Digital System Design

Lecture 8

Boolean Algebra - II

Simplifying Logic Circuits

Objectives:

1. **Deriving of logical expression form truth tables.**
2. **Logical expression simplification methods:**
    a. **Algebraic manipulation.**
    b. **Karnaugh map (k-map).**

1. Deriving of logical expression from truth tables

Definitions:

Literal: ***Non-complemented*** or ***complemented*** version of a variable (A or $\overline{A}$).

Product term: a ***series of literals*** related to one another through the ***AND*** operator (Example:$A \cdot \overline{B} \cdot C$).

Sum term: A ***series of literals*** related to one another through the ***OR*** operator (Example:($A + \overline{B} + \overline{C}$).

SOP form: *(Sum –of-products form)*:

- The ***logic-circuit simplification*** require the logic expression to be in ***SOP*** form , for example:

$$AB + \overline{A}B\overline{C} + \overline{C}\overline{D}$$

POS form ***(product-of-sum form):***

- This form ***sometimes*** used in logic circuit, example:

$$(A + C) \cdot (C + \overline{D}) \cdot (\overline{B} + C)$$

- The methods of circuit simplification and design that will be used are based on ***SOP*** form.

Canonical and standard form

- ***Product terms*** that consist of the variables of function are called "***Canonical product terms***" or "***Minterms***".
- The term $AB\overline{C}$ is a ***minterm*** in a ***three variable logic function***, but will be a non-minterm in a four variable logic function.
- ***Sum terms*** which contain all the variables of a Boolean function are called "***Canonical sum terms***" or ***Maxterms***".

Example: $A + \overline{B} + C$ is a ***maxterm*** in a ***three variable logic function.***

- for two variables A and B, there are four combination: $\overline{A}\overline{B}, \overline{A}B, A\overline{B}, AB$ called minterms or standard products

A	B	Minterm
0	0	$\overline{A}\overline{B}$
0	1	$\overline{A}B$
1	0	$A\overline{B}$
1	1	AB

- for n variables there are 2^n minterms

Example: *minterms and maxterms for 3 variables*:

inputs			Minterm	Designation	Maxterm	Designation
A	B	C				
0	0	0	$\overline{A}\overline{B}\overline{C}$	m0	$A+B+C$	M0
0	0	1	$\overline{A}\overline{B}C$	m1	$A+B+\overline{C}$	M1
0	1	0	$\overline{A}B\overline{C}$	m2	$A+\overline{B}+C$	M2
0	1	1	$\overline{A}BC$	m3	$A+\overline{B}+\overline{C}$	M3
1	0	0	$A\overline{B}\overline{C}$	m4	$\overline{A}+B+C$	M4
1	0	1	$A\overline{B}C$	m5	$\overline{A}+B+\overline{C}$	M5
1	1	0	$AB\overline{C}$	m6	$\overline{A}+\overline{B}+C$	M6
1	1	1	ABC	m7	$\overline{A}+\overline{B}+\overline{C}$	M7

Two important properties:

- *Any Boolean function can be expressed as a sum of minterms.*
- *Any Boolean function can be expressed as a product of minterms.*

Example: ***Majority function: (for 3 variables)***

Output is one whenever majority of inputs is 1

inputs			**Output**	**Minterms SOP Form**
A	*B*	*C*	**F**	
0	0	0	0	------
0	0	1	0	------
0	1	0	0	------
0	1	1	1	$\bar{A}BC$
1	0	0	0	------
1	0	1	1	$A\bar{B}C$
1	1	0	1	$AB\bar{C}$
1	1	1	1	ABC

- Four product terms, because, there are 4 rows with a 1 output.
- ***Final expression***:

$$F = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

inputs			**Output**	**Maxterms POS Form**
A	*B*	*C*	**F**	
0	0	0	0	$A + B + C$
0	0	1	0	$A + B + \bar{C}$
0	1	0	0	$A + \bar{B} + C$
0	1	1	1	------
1	0	0	0	$\bar{A} + B + C$
1	0	1	1	------
1	1	0	1	------
1	1	1	1	------

- Four sum terms, because, there are 4 rows with a 0 output.
- ***Final expression***:

$$F = (A + B + C) \cdot (A + B + \bar{C}) \cdot (A + \bar{B} + C) \cdot (\bar{A} + B + C)$$

Derivation of logical expression form truth tables

Summary

- ✓ *Sum-of-product (SOP) form.*
- ✓ *Product-of- sums (POS) form.*
  - ***SOP form :***

- Write an ***AND*** term for each input combination that produces a 1 output.
- Write the variable if its value is 1 , complement otherwise.
- ***OR*** the ***AND*** terms to get the final expression.

