Special Description & Transformation
Central Topic

**Problem**

Robotic manipulation, by definition, implies that parts and tools will be moving around in space by the manipulator mechanism. This naturally leads to the need of representing positions and orientations of the parts, tools, and the mechanism itself.

**Solution**

Mathematical tools for representing position and orientation of objects / frames in a 3D space.
Coordinate System 1/2
Description of a Position

The location of any point in can be described as a 3x1 \textit{position vector} in a reference coordinate system.

\[
\begin{bmatrix}
p_x \\
p_y \\
p_z \\
\end{bmatrix}
\]
### Description of an Orientation

The orientation of a body is described by attaching a coordinate system to the body \{B\} and then defining the relationship between the body frame and the reference frame \{A\} using the rotation matrix.

The rotation matrix describing frame \{B\} relative to frame \{A\}

\[
A_B R = [\hat{X}_B, \hat{Y}_B, \hat{Z}_B] = \begin{bmatrix}
 r_{11} & r_{12} & r_{13} \\
 r_{21} & r_{22} & r_{23} \\
 r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]
Description of an Frame

The information needed to completely specify where is the manipulator hand is a position and an orientation.

\[ \{B\} = \{A_R, A_P\}_{BORG} \]

The rotation matrix describing frame \{B\} relative to frame \{A\}

The origin of frame \{B\} relative to frame \{A\}
Mapping - Translated Frames

Assuming that frame \{B\} is only \textit{translated} (not rotated) with respect frame \{A\}. The position of the point can be expressed in frame \{A\} as follows.

\[ A_P = B_P + A_P_{BORG} \]
Mapping - Rotated Frames

Assuming that frame \( \{B\} \) is only rotated (not translated) with respect frame \( \{A\} \) (the origins of the two frames are located at the same point), the position of the point in frame \( \{B\} \) can be expressed in frame \( \{A\} \) using the rotation matrix as follows:

\[
\begin{align*}
\mathbf{A} \mathbf{P} &= \mathbf{B} \mathbf{P} \\
\mathbf{B} \mathbf{P} &= \mathbf{A} \mathbf{P} \mathbf{R}
\end{align*}
\]
Mapping - Rotated Frames - Inversion

Given: The rotation matrix from frame \( \{A\} \) to frame \( \{B\} \) - \( \mathbf{A}_B^A \) R
Calculate: The rotation matrix from frame \( \{B\} \) to frame \( \{A\} \) - \( \mathbf{B}_A^B \) R

\[
\mathbf{A}_B^A \mathbf{P} = \mathbf{B}_A^B \mathbf{P}
\]

\[
\mathbf{A}_B^A \mathbf{R}^{-1} \mathbf{A}_B^B \mathbf{P} = \mathbf{A}_B^A \mathbf{R}^{-1} \mathbf{A}_B^B \mathbf{P}
\]

\[
\mathbf{A}_B^A \mathbf{R}^{-1} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\mathbf{P} = \mathbf{I} \mathbf{P}
\]

\[
\mathbf{A}_B^A \mathbf{R}^{-1} \mathbf{A}_B^B \mathbf{P} = \mathbf{I} \mathbf{B}_A^B \mathbf{P} = \mathbf{B}_A^B \mathbf{P}
\]

\[
\mathbf{B}_A^B \mathbf{P} = \mathbf{A}_B^A \mathbf{R}^{-1} \mathbf{A}_B^B \mathbf{P}
\]

\[
\mathbf{B}_A^B \mathbf{P} = \mathbf{A}_B^A \mathbf{R}^{-1} \mathbf{A}_B^B \mathbf{P}
\]

\[
\mathbf{A}_B^A \mathbf{R}^{-1} = \mathbf{A}_B^A \mathbf{R}^T
\]

Orthogonal Coordinate system
Mapping - Rotated Frames - Example

Given:

\[ B_P = \begin{bmatrix} 0 & 0 \\ B_y & 2 \\ 0 & 0 \end{bmatrix} \]

\[ \theta = 30^\circ \]

Compute: \[ A_P \]

Solution:

\[ A_P = B_R \cdot B_P \]
Mapping - Rotated Frames - Example

\[ A_B R = \begin{bmatrix} \hat{X}_B, \hat{Y}_B, \hat{Z}_B \end{bmatrix} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
\[
A_P = A_R \begin{bmatrix}
 c \theta & -s \theta & 0 \\
 s \theta & c \theta & 0 \\
 0 & 0 & 1
\end{bmatrix} B_P = \begin{bmatrix}
 0.866 & -0.500 & 0.000 \\
 0.500 & 0.866 & 0.000 \\
 0.000 & 0.000 & 1.000
\end{bmatrix} \begin{bmatrix}
 0.000 \\
 2.000 \\
 0.000
\end{bmatrix} = \begin{bmatrix}
 -1.000 \\
 1.732 \\
 0.000
\end{bmatrix}
\]
Mapping - Rotated Frames - General Notation

