

# DISCRETE STRUCTURES

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# Discrete Structures<sup>†</sup>

Revision Notes and Problems  
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## Preface

These notes were prepared for students as a revision workbook and are not meant to substitute the in-class notes. No student is expected to really benefit from these notes unless they have regularly attended the lectures.

## Chapter 1

### Propositional Logic

Propositions, Logic Operators, Truth Tables, Tautologies, Quine's Method, Logical Equivalence, Normal Forms, Predicates and Quantifiers

## Chapter 2

### Methods of Proof

Direct Proof, Proof by Contrapositive, Proving Equivalence, Mathematical Induction

## Chapter 3

### The Integers

Binary, Hexadecimal, and Base-n Representations, the Floor and Ceiling Functions, Modulo Operation, Divisibility, GCD and LCM, the Euclidean Algorithm, Sequences, Recurrence Relations

## Chapter 4

### Sets and Counting

Set Operations, Venn Diagrams, Set Identities, Power Set, Cross Product, Cardinality, the Pigeonhole Principle, the Inclusion Exclusion Principles, Permutations and Combinations

## Chapter 5

### Binary Relations

Properties of Relations, Digraphs, Equivalence Relations and Equivalence Classes, Partial Order, Hasse Diagrams, Total Order, Zero-one Matrices, Transitive Closures

## Chapter 6

### Graph Theory

Adjacency and Incidence Matrices, Complete Graphs, Complete Bipartite Graphs, Euler and Hamiltonian Paths/Circuits, Trees, Minimal Spanning Trees, Traversal Algorithms, Planar Graphs and Chromatic Numbers, The Four-Color Theorem, Dual Graphs

## Appendices

1. Personalized Projects
2. Selected Answers

## References

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## Chapter 1 Propositional Logic

A **proposition** is a statement which has a truth value: either true or false.

- Examples: 1) Amman is in Jordan  
 2)  $2 + 2 = 4$   
 3)  $2 + 2 = 5$

Some statements are not a proposition because they have no truth values.

- Examples: 1) Philadelphia University  
 2)  $n + n = 2n$   
 3)  $x + y = 0$

The **negation** of a proposition  $p$  (not  $p$ ) is denoted by  $\neg p$ .

- Examples: 1)  $p$ : Amman is in Jordan  
 $\neg p$ : Amman is not in Jordan  
 2)  $p$ :  $2 + 2 = 5$   
 $\neg p$ :  $2 + 2 \neq 5$

The **conjunction** of two propositions:  $p \wedge q$  ( $p$  and  $q$ ) is one whose value is true only when both are true. The **disjunction**  $p \vee q$  ( $p$  or  $q$ ) is false only when both are false.

1.1 Let  $p$ : Amman is in Jordan and  $q$ :  $2 + 2 = 5$ .

- What is the proposition  $p \wedge \neg q$  ?
- What is the value of  $p \wedge \neg q$  ?
- What is the proposition  $\neg p \vee \neg q$  ?
- What is the value of  $\neg p \vee \neg q$  ?

The **implication** of two propositions:  $p \rightarrow q$  (if  $p$  then  $q$ ) is one whose value is false only when  $p$  is true and  $q$  is false. The **biconditional**  $p \leftrightarrow q$  ( $p$  if and only if  $q$ ) is true only when the values of  $p$  and  $q$  are the same, whereas the **exclusive or**  $p \oplus q$  (either  $p$  or  $q$  but not both) is true only when the values of  $p$  and  $q$  are not the same.

1.2 Let  $p$ : Today is cold,  $q$ : Today is hot, and  $r$ : Today is windy. Write the following propositions using  $p$ ,  $q$ , and  $r$ .

- Today is hot if and only if not windy.
- Either today is cold or not cold, but not both.
- If today is not windy then it is not hot.
- Today is neither cold nor windy.
- If today is windy then either it is hot or cold.

Logic operators can be presented in their **truth tables**:

$p$	$q$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
T	T	T	T	T	T	F
T	F	F	T	F	F	T
F	T	F	T	T	F	T
F	F	F	F	T	T	F

1.3 Draw the truth table for each of the following propositions.

- $\neg p \vee \neg q$
- $\neg(p \wedge q) \rightarrow p$
- $(p \oplus \neg q) \leftrightarrow (\neg p \vee q)$
- $(p \rightarrow q) \rightarrow r$
- $[(p \wedge q) \rightarrow r] \oplus [\neg p \vee (q \leftrightarrow r)]$

Two propositions are **equivalent** if their truth tables are identical, for example exclusive or is equivalent to the negation of biconditional:  $p \oplus q \equiv \neg(p \leftrightarrow q)$

1.4 Prove the following equivalences by drawing the truth tables.

- a)  $\neg p \vee \neg q \equiv \neg(p \wedge q)$
- b)  $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
- c)  $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$

1.5 The **difference** of two propositions is defined by  $p - q \equiv p \wedge \neg q$ . Prove that

$$p \rightarrow q \equiv \neg(p - q).$$

The **contrapositive** of  $p \rightarrow q$  is the proposition  $\neg q \rightarrow \neg p$ . It is not difficult to show that an implication is equivalent to its contrapositive:  $p \rightarrow q \equiv \neg q \rightarrow \neg p$ .

1.6 For each proposition below write an equivalent statement using contrapositive.

- a) If I study hard then I get good mark.
- b) If it rains then it is not hot.
- c) If today is not Sunday then tomorrow is not Monday.
- d) If I am not lazy then I come to the lecture.

A **tautology** is a compound proposition whose truth table is all true, whereas a **contradiction** is all false. A **contingency** is a mix of true and false.

1.7 Identify each proposition as a tautology, contradiction, or contingency.

- a)  $(p \wedge q) \rightarrow p$
- b)  $p \rightarrow (p \vee q)$
- c)  $p \rightarrow (p \rightarrow q)$
- d)  $p \rightarrow (q \rightarrow p)$
- e)  $\neg p \wedge \neg(p \rightarrow q)$

Other than using truth tables, checking for tautologies and contradictions can be done by using the so-called **Quine's method**. We need the following simple rules:

- 1)  $T \rightarrow p \equiv p$   
 $F \rightarrow p \equiv T$   
 $p \rightarrow T \equiv T$   
 $p \rightarrow F \equiv \neg p$
- 2) Similar rules as above, replacing  $\rightarrow$  by  $\wedge$ ,  $\vee$ ,  $\leftrightarrow$ , or  $\oplus$ .