2. logical expression simplification methods:

Two basic methods:

a) Algebraic manipulation:

- Use ***Boolean laws*** to simplify the expression: (difficult to use and don't know if you have the simplified form.

b) Karnaugh map (k-map) method:

- ***Graphical*** method.
- ***Ease*** to use.
- Can be used to simplify logical expressions ***with a few variables***.

a) Algebraic manipulation

Example 1:

Design a logic circuit that has three inputs, A, B and C and whose output will be high only when a majority of the input are high .(complete design procedure).

Solution:

Step 1: ***set up the truth table*** (see previous section)

Step 2: ***write the AND term for each case where the output is a 1*** :(see previous section)

Step 3: ***write the sum-of-products expression for the output***

$$F = \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$$

Step 4: ***simplify the output expression***

- Find the ***common*** term(s): ABC -common term.
- Use the common term (ABC) to ***factor*** with other terms:

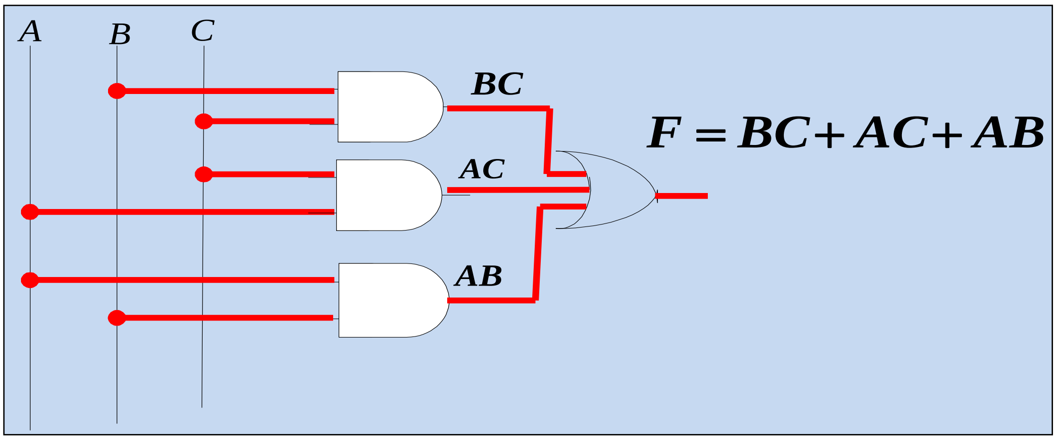
$$F = \overline{A}BC + ABC + A\overline{B}C + ABC + AB\overline{C} + ABC$$

- Factoring the appropriate pairs of terms:

$$F = BC(\bar{A} + A) + AC(B + \bar{B}) + AB(\bar{C} + C)$$

$$F = BC + AC + AB$$

Step 5: ***implement the circuit for the final expression.***



Example 2: ***simplify the Boolean function.***

$$F = AB + \bar{A}C + BC$$

Solution:

$F = AB + \bar{A}C + BC.1$ ***(Common term)***

$= AB + \bar{A}C + BC(A + \bar{A})$

$= AB + ABC + \bar{A}C + \bar{A}BC$

$= AB(1 + C) + \bar{A}C(1 + B)$

$= AB + \bar{A}C$ *(The result)*

Example 3: ***simplify the following Boolean function to a minimum number of literals.***

$$X = BC + A\bar{C} + AB + BCD$$

Solution:

$X = BC(1 + D) + A\bar{C} + AB$

$= BC + A\bar{C} + AB = BC + A(\bar{C} + B)$

Example 4: ***simplify the expression.***

$$\boldsymbol{Z = A\bar{B}\bar{C} + A\bar{B}C + ABC}$$

Solution: *(the expression is in SOP form)*

$$\boldsymbol{Z = A\bar{B}(\bar{C} + C) + ABC}$$

$$\boldsymbol{= A\bar{B}(1) + ABC = A\bar{B} + ABC}$$

$$\boldsymbol{Z = A(\bar{B} + BC) = A(\bar{B} + B)(\bar{B} + C) = A(\bar{B} + C)}$$

