f(x) = \int_{-\infty}^{\infty} F(u) \cdot e^{j2\pi ux} \, du}$$

"Inverse Fourier Transform"

The Fourier Transform can be easily extended by 2 (and n dimensions)

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot e^{-j2\pi(ux+vy)} \, dx\,dy.$$

and the inverse as :

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) \cdot e^{j2\pi(ux+vy)} \, du\, dv.$$

* The Discrete Fourier Transform (DFT) of one Variable

Since we are dealing with digital images,
for $f(x)$, $x = 0,1,\dots, M-1$ (1-D function), we have

$$F(u) = \sum_{x=0}^{M-1} f(x) \cdot e^{-j2\pi ux/M} \quad , \quad u = 0,1,\dots, M-1$$

and the inverse IDFT expression is :

$$f(x) = \frac{1}{M} \cdot \sum_{u=0}^{M-1} F(u) \cdot e^{j2\pi ux/M} \quad , \quad x = 0,1,\dots, M-1$$

For discrete functions, the DFT & IDFT always exist.
Using Euler's Equation : ($e^{jx} = \cos x + j \sin x$)

$$F(u) = \sum_{x=0}^{M-1} f(x) \left[\cos\left(\frac{2\pi ux}{M}\right) - j \sin\left(\frac{2\pi ux}{M}\right) \right], \text{ for } u = 0,\dots, M-1$$

- Each term of $F(u)$ is composed of all values of $f(x)$
- The values of u are the frequency domain
- each $F(u)$ is a frequency component of the transform.
- both the forward and inverse DFT are periodic, with period M, that is:

$$F(u) = F(u+kM) \quad \text{and} \quad f(x) = f(x+kM), \text{ where } k \text{ is integer.}$$

* The 2-D Discrete Fourier Transform :

Since our images are nothing more than 2-D discrete functions, we are interested in 2-D DFT.
Let $f(x,y)$, for $x = 0,1,2,\dots, M-1$ and $y = 0,1,\dots, N-1$, denote an $M \times N$ image, The 2-D discrete Fourier Transform (DFT) of f, denoted by $F(u,v)$ and given by the equation :

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \cdot e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} ,$$

for $u = 0,1,\dots, M-1$ and $v = 0,1,\dots, N-1$.

The $M \times N$ rectangular region defined by $u = 0,1,\dots, M-1$ and $u = 0,1,\dots, N-1$ is often referred to as the frequency Rectangle,
Clearly, the inverse IDFT is defined by :

$$f(x,y) = \frac{1}{MN} \cdot \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \cdot e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}, \text{ for } x = 0,1,\ldots,M$$

and $y = 0,1,\ldots, N-1$.

Thus given $F(u,v)$, we can obtain $f(x,y)$, The values of $F(u,v)$ sometimes referred to as DFT data.

The DFT data are complex numbers, and we can express them in terms of polar coordinates:

$$F(u,v) = |F(u,v)| \cdot e^{-j\theta(u,v)}$$

where the magnitude or spectrum is denoted by

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{\frac{1}{2}}$$ and the

phase angle by $= \theta(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$.

R: real part, I - Imaginary part.

The value of transform at the origin of the frequency domain ($F(0,0)$) is called the dc component of the Fourier Transform

$$F(0,0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$, this is nothing more than the MN times the average value of $f(x,y)$ gray scale of the image.

Note.1: in Matlab: $f(1,1)$ corresponds to the mathematical quantity $f(0,0)$, and $F(1,1)$ to $F(0,0)$.

Note.2: the principal method of visually analyzing a transform is to compute its spectrum (magnitude) and display it as an image.