Example: Is the proposition  $(p \vee q) \rightarrow \neg p$  a tautology or contradiction?

Solution: If  $p \equiv T$  then  $(T \vee q) \rightarrow F \equiv T \rightarrow F \equiv F$ .  
 If  $p \equiv F$  then  $(F \vee q) \rightarrow T \equiv q \rightarrow T \equiv T$ .  
 Hence  $(p \vee q) \rightarrow \neg p$  is a contingency.

1.8 Redo Problem 1.7 using Quine's method.

1.9 An **argument** consists of two components: a set of **premises**  $p_1, p_2, \dots, p_n$  and a **conclusion**  $Q$ . The argument (or its conclusion) is **valid** if  $p_1 \wedge p_2 \wedge \dots \wedge p_n \rightarrow Q$  is a tautology.

Which of the following arguments are valid?

- a)  $p_1$ : I failed my exam today  
 $p_2$ : If I studied last night then I did not fail my exam today  
 $Q$ : I did not study last night
- b)  $p_1$ : If it snows then the school is closed  
 $p_2$ : It is not snowing  
 $Q$ : The school is not closed

The following is a list of some common logical equivalence rules:

- |  |  |
|--|--|
| <p>1) <math>p \wedge q \equiv q \wedge p</math><br/> <math>p \vee q \equiv q \vee p</math></p> <p>2) <math>p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r</math><br/> <math>p \vee (q \vee r) \equiv (p \vee q) \vee r</math></p> <p>3) <math>p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)</math><br/> <math>p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)</math></p> | <p>4) <math>\neg(\neg p) \equiv p</math><br/> <math>\neg(p \wedge q) \equiv \neg p \vee \neg q</math><br/> <math>\neg(p \vee q) \equiv \neg p \wedge \neg q</math></p> <p>5) <math>p \rightarrow q \equiv \neg p \vee q</math><br/> <math>p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)</math><br/> <math>p \oplus q \equiv \neg(p \leftrightarrow q)</math></p> |
|--|--|

1.10 Prove by applying the above rules.

- a)  $\neg(p \rightarrow q) \equiv p \wedge \neg q$   
 b)  $p \rightarrow q \equiv \neg q \rightarrow \neg p$   
 c)  $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$   
 d)  $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$   
 e)  $p \oplus q \equiv (p \wedge \neg q) \vee (q \wedge \neg p)$

1.11 True or False. Prove by any method you like.

- a)  $p \rightarrow (q \rightarrow r) \equiv (p \rightarrow q) \rightarrow r$   
 b)  $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$   
 c)  $p \vee (q \oplus r) \equiv (p \vee q) \oplus (p \vee r)$   
 d)  $\neg(p \oplus q) \equiv \neg p \leftrightarrow \neg q$

A **CNF** (Conjunctive Normal Form) is a compound proposition in the form conjunctions of disjunctions of propositional variables or their negations, for example  $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$ . Similarly a **DNF** (Disjunctive Normal Form) is disjunctions of conjunctions, such as  $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r)$ . We say that the normal form is **full** when no variable is missing in each bracket.

**Theorem:** Every compound proposition is equivalent to a CNF and to a DNF.

**Example:** Convert  $[(p \leftrightarrow q) \oplus \neg p] \rightarrow \neg q$  to a CNF and to a DNF.

**Solution:** First draw the truth table. The result is

p	q	...	$[(p \leftrightarrow q) \oplus \neg p] \rightarrow \neg q$
T	T	...	F
T	F	...	T
F	T	...	F
F	F	...	T

A full CNF can be obtained by selecting the variables with false values from each row of the table whose result is false:  $(\neg p \vee \neg q) \wedge (p \vee \neg q)$  and similarly a full DNF from the true:  $(p \wedge \neg q) \vee (\neg p \wedge \neg q)$ . Both forms are equivalent to the given proposition:  $[(p \leftrightarrow q) \oplus \neg p] \rightarrow \neg q \equiv (\neg p \vee \neg q) \wedge (p \vee \neg q) \equiv (p \wedge \neg q) \vee (\neg p \wedge \neg q)$ .

1.12 Convert each proposition to a CNF and to a DNF.

- a)  $\neg(p \wedge q) \rightarrow p$   
 b)  $(p \oplus \neg q) \leftrightarrow (\neg p \vee q)$   
 c)  $(p \rightarrow q) \rightarrow r$   
 d)  $[(p \wedge q) \rightarrow r] \oplus [\neg p \vee (q \leftrightarrow r)]$

1.13 Convert each CNF to DNF and vice versa.

- a)  $(p \wedge q) \vee (\neg p \wedge q)$   
 b)  $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$   
 c)  $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$   
 d)  $(p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r)$

A **predicate** is a propositional function such as  $P(x): x + 2 = 5$ . For each value of  $x$ ,  $P(x)$  becomes a proposition, for instance,  $P(3): 3 + 2 = 5$  is true and  $P(2): 2 + 2 = 5$  is false.

1.14 Let  $P(x): x^2 < x$ .

- a) What is the value of  $P(1)$ ?
- b) What is the value of  $P(2)$ ?
- c) For which  $x$  is the value of  $P(x)$  true?

1.15 Let  $P(x,y): x^2 + y^2 = (x + y)^2$ . Find the values of the following propositions.

- a)  $P(0,1)$
- b)  $P(0,0)$
- c)  $P(1,1)$
- d) For which  $(x,y)$  is the value of  $P(x,y)$  true?

A predicate can also be made a proposition by adding a **quantifier** such as  $\exists$  (there is / there exists / there is at least one) and  $\forall$  (for all / for any /for each).

Example: Let  $P(x): x + 2 = 5$ .

- 1)  $\exists x P(x)$  means "there is at least one  $x$  such that  $x + 2 = 5$ " which is true.
- 2)  $\forall x P(x)$  means "for all  $x$ ,  $x + 2 = 5$ " which is false.

1.16 Let  $P(x): x < 2x$ .

- a) What is the value of  $\exists x P(x)$ ?
- b) What is the value of  $\forall x P(x)$ ?