Example 5: ***simplify the expression***.

$$\boldsymbol{Z = \bar{A}C\left(\overline{\bar{A}BD}\right) + \bar{A}B\bar{C}\bar{D} + A\bar{B}C}$$

Solution:

- ✓ *First, use Demorgan's theorem on the first term:*

$$\boldsymbol{Z = \bar{A}C\left(\bar{\bar{A}} + \bar{B} + \bar{D}\right) + \bar{A}B\bar{C}\bar{D} + A\bar{B}C}$$

$$\boldsymbol{Z = \bar{A}C(A + \bar{B} + \bar{D}) + \bar{A}B\bar{C}\bar{D} + A\bar{B}C}$$

- ✓ *Multiply, we get:*

$$\boldsymbol{Z = \bar{A}CA + \bar{A}C\bar{B} + \bar{A}C\bar{D} + \bar{A}B\bar{C}\bar{D} + A\bar{B}C}$$

($\bar{A}A = 0$) then

$$\boldsymbol{Z = \bar{A}\bar{B}C + \bar{A}C\bar{D} + \bar{A}B\bar{C}\bar{D} + A\bar{B}C}$$

- ✓ *Check for the largest common factor between any two or more product terms:*

$$\boldsymbol{Z = \bar{B}C(\bar{A} + A) + \bar{A}\bar{D}\,(C + B\bar{C})}$$

$$\boldsymbol{\bar{A} + A \rightarrow 1 \qquad C + B\bar{C} \rightarrow C + B}$$

- ✓ *We get:*

$$\boldsymbol{Z = \bar{B}C + \bar{A}\bar{D}\,(B + C)}$$

- ✓ *The result can be obtained with other choices*

Example 6: ***Find the complement of the following Boolean function:***

$$Z = (B\overline{C} + \overline{A}D)(\,A\overline{B} + C\overline{D})$$

Solution:

- ✓ *Use Demorgan's theorem :*

$$\overline{Z} = \overline{(B\overline{C} + \overline{A}D)(\,A\overline{B} + C\overline{D})}$$

$$= \overline{(B\overline{C} + \overline{A}D)} + \overline{(\,A\overline{B} + C\overline{D})}$$

$$= \overline{B\overline{C}} \cdot \overline{\overline{A}D} + \overline{A\overline{B}} \cdot \overline{C\overline{D}}$$

$$= \left(\overline{B} + \overline{\overline{C}}\right) \cdot \left(\overline{\overline{A}} + \overline{D}\right) + \left(\overline{A} + \overline{\overline{B}}\right)\left(\overline{C} + \overline{\overline{D}}\right)$$

$$= \ (\overline{B} + C) \cdot (A + \overline{D}) + (\overline{A} + B) \cdot (\overline{C} + D)$$

- ✓ *It's not in standard form*

Example 7:

Express the following function in a sum of minterms and product of maxterms.

$$Z = (AB + C)(B + AC)$$

a) ***Sum of minterms****:*

- ✓ *Multiply, we get:*

$$Z = ABB + AABC + BC + ACC$$

$$= AB + ABC + BC + AC$$

$$= ABC + AB(C + \overline{C}) + BC(A + \overline{A}) + AC(B + \overline{B})$$

$$= ABC + ABC + AB\overline{C} + ABC + \overline{A}BC + ABC + A\overline{B}C$$

$$Z = \ AB\overline{C} + ABC + \overline{A}BC + A\overline{B}C$$

m6 m7 m3 m5

$C + \overline{C} = 1$
$A + \overline{A} = 1$
$B + \overline{B} = 1$
$A + A = A$

- ✓ *So, we can write:*

$$Z(A, B, C) = \sum m(3, 5, 6, 7)$$

inputs			Minterm	z
A	B	C		
0	0	0	m0	0
0	0	1	m1	0
0	1	0	m2	0
0	1	1	m3	1
1	0	0	m4	0
1	0	1	m5	1
1	1	0	m6	1
1	1	1	m7	1