- Power spectrum $P(u,v) = |F(u,v)|^2$

$$= R^2(u,v) + I^2(u,v)$$, we can display $|F(u,v)|$ or $P(u,v)$,

Some properties of the 2-D DFT

1. Translation and Rotation:

$$f(x,y) \cdot e^{j2\pi\left(\frac{u_0 x}{M} + \frac{v_0 y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)$$

and :

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v) \cdot e^{-j2\pi\left(\frac{x_0 u}{M} + \frac{y_0 v}{N}\right)}.$$

multiplying $f(x,y)$ by the exponential shifts the origin of the DFT to (u_0, v_0), and multiplying $F(u,v)$ by the negative of that exponential shifts the origin of $f(x,y)$ to (x_0, y_0).

translation has no effect on the magnitude of $F(u,v)$.

using the Polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad u = \omega\cos\varphi, \quad v = \omega\sin\varphi$$

results in the following transform pair :

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

which indicates the rotating $f(x,y)$ by an angle θ_0 rotates $F(u,v)$ by the same angle, Conversely, rotating $F(u,v)$ rotates $f(x,y)$ by the same angle.

example :

for $u_0 = M/2$ and $v_0 = N/2$ $\left[\overset{\text{old origin}}{f(0,0)} \longrightarrow \overset{\text{new origin}}{f(\tfrac{M}{2}, \tfrac{N}{2})}\right]$

$$e^{-j2\pi\left(\frac{u_0 x}{M} + \frac{v_0 y}{N}\right)} = e^{-j\pi(x+y)} = (-1)^{x+y}$$

$$\Rightarrow \mathcal{F}\left(f(x,y) \cdot (-1)^{x+y}\right) = F\left(u - \tfrac{M}{2}, v - \tfrac{N}{2}\right)$$

- In image processing, it is common to multiply the input image by $(-1)^{x+y}$ prior to computing $F(u,v)$, this has the effect of centering the transform, since $F(0,0)$ is now located at $u = M/2$, $v = N/2$,

2. Periodicity:

$$F(u,v) = F(u + k_1 \cdot M, v) = F(u, v + k_2 \cdot N) = F(u + k_1 M, v + k_2 \cdot N)$$

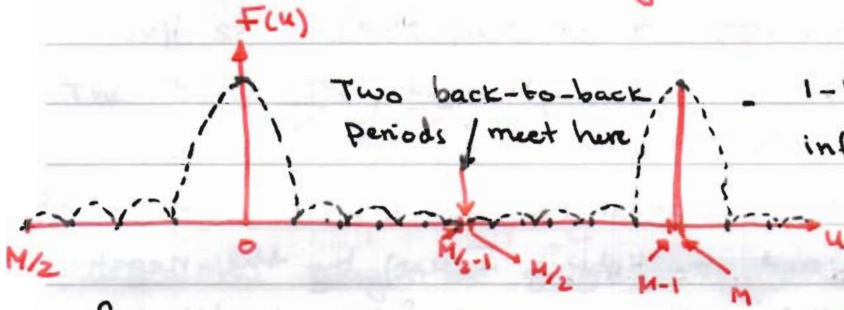
and .

$$f(x,y) = f(x + k_1 \cdot M, y) = f(x, y + k_2 \cdot N) = f(x + k_1 \cdot M, y + k_2 \cdot M)$$

where k_1, k_2 - are integers.

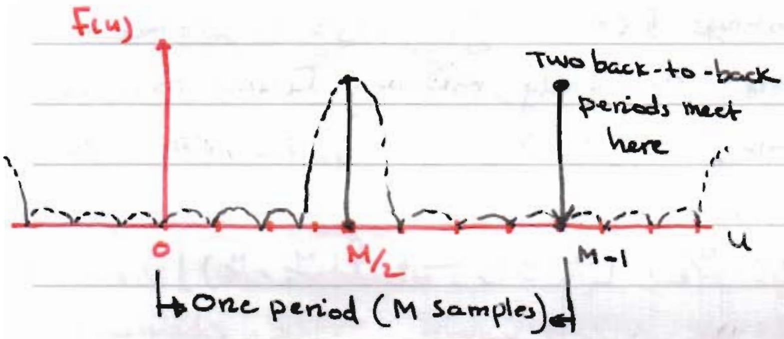
The periodicity of the transform and its inverse are important issues in the implementation of the image:

a- 1-D case periodicity:



1-D DFT showing an infinite number of period, the transformed data in the interval from 0 to M-1 Consists of two back-to-back half periods meeting at point M/2.