1.17 Let  $P(x,y): x^2 + y^2 = (x + y)^2$ . Find the values of the following propositions.

- a)  $\exists x \exists y P(x,y)$
- b)  $\forall x \forall y P(x,y)$
- c)  $\exists x \forall y P(x,y)$
- d)  $\forall x \exists y P(x,y)$
- e)  $\exists y \forall x P(x,y)$

1.18 Repeat Problem 1.17 using the following predicates.

- a)  $P(x,y): x^2 + y^2 > 0$
- b)  $P(x,y): x^2 + y^2 \geq 0$
- c)  $P(x,y): x^2 - y^2 \geq 0$

## Chapter 2 Methods of Proof

### Direct Proof:

To prove a proposition in the form  $p \rightarrow q$ , we begin by assuming that  $p$  is true and then show that  $q$  must be true.

**Example:** An **even** number is of the form  $2n$  where  $n$  is an integer, whereas an **odd** number is  $2n + 1$ . Prove that if  $x$  is an odd integer then  $x^2$  is also odd.

**Solution:** Let  $p$ :  $x$  is odd, and  $q$ :  $x^2$  is odd. We want to prove  $p \rightarrow q$ .

Start:  $p$ :  $x$  is odd

$\rightarrow x = 2n + 1$  for some integer  $n$

$\rightarrow x^2 = (2n + 1)^2$

$\rightarrow x^2 = 4n^2 + 4n + 1$

$\rightarrow x^2 = 2(2n^2 + 2n) + 1$

$\rightarrow x^2 = 2m + 1$ , where  $m = (2n^2 + 2n)$  is an integer

$\rightarrow x^2$  is odd

$\rightarrow q$

2.1 Prove the following propositions.

- If  $x$  is an even integer then  $x^3$  is also even.
- If  $x$  is an odd integer then  $x^3$  is also odd.
- If  $x$  and  $y$  are odd integers then  $x + y$  is even.
- If  $x$  and  $y$  are odd integers then  $xy$  is also odd.
- If  $x$  is an odd integer then  $x^2 - 3x$  is even.

### Proof by Contrapositive:

To prove a proposition in the form  $p \rightarrow q$ , we may instead prove its contrapositive:  $\neg q \rightarrow \neg p$ . This works because  $p \rightarrow q \equiv \neg q \rightarrow \neg p$ .

**Example:** Prove that if  $x^2$  is odd then  $x$  must be odd.

**Solution:** Let  $p$ :  $x^2$  is odd, and  $q$ :  $x$  is odd. We will prove  $p \rightarrow q$  by proving  $\neg q \rightarrow \neg p$ .

Start:  $\neg q$ :  $x$  is even

$\rightarrow x = 2n$  for some integer  $n$

$\rightarrow x^2 = (2n)^2$

$\rightarrow x^2 = 4n^2$

$\rightarrow x^2 = 2(2n^2)$

$\rightarrow x^2 = 2m$ , where  $m = 2n^2$  is an integer

$\rightarrow x^2$  is even

$\rightarrow \neg p$

2.2 Prove the following propositions.

- If  $x^2$  is even then  $x$  must be even.
- If  $x^3$  is even then  $x$  must be even.
- If  $x^2 - 2x$  is even then  $x$  must be even.
- If  $x^3 - 4x + 2$  is odd then  $x$  must be odd.

### Proving Equivalence:

To prove a proposition in the form  $p \leftrightarrow q$ , we prove its equivalence:

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ .

**Example:** Prove that  $x^2$  is odd if and only if  $x$  is odd.

**Solution:** Let  $p$ :  $x^2$  is odd, and  $q$ :  $x$  is odd. We will prove  $p \leftrightarrow q$  by proving  $p \rightarrow q$  and  $q \rightarrow p$

Step 1) Prove  $p \rightarrow q$  ... (Like Example 2)

Step 2) Prove  $q \rightarrow p$  ... (Like Example 1)

Proof is complete.

2.3 Prove the following propositions.

- a)  $x^2$  is even if and only if  $x$  is even.
- b)  $x^3$  is even if and only if  $x$  is even.
- c)  $x^2 - 2x$  is even if and only if  $x$  is even.
- d)  $x^3 - 4x + 2$  is odd if and only if  $x$  is odd.

**Mathematical Induction:**

To prove a proposition in the form  $\forall n P(n)$  where  $n$  is a positive integer, it suffices to prove the following two propositions.

- 1)  $P(1)$
- 2)  $P(n) \rightarrow P(n+1)$

Example: Prove the following formula for all positive integers  $n$ .  
 $1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2$

Solution: Let  $P(n): 1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2$   
We shall prove  $\forall n P(n)$  in two steps:  
1)  $P(1): 1 = 1^2$  so this proposition is true.  
2)  $P(n): 1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2$   
 $\rightarrow 1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) + (2n + 1) = n^2 + (2n + 1)$   
 $\rightarrow 1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) + (2n + 1) = (n + 1)^2$   
 $\rightarrow P(n+1)$

2.4 Prove the following formulas for all positive integers  $n$ .

- a)  $1 + 2 + 3 + 4 + 5 + \dots + n = n(n + 1) \div 2$
- b)  $2 + 4 + 6 + 8 + 10 + \dots + 2n = n^2 + n$
- c)  $1 + 2 + 4 + 8 + 16 + \dots + 2^{n-1} = 2^n - 1$
- d)  $1 + 3 + 9 + 27 + 81 + \dots + 3^{n-1} = (3^n - 1) \div 2$
- e)  $1 + 4 + 9 + 16 + 25 + \dots + n^2 = n(n + 1)(2n + 1) \div 6$

2.5 Prove the following propositions.

- a)  $n < 2^n \forall n \geq 1$
- b)  $2^n < n! \forall n \geq 4$
- c)  $3^n < n! \forall n \geq 7$
- d)  $2^n > n^2 \forall n \geq 5$
- e)  $n! < n^n \forall n \geq 2$

## Chapter 3 The Integers

In the **binary** number system we use only 0 and 1 to count. The positive integers go like this: 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, ...

Note that the digits of a decimal number represent powers of 10, for example  $59012 = 5 \times 10^4 + 9 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 2 \times 10^0$ . Similarly, the digits of a binary number represent powers of 2.

Examples: 1)  $1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 8 + 4 + 0 + 1 = 13_{10}$   
 2)  $101110_2 = 2^1 + 2^2 + 2^3 + 2^5 = 2 + 4 + 8 + 32 = 46_{10}$

3.1 Convert these binary numbers to decimal.

- 101010
- 101001000
- 10110111
- 1000001

Example: Convert the number 54 to binary.