Sum of minterms

b) Sum of maxterms

- ✓ *Using distributive law:*

$$(A + BC) = (A + B)(A + C)$$

- ✓ *In the*

$$Z = (AB + C)(B + AC)$$

- ✓ *We get:*

$$= (A + C)(B + C)(B + A)(B + C)$$

$$= (A + B)(B + C)(C + A)$$

$$Z = (A + B)(B + C)(C + A)$$

  - *In* $A + B$ *we need* C
  - *In* $B + C$ *we need* A
  - *In* $C + A$ *we need* B

- ✓ *We can write*

$$(A + B) = (A + B + 0) = (A + B + C\bar{C})$$

$$(A + B) = (A + B + C)(A + B + \bar{C})$$

$$(B + C) = (B + C + 0) = (B + C + A)(B + C + \bar{A})$$

$$(C + A) = (C + A + B)(C + A + \bar{B})$$

- ✓ *Substitute all of these term in* Z *,we get:*

$$Z = (A + B)(B + C)(C + A)$$

$$= (A+B+C)(A+B+\overline{C})\,(B+C+A)(B+C+\overline{A})\,(C+A+B)(C+A+\overline{B})$$

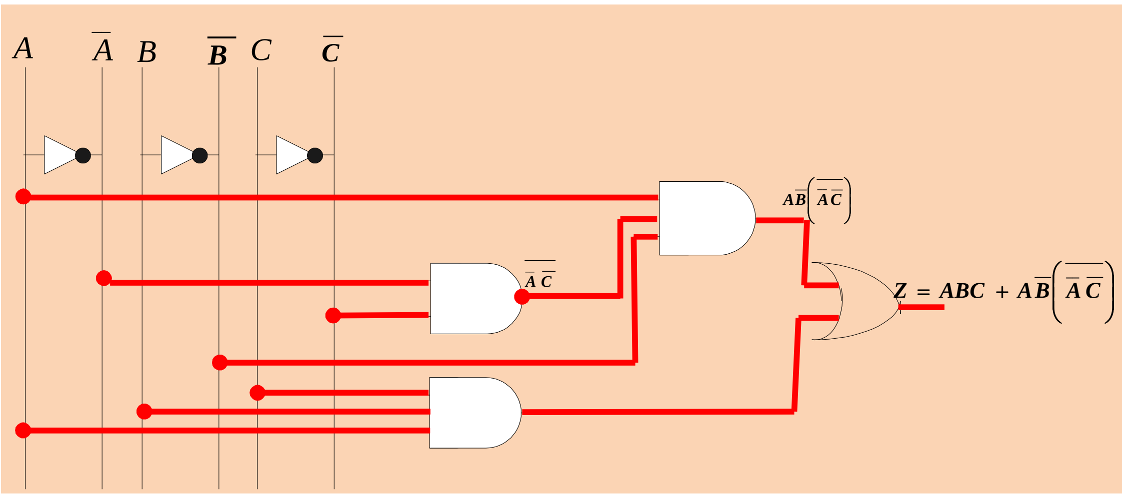
$$Z = (A+B+C)(A+B+\overline{C})\,(B+C+\overline{A})(C+A+\overline{B})$$

$$Z = (A+B+C)(A+B+\overline{C})\,(\overline{A}+B+C)(A+\overline{B}+C)$$

$$Z = M0 \cdot M1 \cdot M4 \cdot M2$$

inputs			Z	Maxterm
A	B	C		
0	0	0	0	M0
0	0	1	0	M1
0	1	0	0	M2
0	1	1	1	M3
1	0	0	0	M4
1	0	1	1	M5
1	1	0	1	M6
1	1	1	1	M7

***Example 8**: simplify the logic circuit shown in the following figure.*



Solution:

- *The output of the circuit is*

$$Z = abc + a\overline{b}.(\overline{\overline{a}\,\overline{c}})$$

- *Using Demorgan's law and multiply out all terms:*

$$Z = abc + a\overline{b}(\overline{\overline{a}} + \overline{\overline{c}})$$

$$Z = abc + a\bar{b}(a + c)$$

$$Z = abc + a\bar{b}a + a\bar{b}c$$

$$Z = abc + a\bar{b}a + a\bar{b}c$$

$$Z = abc + a\bar{b} + a\bar{b}c$$

SOP Form

$$Z = abc + a\bar{b} + a\bar{b}c$$

$$Z = ac(b + \bar{b}) + a\bar{b}$$

$$Z = ac.1 + a\bar{b} = ac + a\bar{b} = a(c + \bar{b})$$