The rotation matrices with respect to the reference frame are defined as follows:

\[
R_x(\gamma) = \begin{bmatrix}
1 & 0 & 0 \\
0 & c\gamma & -s\gamma \\
0 & s\gamma & c\gamma
\end{bmatrix}
\]

\[
R_y(\beta) = \begin{bmatrix}
c\beta & 0 & s\beta \\
0 & 1 & 0 \\
-s\beta & 0 & c\beta
\end{bmatrix}
\]

\[
R_z(\alpha) = \begin{bmatrix}
c\alpha & -s\alpha & 0 \\
s\alpha & c\alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
Mapping - Rotated Frames - Methods

- X-Y-Z Fixed Angles

  The rotations perform about an axis of a **fixed** reference frame

- Z-Y-X Euler Angles

  The rotations perform about an axis of a **moving** reference frame
Mapping - Rotated Frames - X-Y-Z Fixed Angles

Start with frame \{B\} coincident with a known reference frame \{A\}.

- Rotate frame \{B\} about \(\hat{X}_A\) by an angle \(\gamma\)
- Rotate frame \{B\} about \(\hat{Y}_A\) by an angle \(\beta\)
- Rotate frame \{B\} about \(\hat{Z}_A\) by an angle \(\alpha\)

**Note** - Each of the three rotations takes place about an axis in the fixed reference frame \{A\}
Mapping - Rotated Frames - X-Y-Z Fixed Angles

\[
_\mathcal{A}^B R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) =
\begin{bmatrix}
c\alpha & -s\alpha & 0 \\
s\alpha & c\alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
c\beta & 0 & s\beta \\
0 & 1 & 0 \\
-s\beta & 0 & c\beta
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & c\gamma & -s\gamma \\
0 & s\gamma & c\gamma
\end{bmatrix}
\]

\[
_\mathcal{A}^B R_{XYZ}(\gamma, \beta, \alpha) =
\begin{bmatrix}
c\alpha c\beta & c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\
s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\
-s\beta & c\beta s\gamma & c\beta c\gamma
\end{bmatrix}
\]
Mapping - Rotated Frames - X-Y-Z Fixed Angles

\[ \mathcal{A}_B^R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta \gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta \gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta c\gamma & c\beta s\gamma \end{bmatrix} \]

\[ \beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}) \quad \text{for} \quad -90^\circ \leq \beta \leq 90^\circ \]

\[ \alpha = \text{Atan2}(r_{21}/c\alpha, r_{11}/c\alpha) \]

\[ \gamma = \text{Atan2}(r_{32}/c\alpha, r_{33}/c\alpha) \]

\[ \beta = \pm 90^\circ \]

\[ \alpha = 0 \]

\[ \gamma = \text{Atan2}(r_{12}, r_{22}) \]
Atan2 - Definition

Four-quadrant inverse tangent (arctangent) in the range of

\[
\text{Atan} 2(y, x) = \tan^{-1}(y/x)
\]

For example

\[
\begin{align*}
\text{Atan}(+1,+1) &= 45^\circ \\
\text{Atan} 2(+1,+1) &= 45^\circ \\
\text{Atan}(-1,-1) &= 45^\circ \\
\text{Atan} 2(-1,-1) &= -135^\circ 
\end{align*}
\]
Mapping - Rotated Frames - Z-Y-X Euler Angles

Start with frame \{B\} coincident with a known reference frame \{A\}.

- Rotate frame \{B\} about \hat{Z}_A by an angle \(\alpha\)
- Rotate frame \{B\} about \hat{Y}_B by an angle \(\beta\)
- Rotate frame \{B\} about \hat{X}_B by an angle \(\gamma\)

**Euler Angles**

Note - Each rotation is performed about an axis of the moving reference frame \{B\}, rather than a fixed reference frame \{A\}.

Instructor: Jacob Rosen Ph.D.
Models of Robot Manipulation - EE 543 - Department of Electrical Engineering - University of Washington
Mapping - Rotated Frames - X-Y-Z Euler Angles

\[ A^B R_{X'Y'Z'}(\alpha, \beta, \gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c \alpha & -s \alpha & 0 \\ s \alpha & c \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c \beta & 0 & s \beta \\ 0 & 1 & 0 \\ -s \beta & 0 & c \beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c \gamma & -s \gamma \\ 0 & s \gamma & c \gamma \end{bmatrix} \]

\[ A^B R_{X'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} c \alpha c \beta & c \alpha s \beta s \gamma - s \alpha c \gamma & c \alpha s \beta c \gamma + s \alpha s \gamma \\ s \alpha c \beta & s \alpha s \beta s \gamma + c \alpha c \gamma & s \alpha s \beta c \gamma - c \alpha s \gamma \\ -s \beta & c \beta s \gamma & c \beta c \gamma \end{bmatrix} \]
Mapping - Rotated Frames