Solution: To find the appropriate powers of 2, divide this number by 2 repeatedly.

$54 \div 2 = 27$  remain 0  
 $27 \div 2 = 13$  remain 1  
 $13 \div 2 = 6$  remain 1  
 $6 \div 2 = 3$  remain 0  
 $3 \div 2 = 1$  remain 1  
 $1 \div 2 = 0$  remain 1

The answer is these remainders from the last one up: 110110

3.2 Convert these decimal numbers to binary.

- 37
- 99
- 500
- 999

The **hexadecimal** number system uses 16 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. Again, the digits represent powers of the base, in this case 16.

Example:  $1A5E_{16} = 1 \times 16^3 + 10 \times 16^2 + 5 \times 16^1 + 14 \times 16^0 = 4096 + 2560 + 80 + 14 = 6750_{10}$

3.3 Convert these hexadecimal numbers to decimal.

- 5FE
- BCD
- A0A0
- 11111

3.4 Convert the decimal numbers in Problem 3.2 to hexadecimal.

Because  $16 = 2^4$ , every 1 hexadecimal digit corresponds to 4 binary digits.

Examples: 1)  $7BF8_{16} = 0111\ 1011\ 1111\ 1000 = 111101111111000_2$   
 2)  $1111011111100010110100_2 = 0011\ 1101\ 1111\ 1000\ 1011\ 0100 = 3DF8B4_{16}$

The following table is useful when converting between binary and hexadecimal.

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

3.5 Convert the hexadecimal numbers in Problem 3.3 to binary.

3.6 Convert the binary numbers in Problem 3.1 to hexadecimal.

3.7 Convert the decimal numbers in Problem 3.2 to **octal** (base 8).

3.8 A real number between 0 and 1 is represented by negative powers of the base. For example, in decimal  $0.125 = 1 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3}$ . Convert the following numbers to decimal.

- a)  $0.1101_2$
- b)  $0.000001_2$
- c)  $111.111_2$
- d)  $0.A8_{16}$
- e)  $111.111_{16}$

3.9 Convert these decimal numbers to binary and then to hexadecimal.

- a) 0.03125
- b) 0.765625
- c)  $5/8$
- d)  $1/3$
- e) 25.25

The **floor** function  $\lfloor x \rfloor$  of a real number  $x$  is the greatest integer  $n \leq x$  whereas the **ceiling** function  $\lceil x \rceil$  is the smallest integer  $n \geq x$ .

- Examples:
- |  |                                  |
|--|----------------------------------|
| 1) $\lfloor 1.99 \rfloor = 1$          | $\lceil 1.99 \rceil = 2$         |
| 2) $\lfloor 7.01 \rfloor = 7$          | $\lceil 7.01 \rceil = 8$         |
| 3) $\lfloor 5 \rfloor = 5$             | $\lceil 5 \rceil = 5$            |
| 4) $\lfloor -\frac{1}{2} \rfloor = -1$ | $\lceil -\frac{1}{2} \rceil = 0$ |

For two integers  $m$  and  $n > 0$  define the modulo operation  $m \bmod n = m - \lfloor m/n \rfloor \times n$  which is the same as the remainder upon dividing  $m$  by  $n$ .

Example: Evaluate  $217 \bmod 5$ .

Solution:  $217 \div 5 = 43.4$  hence  $217 \bmod 5 = 217 - (43 \times 5) = 2$ .  
Equivalently,  $217 = (43) \times 5 + (2)$  hence  $217 \bmod 5 = 2$ .

3.10 Evaluate the following.

- a)  $123 \bmod 3$
- b)  $2000 \bmod 7$
- c)  $25 \bmod 11$
- d)  $11 \bmod 25$

The following statements are all equivalent, meaning that  $m \bmod n = 0$ .

- »  $n$  divides  $m$
- »  $n$  is a divisor of  $m$
- »  $n$  is a factor of  $m$
- »  $m$  is divisible by  $n$
- »  $m$  is a multiple of  $n$
- »  $m/n$  is an integer
- »  $m = nk$  for some integer  $k$

Examples: a) 3 divides 12 and 21 but not 32  
b) even numbers are multiples of 2  
c) 0 is divisible by all integers except itself

3.11 Prove that if  $a \bmod n = b \bmod n$  then  $n$  divides  $a - b$ .

3.12 Prove by induction for all positive integers  $n$ .

- a)  $2^{2n} - 1$  is divisible by 3
- b) 7 divides  $2^{3n} - 1$
- c)  $n^3 + 2n$  is a multiple of 3
- d)  $n^5 - n \pmod{5} = 0$
- e)  $2^{n+2} + 3^{2n+1}$  is divisible by 7

The **GCD** (greatest common divisor) of two integers is the biggest integer that divides both. Similarly the **LCM** (least common multiple) is the smallest integer divisible by both.

Examples: GCD (12, 16) = 4 since 4 divides 12 and 16 and is the biggest of such.  
LCM (12, 16) = 48 since 48 is divisible by 12 and 16 and the smallest of such.

The **Euclidean algorithm** gives an efficient way to compute GCD by iteration:

$$\text{GCD}(m, n) := \text{GCD}(n, m \bmod n)$$

Example: Find GCD (278, 144) using the algorithm.

Solution: GCD (278, 144)  
= GCD (144, 134) because  $278 \bmod 144 = 134$   
= GCD (134, 10) because  $144 \bmod 134 = 10$   
= GCD (10, 4) because  $134 \bmod 10 = 4$   
= GCD (4, 2) because  $10 \bmod 4 = 2$   
= GCD (2, 0) because  $4 \bmod 2 = 0$   
= 2

3.13 Find the GCD of each pair using the Euclidean algorithm.

- a) 275 and 115
- b) 999 and 123
- c) 456 and 144
- d) 725 and 1000

**Theorem:**  $\text{GCD}(m, n) \times \text{LCM}(m, n) = m \times n$

3.14 Find the LCM of each pair in Problem 3.13.

A **sequence** is a function  $f(n)$  defined over the (non-negative) integers, hence can be ordered  $f(0), f(1), f(2), f(3), \dots$

Examples: 1)  $f(n) = n^2$  is the sequence 0, 1, 4, 9, 16, 25, 36, ...  
2)  $f(n) = 2n + 1$  is the sequence 1, 3, 5, 7, 9, 11, 13, ...

A sequence is **recursive** if  $f(n)$  depends on  $f(0), f(1), \dots, f(n-1)$ .

