Philadelphia University
Faculty of Engineering

Marking Scheme

Examination Paper

Department of Communication & Electronics Engineering

Probability and Random Variables

(650364)

Final Exam First semester Date: 21/01/2020

Section 1

Weighting 40% of the module total

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Marking Scheme

Probability and Random Variables (650364)

The presented exam questions are organized to overcome course material through 6 questions.
The *all questions* are compulsory requested to be answered.

Marking Assignments

Question 1: This question is attributed with 10 marks if answered properly, the answer is the following:

1) A box of **8 marbles** has **4 red**, **2 green**, and **2 blue** marbles. If you select one marble, what is the probability that it is a **red or blue** marble?
   - **a)** 0.60 **b)** **0.75**
   - **c)** 6.00 **d)** 0.80
2) Compute $\binom{9}{4}$
   - **a)** 84 **b)** **126**
   - **c)** 3024 **d)** 15,120
3) About the independent events **A** and **B** it is known that $P(A|B) = 0.2$ and $P(B|A) = 0.5$. Compute the probability $P(A \cup B)$
   - **a)** 0.5 **b)** 0.7
   - **c)** 0.1 **d)** **0.6**
4) In a sample of **10** telephones, **4** are defective. If **3** are selected at random and tested, what is the probability that all will be nondefective?
   - **a)** 1/30 **b)** 8/125
   - **c)** **1/6** **d)** 27/125
5) The corresponding **z** value (**standard normal value – Z score**) for a value of **9** if the **mean** of a variable is **12** and the standard deviation is **4**.
   - **a)** **-0.75** **b)** 0.75
   - **c)** 0.5 **d)** -0.5
6) Which of the following is **NOT** required of a **binomial distribution**

a)	**Each trial has exactly two outcomes.**
b)	**There is a fixed number of trials.**
c)	**The probability of success remains fixed for all trials.**
d)	**There are more than 30 trials.**

7) Formula of **variance** of **uniform distribution** is as
   - **a)** $(b - a)^2 / 6$ **b)** $(b + a)^2 / 12$
   - **c)** $(b - a)^3 / 8$ **d)** $(b + a)^2 / 2$
8) The number of traffic accidents that occur on a particular stretch of road during a month follows a **Poisson distribution** with a **mean** of **7.9**. Find the probability of observing exactly **five** accidents on this stretch of road next month.
   - **a)** **0.095067** **b)** 1.727754
   - **c)** 18.672798 **d)** 1.027438
9) Consider the given discrete probability distribution. Find the probability that x equals **5**.

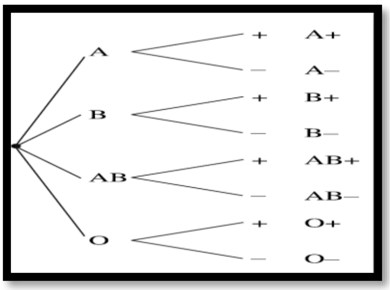
x	2	5	6	9
$P(x)$	0.09	?	0.23	0.21

   - **a)** 0.53 **b)** 2.65
   - **c)** **0.47** **d)** 2.35
10) If **X** is a discrete random variable and **f(x)** is the probability function of **X**, then the expected value of this random variable is equal to:
   - **a)** $\sum f(x)$ **b)** $\sum[x + f(x)]$
   - **c)** $\sum f(x) + x$ **d)** $\sum x f(x)$

Question 2: This question is attributed with 5 marks if answered properly, the answer is the following:

a) *(2 marks)*

Solution



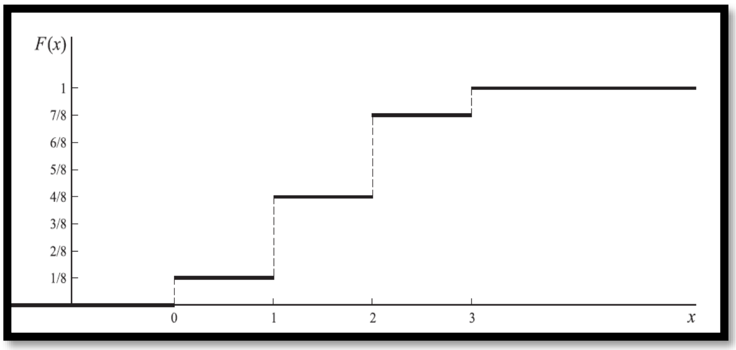
b) *(3 marks)*

Solution

$$P(\text{exactly } x \text{ boys}) = P(X = x) = {}_3C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{3-x} = {}_3C_x\left(\frac{1}{2}\right)^3$$

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 1/8 & 0 \le x < 1 \\ 1/2 & 1 \le x < 2 \\ 7/8 & 2 \le x < 3 \\ 1 & 3 \le x < \infty \end{cases}$$



Question 3: This question is attributed with 5 marks if answered properly, the answer is the following:

Solution

a) *(1.5 marks)*

$$P(X \le 2, Y \le 4) = P_{XY}(1,2) + P_{XY}(1,4) + P_{XY}(2,2) + P_{XY}(2,4)$$
$$= \frac{1}{12} + \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{3}{8}.$$

b) *(1.5 marks)*

$$P_X(x) = \begin{cases} \frac{1}{6} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{11}{24} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{1}{2} & y = 2 \\ \frac{1}{4} & y = 4 \\ \frac{1}{4} & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

c) ***(1.5 marks)***

$$\begin{aligned} P(Y = 2|X = 1) &= \frac{P(X = 1, Y = 2)}{P(X = 1)} \\ &= \frac{P_{XY}(1,2)}{P_X(1)} \\ &= \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}. \end{aligned}$$

d) X and Y are not independent ***(1.5 marks)***

$$P(X = 2, Y = 2) = \frac{1}{6} \neq P(X = 2)P(Y = 2) = \frac{3}{16}.$$

<u>Question 4:</u> This question is attributed with 8 marks if answered properly, the answer is the following:

a) ***(5 marks)***

Solution

i.