Fixed Angles Versus Euler Angles

$$\begin{align*}
&A_B R_{XYZ}(\gamma, \beta, \alpha) = A_B R_{XYZ}(\beta, \gamma, \alpha) = A_B R_{XYZ}(\gamma, \alpha, \beta) \\
&= A_B R_{XYZ}(\alpha, \beta, \gamma)
\end{align*}$$

Three rotations taken about fixed axes (Fixed Angles) yield the same final orientation as the same three rotation taken in an opposite order about the axes of the moving frame (Euler Angles)
Operator - Rotating Vector

- Rotational Operator - Operates on a vector \( \mathbf{A}_1 \) and changes that vector to a new vector \( \mathbf{B}_1 \), by means of a rotation \( \mathbf{R} \).

\[
\mathbf{A}_2 = \mathbf{R} \mathbf{A}_1 \]

- Note: The rotation matrix which rotates vectors through same the rotation \( \mathbf{R} \), is the same as the rotation which describes a frame rotated by \( \mathbf{R} \) relative to the reference frame.

\[
\mathbf{A}_2 = \mathbf{R} \mathbf{A}_1 \iff \mathbf{A}_1 = \mathbf{R}^{-1} \mathbf{A}_2
\]
Operator - Rotating Vector - Example

Given:

\[ A_P^1 = \begin{bmatrix} 0 \\ A_{P_{1y}} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \]

Compute: The vector $B_P^1$ obtained by rotating this vector about $\hat{Z}$ by 30 degrees

Solution:

\[
A_P^1 = R(30^\circ) \quad A_P^2 = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ A_{P_{1y}} \end{bmatrix} = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \end{bmatrix} = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}
\]
Operator - Rotating Vector - Example
Mapping - General Frames

Assuming that frame \{B\} is both *translated* and *rotated* with respect frame \{A\}. The position of the point expressed in frame \{B\} can be expressed in frame \{A\} as follows.

\[
\{B\} = \{B_R, P_{B\text{ORG}}\}
\]

\[
^A P = ^A B R \cdot ^B P + P_{B\text{ORG}}
\]

\[
^A P = ^A B T \cdot ^B P
\]
Mapping - Homogeneous Transform

The homogeneous transform is a 4x4 matrix casting the rotation and translation of a general transform into a single matrix. In other fields of study it can be used to compute perspective and scaling operations when the last raw is other then [0001] or the rotation matrix is not orthonormal.

\[
^A P = ^B R \cdot ^B P + ^A P_{BORG}
\]

\[
^A P = ^B T \cdot ^B P
\]
Homogeneous Transform - Special Cases

Translation

\[
\begin{bmatrix}
A \mathbf{T}_B^A & \mathbf{A P}_{BORGx} \\
A \mathbf{T}_B^A & \mathbf{A P}_{BORGy} \\
A \mathbf{T}_B^A & \mathbf{A P}_{BORGz} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Rotation

\[
\begin{bmatrix}
r_{11} & r_{12} & r_{13} & 0 \\
r_{21} & r_{22} & r_{23} & 0 \\
r_{31} & r_{32} & r_{33} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Homogeneous Transform - Example (1/3)

Given:

\[
^B P = \begin{bmatrix}
0 \\
B p_y \\
0
\end{bmatrix} = \begin{bmatrix} 0 \\
2 \\
0 \end{bmatrix}
\]

Frame \(\{B\}\) is rotated relative to frame \(\{A\}\) about \(\hat{Z}_A\) by 30 degrees, and translated 10 units in \(\hat{X}_A\) and 5 units in \(\hat{Y}_A\).

Calculate: The vector \(^A p\) expressed in frame \(\{A\}\).
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Homogeneous Transform - Example (3/3)

\[
^A P = ^A T_B ^B P = \begin{bmatrix}
^A P \\
1
\end{bmatrix} = \begin{bmatrix}
^A R \\
0 \\
0 \\
0 \\
1 \\
1
\end{bmatrix} \begin{bmatrix}
^A P_{BORG} \\
1
\end{bmatrix}
\]

\[
^A P = \begin{bmatrix}
0.866 & -0.500 & 0.000 & 10.0 \\
0.500 & 0.866 & 0.000 & 5.0 \\
0.000 & 0.000 & 1.000 & 0.0 \\
0.000 & 0.000 & 0.000 & 1
\end{bmatrix} \begin{bmatrix}
3.0 \\
7.0 \\
0.0 \\
1
\end{bmatrix} = \begin{bmatrix}
9.098 \\
12.562 \\
0.0 \\
1
\end{bmatrix}
\]
Transformation Arithmetic - Compound Transformations