Example: The **Fibonacci** sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... is recursive with a **recurrence relation** given by  $f(n) = f(n-1) + f(n-2)$  for all  $n \geq 2$ .

3.15 Find a recurrence relation for each given sequence.

- a) 1, 3, 5, 9, 17, 31, 57, 105, ...
- b) 7, 17, 27, 37, 47, 57, 67, ...
- c) 1, 1, 2, 6, 24, 120, 720, ...
- d) 2, 4, 5, 7, 9, 12, 16, 22, ...

A recurrence relation of the form  $f(n) = A f(n-1) + B f(n-2)$  can be expressed explicitly in one of two ways, depending whether the quadratic equation  $x^2 - Ax - B = 0$  has one solution or two, respectively:

- 1)  $f(n) = C x^n + D n x^n$
- 2)  $f(n) = C x_1^n + D x_2^n$

Example: Find an explicit formula for the sequence given by  
 $f(0) = 4$ ,  $f(1) = 7$ ,  $f(n) = f(n-1) + 6f(n-2)$  for all  $n \geq 2$ .

Solution: The equation  $x^2 - x - 6 = 0$  has two solutions  $x_1 = -2$  and  $x_2 = 3$  (How?)

Hence the explicit formula is  $f(n) = C(-2)^n + D(3)^n$

To find C and D substitute the values of  $f(0)$  and  $f(1)$ :

$$f(0) = 4 = C + D$$

$$f(1) = 7 = -2C + 3D$$

The solution is  $C = 1$  and  $D = 3$  (How?) therefore  $f(n) = (-2)^n + 3^{n+1}$ .

3.16 Find an explicit formula for each given sequence.

a)  $f(0) = 1$ ,  $f(1) = 8$ ,  $f(n) = f(n-1) + 2f(n-2)$

b)  $f(0) = 1$ ,  $f(1) = 3$ ,  $f(n) = 4f(n-1) - 4f(n-2)$

c)  $a_0 = 1$ ,  $a_1 = 2$ ,  $a_n = 2a_{n-1} + 3a_{n-2}$

d)  $a_0 = 1$ ,  $a_1 = 4$ ,  $a_n = 2a_{n-1} - a_{n-2}$

3.17 Find an explicit formula for the Fibonacci sequence.

3.18 Prove that  $\text{GCD}[f(n), f(n+1)] = 1$  for all  $n \geq 0$  in the Fibonacci sequence.

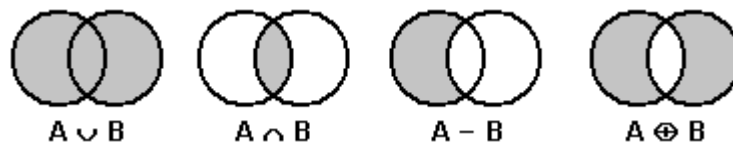
## Chapter 4 Sets and Counting

A **set** is a collection of objects called the **elements** of the set. The ordering of the elements is not important and repetition of elements is ignored, for example  $\{1, 3, 1, 2, 2, 1\} = \{1, 2, 3\}$ . A set may also be empty and it is denoted by  $\phi$  or  $\{\}$ . If  $x$  is an element of the set  $A$  then we write  $x \in A$ , otherwise  $x \notin A$ .

For any two sets  $A$  and  $B$ , define the following set operations.

- 1) The **union**  $A \cup B = \{x \mid x \in A \vee x \in B\}$
- 2) The **intersection**  $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- 3) The **difference**  $A - B = \{x \mid x \in A \wedge x \notin B\}$
- 4) The **symmetric difference**  $A \oplus B = \{x \mid x \in A \oplus x \in B\}$

These set operations can be illustrated using **Venn diagrams**:



Example: If  $A = \{1, 2, 3, 4, 5\}$  and  $B = \{0, 2, 4, 6\}$  then  
 $A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$   
 $A \cap B = \{2, 4\}$   
 $A - B = \{1, 3, 5\}$   
 $B - A = \{0, 6\}$   
 $A \oplus B = \{0, 1, 3, 5, 6\}$

4.1 Let  $A = \{1, 2, 3, 4, 5\}$ ,  $B = \{0, 2, 4, 6\}$  and  $C = \{1, 3, 5\}$ . Find the following set.

- a)  $(A \cup C) \oplus (A \cap C)$
- b)  $A \oplus (B \cup C)$
- c)  $(A \oplus B) - (A \oplus C)$
- d)  $(A - B) \oplus (A - C)$

Define the **complement** of a set  $A$  to be  $A^c = \{x \mid x \notin A\}$ . The following set identities are the analog of logical equivalences.

$$\begin{array}{ll} (A^c)^c = A & A - B = A \cap B^c \\ A \cup B = B \cup A & A \cup (B \cap C) = (A \cup B) \cap C \\ A \cap B = B \cap A & A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ (A \cup B)^c = A^c \cap B^c & A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \\ (A \cap B)^c = A^c \cup B^c & A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \end{array}$$

4.2 True or False? Use Venn diagrams to verify each one.

- a)  $(A \cup B) - (A \cap B) = A \oplus B$
- b)  $(A - B) \cup (B - A) = A \oplus B$
- c)  $(A \oplus B) - B = A$
- d)  $(A \oplus B) \oplus B = A$
- e)  $A \oplus A = A - A$

A set  $S$  is a **subset** of a set  $A$ , written  $S \subseteq A$ , if  $x \in S \rightarrow x \in A$ . For example  $A = \{1, 2\}$  has a total of 4 subsets:  $\{1\}$ ,  $\{2\}$ ,  $\phi$ , and  $A$  itself. The **power set** of a set  $A$ , written  $P(A)$ , is the set whose elements are all the subsets of  $A$ . So for this example  $P(A) = \{\phi, \{1\}, \{2\}, A\}$ . The **cardinality** of a set  $A$  is the number of elements in  $A$ , denoted by  $|A|$ .

**Theorem:** If  $|A| = n$  then  $|P(A)| = 2^n$  (Every set with  $n$  elements has  $2^n$  subsets.)

4.3 Find  $P(A)$  for each set  $A$  to verify the above theorem.

- a)  $A = \{1, 2, 3\}$
- b)  $A = \{1, 2, 3, 4\}$
- c)  $A = \phi$
- d)  $A = P(\phi)$
- e)  $A = P(P(\phi))$

4.4 Prove the above theorem by mathematical induction.

The **cross product** of  $A$  and  $B$  is the set  $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ .

Example: If  $A = \{1, 2, 3\}$  and  $B = \{x, y\}$  then  
 $A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$   
 $B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

**Theorem:** If  $|A| = m$  and  $|B| = n$  then  $|A \times B| = m \times n$

4.5 True or False?

- a)  $P(A \cup B) = P(A) \cup P(B)$
- b)  $P(A \cap B) = P(A) \cap P(B)$
- c)  $P(A \times B) = P(A) \times P(B)$

**Theorem:** If there are  $k$  sets with  $n$  elements in all then one of the sets must contain at least  $\lceil n/k \rceil$  elements. (The Pigeonhole Principle)

Example: The University has 8 faculties. Given any group of 9 students, at least  $\lceil 9/8 \rceil = 2$  of them must belong in the same faculty. And with 42 students at least  $\lceil 42/8 \rceil = 6$  must belong in the same faculty

4.6 What is the minimum number of students to ensure the following is true?

- a) 13 of them must be in the same faculty
- b) 2 of them have their birthdays in the same month
- c) 5 of them have their birthdays in the same month
- d) 5 of them have the same birthdays

**Theorem:**  $|A \cup B| = |A| + |B| - |A \cap B|$  (The Inclusion-Exclusion Principle)

Example: How many positive integers  $\leq 100$  are multiples of 2 or multiples of 3?

Solution:  $A = \langle 2 \rangle = \{2, 4, 6, \dots, 100\}$ ,  $|A| = \lfloor 100/2 \rfloor = 50$   
 $B = \langle 3 \rangle = \{3, 6, 9, \dots, 99\}$ ,  $|B| = \lfloor 100/3 \rfloor = 33$   
 $A \cap B = \langle \text{LCM}(2,3) \rangle = \langle 6 \rangle = \{6, 12, 18, \dots, 96\}$ ,  $|A \cap B| = \lfloor 100/6 \rfloor = 16$   
 $|A \cup B| = 50 + 33 - 16 = 67$

4.7 How many positive integers  $\leq 200$  are multiples of

- a) 3 or 5?
- b) 4 or 6?
- c) not multiples of 2 or 17?
- d) not multiples of 12 or 16?

**Theorem:**  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Example: How many positive integers  $\leq 100$  are multiples of 4, 5, or 6?

Solution:  $A = \langle 4 \rangle$ ,  $|A| = \lfloor 100/4 \rfloor = 25$   
 $B = \langle 5 \rangle$ ,  $|B| = \lfloor 100/5 \rfloor = 20$   
 $C = \langle 6 \rangle$ ,  $|C| = \lfloor 100/6 \rfloor = 16$   
 $A \cap B = \langle \text{LCM}(4,5) \rangle = \langle 20 \rangle$ ,  $|A \cap B| = \lfloor 100/20 \rfloor = 5$

$$\begin{aligned}
A \cap C &= \langle \text{LCM}(4,6) \rangle = \langle 12 \rangle, |A \cap C| = \lfloor 100/12 \rfloor = 8 \\
B \cap C &= \langle \text{LCM}(5,6) \rangle = \langle 30 \rangle, |B \cap C| = \lfloor 100/30 \rfloor = 3 \\
A \cap B \cap C &= \langle \text{LCM}(4,5,6) \rangle = \langle 60 \rangle, |A \cap B \cap C| = \lfloor 100/60 \rfloor = 1 \\
|A \cup B \cup C| &= 25 + 20 + 16 - 5 - 8 - 3 + 1 = 46
\end{aligned}$$

4.8 How many positive integers  $\leq 1000$  are multiples of

- 2, 3, or 5?
- 4, 6, or 20?
- not multiples of 4, 6, or 20?
- not multiples of 8, 12, or 20?

4.9 Generalize the above theorem for four sets:  $|A \cup B \cup C \cup D|$

A **combination** of elements is the set containing those elements, whereas a **permutation** is like a set but with specific ordering of the elements. For example there are 6 different permutations of the elements A, B, C, namely ABC, ACB, BAC, BCA, CAB, and CBA.

**Theorem:** There are  $n!$  different permutations of  $n$  elements.

4.10 How many different permutations of the alphabet  $\{A, B, C, \dots, Z\}$  which

- contain the word CAR or BYTE?
- contain the word NO or YES or WHAT?
- do not contain the word AND or OR or XOR?
- do not contain the word BY or DNA or COMPUTER?

4.11 A **multiset** is like a set but with repetition of elements allowed. How many different permutations are there of the elements taken from

- the multiset  $\{A, B, B, C\}$ ?
- the word DISCRETE?
- the word MATHEMATICS?
- the word UNUSUAL?

4.12 A **string** over a set  $\Sigma$  is a sequence of elements of  $\Sigma$ . For example over  $\Sigma = \{0, 1\}$  the sequence 101101101101... is a string, whether or not the length is finite. Let  $\Sigma^n$  denote the set of all strings of length  $n$  over  $\Sigma$ .

- Find  $|\Sigma^3|$  for  $\Sigma = \{0, 1\}$ .
- Find  $|\Sigma^2|$  for  $\Sigma = \{a, b, c\}$ .
- If  $|\Sigma| = m$ , what is  $|\Sigma^n|$ ?

$C(n, k)$  denotes the number of subsets which contain  $k$  elements from a set with  $n$  elements. For example  $C(3, 2) = 3$  because there are 3 subsets of  $\{a, b, c\}$  which have 2 elements, namely  $\{a, b\}$ ,  $\{a, c\}$ , and  $\{b, c\}$ .

**Theorem:**  $C(n, k) = \frac{n!}{k!(n-k)!}$ .

Example: If  $|S| = 10$  how many subsets of  $S$  are there that contain 8 elements?

Solution:  $C(10, 8) = 10! / (8! 2!) = (1 2 3 4 5 6 7 8 9 10) / (1 2 3 4 5 6 7 8 1 2) = 45$ .

4.13 Let  $|S| = 7$ . How many subsets does  $S$  have which contain

- 4 elements?
- 3 elements?
- 7 elements?
- more than 5 elements?
- at least 1 element?