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^2 x\left(\frac{1}{2}x\right)dx = \int_0^2 \frac{x^2}{2}\,dx = \frac{x^3}{6}\bigg|_0^2 = \frac{4}{3}$$

ii.

$$E(3X^2 - 2X) = \int_{-\infty}^{\infty} (3x^2 - 2x)f(x)\,dx = \int_0^2 (3x^2 - 2x)\left(\frac{1}{2}x\right)dx = \frac{10}{3}$$

iii.

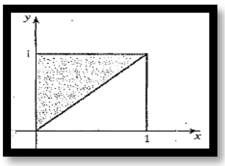
$$\sigma^2 = E\left[\left(X - \frac{4}{3}\right)^2\right] = \int_{-\infty}^{\infty}\left(x - \frac{4}{3}\right)^2 f(x)\,dx = \int_0^2\left(x - \frac{4}{3}\right)^2\left(\frac{1}{2}x\right)dx = \frac{2}{9}$$

$$\sigma = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}$$

b) ***(3 marks)***

Solution

The joint pdf is given to be a constant in the shaded region in Fig.



$$\iint f_{xy}(x,y)\,dx\,dy = \int_0^1\int_0^y k\,dx\,dy = \int_0^1 ky\,dy = \frac{k}{2} = 1 \Rightarrow k = 2$$

$$f_x(x) = \int f_{xy}(x,y)\,dy = \int_x^1 k\,dy = k(1 - x) \qquad 0 < x < 1$$

$$f_y(y) = \int f_{xy}(x, y)\,dx = \int_0^y k\,dx = ky \qquad 0 < y < 1$$

$$f_{x|y}(x \mid y) = \frac{f_{xy}(x, y)}{f_y(y)} = \frac{1}{y} \qquad 0 < x < y < 1$$

$$f_{y|x}(y \mid x) = \frac{f_{xy}(x, y)}{f_x(x)} = \frac{1}{1-x} \qquad 0 < x < y < 1$$

Question 5: This question is attributed with 6 marks if answered properly, the answer is the following:

Solution

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x, y)\,dx\,dy = \int_{x=0}^{4}\int_{y=1}^{5} x\left(\frac{xy}{96}\right)dx\,dy = \frac{8}{3}$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x, y)\,dx\,dy = \int_{x=0}^{4}\int_{y=1}^{5} y\left(\frac{xy}{96}\right)dx\,dy = \frac{31}{9}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy)f(x, y)\,dx\,dy = \int_{x=0}^{4}\int_{y=1}^{5} (xy)\left(\frac{xy}{96}\right)dx\,dy = \frac{248}{27}$$

$$E(2X + 3Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (2x + 3y)f(x, y)\,dx\,dy = \int_{x=0}^{4}\int_{y=1}^{5} (2x + 3y)\left(\frac{xy}{96}\right)dx\,dy = \frac{47}{3}$$

Question 6: This question is attributed with 6 marks if answered properly, the answer is the following:

a) ***(2 marks)***

Solution

Autocorrelation Function Properties:

1. $|R_{xx}(\tau)| \le R_{xx}(0)$
2. $R_{xx}(-\tau) = R_{xx}(\tau)$
3. $R_{xx}(0) = E[X^2(t)]$
4. If $E[X(t)] = \overline{X} \ne 0$ and $X(t)$ is ergodic with no periodic components, then $\lim_{|\tau|\to\infty} R_{xx}(\tau) = \overline{X}^2$
5. If $X(t)$ has a periodic component, then $R_{xx}(\tau)$ will have a periodic component with the same period.
6. If $X(t)$ is ergodic, zero mean, and has no periodic components, then $\lim_{|\tau|\to\infty} R_{xx}(\tau) = 0$
7. $R_{xx}(\tau)$ cannot have an arbitrary shape.

b) ***(4 marks)***

Solution

A random process is called wide-sense stationary (WSS) if the two following conditions are true

$$E[X(t)] = \overline{X} = \text{constant}$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X(t) = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta = 0$$

$$\begin{aligned} R_{XX}(t, t+\tau) &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\cos[\omega(t+\tau) + \theta]\, d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos\omega\tau + \cos(2\omega t + 2\theta + \omega\tau)]\, d\theta \\ &= \frac{A^2}{2}\cos\omega\tau \end{aligned}$$

The autocorrelation function

$$\begin{aligned} R_{xx}(t, t+\tau) &= E[A\cos(\omega_0 t + \Theta)A\cos(\omega_0 t + \omega_0\tau + \Theta)] \\ &= \frac{A^2}{2}E[\cos\omega_0\tau + \cos(2\omega_0 t + \omega_0\tau + 2\Theta)] \\ &= \frac{A^2}{2}\cos\omega_0\tau + \frac{A^2}{2}E[\cos(2\omega_0 t + \omega_0\tau + 2\Theta)] \\ &= \frac{A^2}{2}\cos\omega_0\tau \end{aligned}$$

Since

$$E[X(t)] = \text{constant}$$

And

$$R_{XX}(t, t+\tau) = R_{XX}(\tau)$$

Then the random process is wide-sense stationary