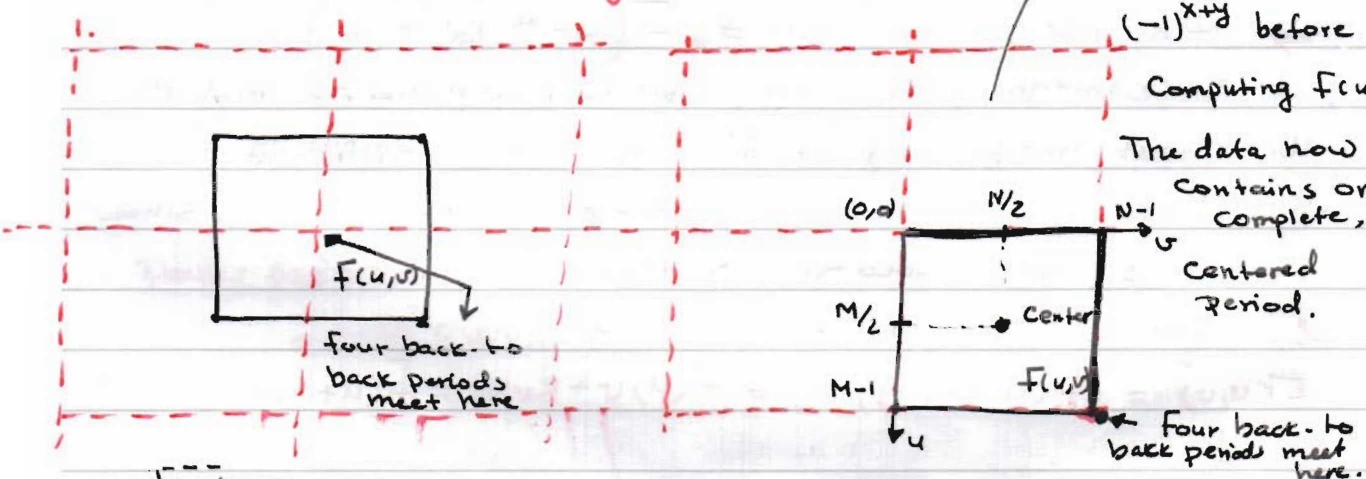
For display and filtering processing, it must be a complete period of the transform in which the data are contiguous, We use translation property to shift the origin to $F(M/2, N/2)$.



* shifted DFT obtained by multiplying $f(x)$ by $(-1)^x$ before computing $F(u)$.

b- 2-D case Periodicity:

Shifted DFT obtained by multiplying $f(x,y)$ by $(-1)^{x+y}$ before Computing $F(u,v)$, The data now contains one complete, Centered Period.



= Periods of the DFT = M×N data array, F(u,v)

3. Conjugate symmetric :

if $f(x,y)$ is real, its Fourier transform is conjugate symmetric about the image, that is

$F(u,v) = F^*(-u,-v)$, which implies that the Fourier spectrum also is symmetric about the origin :

$$|F(u,v)| = |F(-u,-v)|$$

4. The 2-D Convolution Theorem :

2-D Circular convolution equation :

$$f(x,y) * h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, h(x-m, y-n)$$

for $x = 0, 1, \ldots, M-1$ and $y = 0, 1, \ldots, N-1$,

the 2-D Convolution theorem is given by the expressions

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v) \cdot H(u,v) \qquad (1)$$

and, conversely,

$$f(x,y) \cdot h(x,y) \Leftrightarrow F(u,v) * H(u,v) \qquad (2)$$

where F and H are the DFT of f and h respectively.

equation 1 states that the inverse DFT of the product $F(u,v)\,H(u,v)$ yields $f(x,y) * h(x,y)$, the 2-D spatial convolution of f and h,

Similary, the DFT of the spatial convolution yields the product of the transforms in the frequency domain.

Equation 1 is the foundation of the linear filtering, and it is the basis for all the filtering techniques.

Summary of 2-D Discrete Fourier Transform Properties.