4.14 Evaluate  $C(n, 0) + C(n, 1) + C(n, 2) + \dots + C(n, n)$ .

**Theorem:** There are  $C(n+k-1, k)$  non-negative integer solutions of the equation  $x_1 + x_2 + x_3 + \dots + x_n = k$

**Example:** How many non-negative integer solutions of  $a + b + c = 8$ ?

**Solution:**  $C(3+8-1, 8) = C(10,8) = 45$ .

4.15 How many integer solutions of  $x + y + z = 11$  with each given condition?

- a)  $x, y, z$  are non-negative
- b)  $x, y, z$  are positive
- c)  $x \geq 1, y \geq 2, \text{ and } z \geq 3$
- d)  $x \leq 3, y \leq 4, \text{ and } z \leq 6$
- e)  $x \leq 5, y \leq 2, \text{ and } z \leq 7$

## Chapter 5 Binary Relations

Any subset of  $A \times B$  is a **binary relation** from  $A$  to  $B$ .

Examples: The following are some, but not all, binary relations from  $\{1, 2\}$  to  $\{x, y\}$ .

- 1)  $\{(1, x), (2, x)\}$
- 2)  $\{(2, y)\}$
- 3)  $\{(1, x), (1, y), (2, x), (2, y)\}$
- 4)  $\phi$

5.1 If  $|A| = m$  and  $|B| = n$  how many different relations are there from  $A$  to  $B$ ?

If  $R \subseteq A \times B$  is a relation then the **inverse**  $R^{-1} = \{(b, a) \mid (a, b) \in R\}$  is a relation from  $B$  to  $A$ .  
And if  $S \subseteq B \times C$  is another relation then the **composition** of  $R$  and  $S$  is a relation from  $A$  to  $C$  defined by  $S \circ R = \{(a, c) \mid (a, b) \in R \text{ and } (b, c) \in S\}$ .

Example: Let  $A = \{1, 2, 3\}$ ,  $B = \{x, y\}$ ,  $C = \{P, Q, R\}$   
and  $R = \{(1, x), (2, x), (3, y)\} \subseteq A \times B$   
and  $S = \{(x, R), (y, P), (y, Q)\} \subseteq B \times C$ .  
Then  $R^{-1} = \{(x, 1), (x, 2), (y, 3)\} \subseteq B \times A$   
and  $S^{-1} = \{(R, x), (P, y), (Q, y)\} \subseteq C \times B$   
and  $S \circ R = \{(1, R), (2, R), (3, P), (3, Q)\} \subseteq A \times C$ .

In the case  $R \subseteq A \times A$  define  $R^2 = R \circ R$  and  $R^3 = R \circ R \circ R$  and so on  $R^n$  in this way.

5.2 Let  $A = \{1, 2, 3, 4\}$  and  $R = \{(1, 2), (2, 1), (2, 4), (3, 3), (4, 1), (4, 3)\} \subseteq A \times A$ .

- a) Find  $R^{-1}$  and  $(R^{-1})^{-1}$
- b) Find  $R^2$  and  $R^3$
- c) Find  $R \circ R^{-1}$  and  $R^{-1} \circ R$
- d) Find  $(R^{-1})^2$  and  $(R^2)^{-1}$

Certain properties of a relation  $R \subseteq A \times A$  are important:

- 1) **reflexive:**  $(a, a) \in R \forall a \in A$
- 2) **symmetric:**  $(a, b) \in R \rightarrow (b, a) \in R \forall a \wedge b \in A$
- 3) **anti-symmetric:**  $(a, b) \in R \rightarrow (b, a) \notin R \forall a \neq b \in A$
- 4) **transitive:**  $(a, b) \wedge (b, c) \in R \rightarrow (a, c) \in R \forall a, b, c \in A$

Note that  $R$  is symmetric when  $R = R^{-1}$  and anti-symmetric when  $R \cap R^{-1} = \phi$  or contains only elements of the form  $(a, a)$ , whereas  $R$  is transitive when  $R^2 \subseteq R$ .

Example: Let  $A = \{1, 2, 3\}$  and consider three relations on  $A$ :  
 $R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$   
 $R_2 = \{(1, 1), (1, 3), (2, 2), (3, 2)\}$   
 $R_3 = \{(1, 2), (1, 3), (2, 3)\}$   
 $R_1$  is reflexive, symmetric, and transitive, but not anti-symmetric.  
 $R_2$  is anti-symmetric, but not reflexive, not symmetric, and not transitive.  
 $R_3$  is anti-symmetric and transitive, but not reflexive and not symmetric.

5.3 Let  $A = \{1, 2, 3, 4\}$ . Which properties above are true for each  $R \subseteq A \times A$ ?

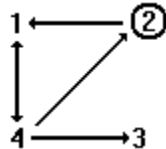
- a)  $R = \{(a, b) \mid a \leq b\}$
- b)  $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
- c)  $R = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 4)\}$
- d)  $R = \{(a, b) \mid a + b \geq 5\}$

5.4 Let  $A = \{1, 2, 3, 4\}$ . Give any example of a relation  $R \subseteq A \times A$  which is

- reflexive, not anti-symmetric, not transitive.
- not reflexive, not symmetric, not transitive.
- symmetric and transitive.
- neither symmetric nor anti-symmetric.
- both symmetric and anti-symmetric.

A relation  $R \subseteq A \times A$  can be represented by a **digraph** in which each element of  $A$  is represented by a **vertex** and each element  $(a, b)$  of  $R$  is represented by an **edge** with direction from  $a$  to  $b$ . In the case  $a = b$  the edge is a **loop**.

Example:  $A = \{1, 2, 3, 4\}$  and  $R = \{(1, 4), (2, 1), (2, 2), (4, 1), (4, 2), (4, 3)\}$ .



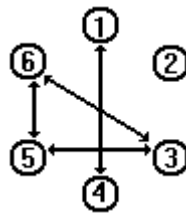
5.5 Draw the digraph for each of the relations in Problem 5.3.

5.6 What characterizes the digraph of a relation with each of the following properties?

- reflexive
- anti-reflexive** [meaning that  $(a, a) \notin R \forall a \in A$ ]
- symmetric
- anti-symmetric
- transitive

$R \subseteq A \times A$  is an **equivalence relation** if it is reflexive, symmetric, and transitive. If  $R$  is an equivalence relation then  $A$  is partitioned into subsets such that in each subset every two vertices are connected by an edge. These subsets are the **equivalence classes** of  $A$  under the relation  $R$ .

Example: The following digraph shows that  $R$  is an equivalence relation. (Why?)  
The equivalence classes are  $\{1, 4\}$ ,  $\{2\}$ , and  $\{3, 5, 6\}$ . (Why?)