Given: Vector $^CP$
Frame $\{C\}$ is known relative to frame $\{B\}$ - $^CT^B$
Frame $\{B\}$ is known relative to frame $\{A\}$ - $^AT^B$

Calculate: Vector $^AP$

\[
^BP = ^CT^CP \\
^AP = ^BT^BP \\
^AP = ^BT^CT^CP
\]
Transformation Arithmetic - Inverted Transformation

Given: Description of frame \{B\} relative to frame \{A\} - $^A_B T$ ($^A_B R, ^A_B P_{BORG}$)

Calculate: Description of frame \{A\} relative to frame \{B\} -

Homogeneous Transform $^B_A T$ ($^B_A R, ^B_A P_{AORG}$)

$^B_A R = ^A_B R^T$

$^B_A T = \begin{bmatrix} \begin{bmatrix} ^A_B R^T & -^A_B R^T A P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$

Note: $^B_A T = ^A_B T^{-1}$
Inverted Transformation - Example (1/2)

Given: Description of frame \{B\} relative to frame \{A\} - $^{A}T_{B} \left( ^{A}R, ^{A}P_{BORG} \right)$
Frame \{B\} is rotated relative to frame \{A\} about $\hat{Z}$ by 30 degrees, and translated 4 units in $\hat{X}$, and 3 units in $\hat{Y}$

Calculate: Homogeneous Transform $^{B}T_{A} \left( ^{B}R, ^{B}P_{AORG} \right)$
Inverted Transformation - Example (2/2)

\[
A^T_B = \begin{bmatrix}
  c \theta & -s \theta & 0 & A^P_{BORGX} \\
s \theta & c \theta & 0 & A^P_{BORGY} \\
0 & 0 & 1 & A^P_{BORGZ} \\
0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
  0.866 & -0.500 & 0.000 & 4.000 \\
  0.500 & 0.866 & 0.000 & 3.000 \\
  0.000 & 0.000 & 1.000 & 0.000 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A^T_B = \begin{bmatrix}
  A^R_B^T & -A^R_B^T A^P_{BORG} \\
  0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
  0.866 & 0.500 & 0.000 & -4.964 \\
  -0.500 & 0.866 & 0.000 & -0.598 \\
  0.000 & 0.000 & 1.000 & 0.000 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]
Operator - Transforming Vector

- Transformation Operator - Operates on a vector $A P_1$ and changes that vector to a new vector $B P_1$, by means of a rotation by $R$ and translation by $Q$

$$A P_2 = T A P_1$$

- Note: The matrix of the transform operator $T$ which rotates vectors by $R$ and translation by $Q$, is the same as the transformation matrix which describes a frame rotated by $R$ and translated by $Q$ relative to the reference frame

$$A P_2 = T A P_1 \iff A P = A T B P$$

Operator \hspace{1cm} Mapping
Homogeneous Transform - Summary of Interpretation

- As a general tool to represent a frame we have introduced the **homogeneous transformation**, a 4x4 matrix containing orientation and position information.

- **Three interpretation of the homogeneous transformation**

  1. **Description of a frame** - \( A_T^B \) describes the frame \( B \) relative to frame \( A \)

     \[
     A_T^B = \begin{bmatrix}
     A_R^B & A_{B\text{ORG}}^B \\
     0 & 0 & 0 & 1
     \end{bmatrix}
     \]

  2. **Transform mapping** - \( A_T^B \) maps \( B \) to \( A \)

     \[
     A_P^B = A_T^B \cdot B_P
     \]

  3. **Transform operator** - \( T \) operates on \( A_P^1 \) to create \( A_P^2 \)

     \[
     A_P^2 = A_T \cdot B_P
     \]
Transform Equations

Given: $^{U}_A T$, $^{A}_D T$, $^{U}_B T$, $^{C}_D T$

Calculate: $^{B}_C T$

$$^{U}_D T = ^{U}_A T ^A_D T$$
$$^{U}_D T = ^{U}_B T ^B_C T ^C_D T$$
$$^{U}_A T ^A_D T = ^{U}_B T ^B_C T ^C_D T$$
$$^{U}_B T ^{-1} ^A_T ^A_D T ^C_D T ^{-1} = ^{U}_B T ^{-1} ^B_T ^B_C T ^C_D T ^C_D T ^{-1}$$
$$^{B}_C T = ^{U}_B T ^{-1} ^A_T ^A_D T ^C_D T ^{-1}$$
Presentation

Give a 10 min presentation of a robot

Name
Application
Size - Height, Length, Weight,
Sensors
Power Source
Cost
Project Status
Web Site
Problem Set No. 1


- Exercises (pp. 60-67)
  - 2.1
  - 2.13
  - 2.27
  - 2.28
  - 2.29
  - 2.30