1) Discrete Fourier transform (DFT) of $f(x,y)$

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \cdot e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

2) Inverse DFT (IDFT) of $F(u,v)$

$$f(x,y) = \frac{1}{MN} \cdot \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \cdot e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)}$$

3) Polar Representation :

$$F(u,v) = |F(u,v)| \cdot e^{j\phi(u,v)}$$

4) Spectrum : $|F(u,v)| = \left[R^2(u,v) + I^2(u,v) \right]^{\frac{1}{2}}$

$R = Real(F)$, $I = Imag(F)$

5) Phase angle : $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)} \right]$

6) Power spectrum:

$$P(u,v) = |F(u,v)|^2$$

7) Average Value : $\bar{f}(x,y) = \frac{1}{MN} \cdot \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) = \frac{1}{MN} \cdot F(0,0)$

8) Periodicity (k_1 and k_2 are integers) :

$$F(u,v) = F(u + k_1 \cdot M, v) = F(u, v + k_2 \cdot N) = F(u + k_1 \cdot M + v + k_2 \cdot N)$$

$$f(x,y) = f(x + k_1 \cdot M, y) = f(x, y + k_2 \cdot N) = f(x + k_1 \cdot M, y + k_2 \cdot N)$$

9) Convolution :

$$f(x,y) * h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

10) Correlation :

$$f(x,y) \circ h(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m,n) h(x+m, y+n)$$

11) Translation (general) :

$$f(x,y) \cdot e^{j2\pi(u_0 x/M + v_0 y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v) \cdot e^{-j2\pi(u x_0/M + v y_0/N)}$$

12) Translation to center of the frequency rectangle $(M/2, N/2)$.

$$f(x,y) \cdot (-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$$

$$f(x - M/2, y - N/2) \Leftrightarrow F(u,v) \cdot (-1)^{u+v}$$

13) Rotation :

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

$x = r\cos\theta$, $y = r\sin\theta$, $u = \omega\cos\varphi$, $v = \omega\sin\varphi$

14) Convolution Theorem :

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v) \cdot H(u,v)$$

$$f(x,y) \cdot h(x,y) \Leftrightarrow F(u,v) * H(u,v)$$

15) Correlation Theorem :

$$f(x,y) \circ h(x,y) \Leftrightarrow F^*(u,v) \, H(u,v)$$

$$f^*(x,y) \cdot h(x,y) \Leftrightarrow F(u,v) \circ H(u,v).$$

Procedure for filtering in the frequency domain

1) multiply the image by $(-1)^{x+y}$ to center the transform.
2) Compute the DFT for $f(x,y)$ and $g(x,y)$ to get $F(u,v)$ of the resulting image (from 1) and the spectrum of the filter $G(u,v)$.
3) multiply $F(u,v)$ by a filter $G(u,v)$.
4) Comupte the inverse DFT transform $H(u,v)$.
5) obtain the real part of 4. ($Real(H(u,v))$)
6) multiply the result by $(-1)^{x+y}$.

Example : Smooth an image with a Gaussian Kernel

- Traditionally, we just convolve the image with a Gaussian Filter Mask (KerneL).

$$\boxed{\text{Image}} * \boxed{\text{Gaussian Filter}} \Longrightarrow \boxed{\text{Smoothed image}}$$

- instead, we will perform multiplication in frequency domain to achieve the same effect.

1) multiply the input image by $(-1)^{x+y}$ to Center the transform.

$$\boxed{\text{original Image}} \xrightarrow{(-1)^{x+y}} \boxed{\text{Centered image}}$$

2) Compute the DFT $F(u,v)$ of the resulting image

$$\boxed{\text{Centered image}} \xrightarrow{\text{DFT}} \boxed{\text{Log Transform } \{F(u,v)\}}$$

Log operation : to scale the output.

3) Compute the DFT $G(u,v)$ of the Filter mask.

$$\boxed{\text{filter mask}} \xrightarrow{\text{DFT}} \boxed{\text{DFT } \{g(x,y)\}} \Rightarrow G(u,v)$$

4) multiply $F(u,v)$ by a fiter $G(u,v)$ (element-by-element)

$$\boxed{F(u,v)} \cdot \boxed{G(u,v)} \Longrightarrow \boxed{H(u,v)}$$

5) $\boxed{H(u,v)} \xrightarrow{\text{IDFT}} \boxed{h(x,y)}$ " Compute $h(x,y)$

6) obtain real part : $Real(h(x,y))$

7) multiply the result of 6 by $(-1)^{x+y}$