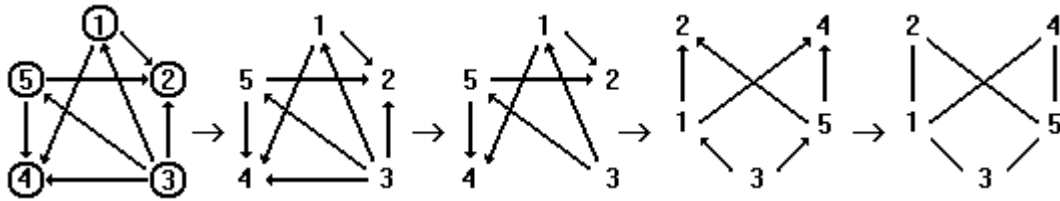
5.7 Prove that  $R$  is an equivalence relation and then find the equivalence classes.

- $A = \{0, 1, 2, 3, 4, 5, 6\}$  and  $R = \{(a, b) \mid a + b \text{ is even}\}$
- $A = \{1, 2, 3, 4\}$  and  $R = \{(a, b) \mid a = b\}$
- $A = \{0, 5, 8, 9, 10, 11\}$  and  $R = \{(a, b) \mid a - b \text{ is a multiple of } 3\}$
- $A = \{1, 2, 3, 6, 7, 11\}$  and  $R = \{(a, b) \mid a \bmod 5 = b \bmod 5\}$
- $A = \{1, 9, 21, 44, 50, 99, 101\}$  and  $R = \{(a, b) \mid (a - b) \bmod 10 = 0\}$

$R \subseteq A \times A$  is a **partial order** relation if it is reflexive, anti-symmetric, and transitive. In this case the digraph of  $R$  can be simplified into a **Hasse diagram** after these 4 steps:

- Do not draw loops.
- Do not draw  $(a, c)$  whenever there are  $(a, b)$  and  $(b, c)$ .
- Redraw the remaining graph so that all edges point upward.
- Do not draw the directions.

Example: The following digraph shows that  $R$  is a partial order relation. (Why?)  
The 4 steps above lead to the Hasse diagram of  $R$ .



5.8 Prove that  $R$  is a partial order relation and then draw the Hasse diagram.

- $A = \{1, 2, 3, 4\}$  and  $R = \{(a, b) \mid a \leq b\}$
- $A = \{1, 2, 6, 12, 24\}$  and  $R = \{(a, b) \mid a \text{ divides } b\}$
- $A = \{1, 2, 6, 10, 20, 30\}$  and  $R = \{(a, b) \mid a \text{ divides } b\}$
- $A = \{1, 5, 7, 10, 35, 70\}$  and  $R = \{(a, b) \mid a \text{ divides } b\}$

$R \subseteq A \times A$  is a **total order** relation if it is a partial order relation in which every two vertices are connected by an edge. The partial order relation in the previous example is not a total order because there is no edge between 2 and 4. Moreover the Hasse diagram of a total order relation can always be drawn as a vertical line.

5.9 Which ones of the partial order relations in Problem 5.8 are total order?

If  $A = \{1, 2, 3, \dots, n\}$  then a binary relation  $R \subseteq A \times A$  can be represented by a **zero-one matrix**  $(m_{ij})$  of size  $n \times n$  where  $m_{ij} = 0$  if  $(i, j) \notin R$  and  $m_{ij} = 1$  if  $(i, j) \in R$ .

Example:  $A = \{1, 2, 3\}$ . Find the zero-one matrix of  $R = \{(1,1), (1,3), (2,1), (3,2), (3,3)\}$ .

Solution: 
$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

5.10 Represent the relations given in Problem 5.3 using zero-one matrices.

5.11 Convert these zero-one matrices to digraphs.

$$\text{a) } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{c) } \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{d) } \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The **transitive closure** of  $R \subseteq A \times A$  is the smallest transitive relation containing  $R$ .

**Theorem:** The transitive closure of  $R$  is given by  $R \cup R^2 \cup R^3 \cup \dots \cup R^n$  where  $n = |A|$ .

5.12 Let  $A = \{1, 2, 3, 4\}$ . Use the above theorem to find the transitive closure of  $R \subseteq A \times A$ .

- $R = \{(1, 2), (2, 1), (2, 3), (3, 4)\}$
- $R = \{(1, 1), (1, 2), (2, 1), (4, 3)\}$
- $R = \{(1, 1), (1, 4), (2, 1), (2, 2), (3, 3), (4, 4)\}$
- $R = \{(1, 4), (2, 1), (2, 4), (3, 2), (3, 4), (4, 3)\}$

5.13 Find the zero-one matrix of the transitive closure for each relation in Problem 5.11.

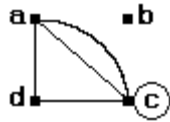
5.14 Given  $R$  and its zero-one matrix  $M$ , discover a way to compute  $M^2$ , the corresponding matrix of  $R^2$ , then redo Problem 5.13 using only matrices.

5.15 Discuss the obvious definitions of **reflexive closure** and **symmetric closure** and how we might find them.

## Chapter 6 Graph Theory

A **graph** consists of two components: a set of vertices and a multiset of edges and loops.

Example:  $G = \{a, b, c, d\} \cup \{ac, ac, ad, cc, cd\}$  which we represent graphically as follows.



There are two matricial representations of a graph  $G$  with  $m$  vertices and  $n$  edges:

- 1) The **adjacency matrix** of  $G$  is an  $m \times m$  matrix defined by  $m_{ij}$  = the number of edges between vertex  $i$  and vertex  $j$ .
- 2) The **incidence matrix** of  $G$  is an  $m \times n$  matrix defined by  $m_{ij} = 1$  if edge  $j$  is incidence on vertex  $i$  and  $m_{ij} = 0$  otherwise.

Example: The adjacency matrix and the incidence matrix of  $G$  above are respectively

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

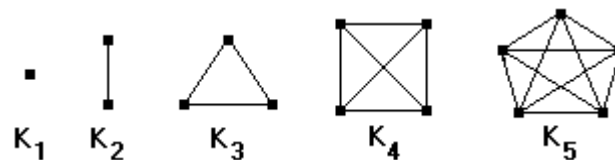
6.1 Convert these adjacency matrices to incidence matrices.

$$\text{a) } \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{c) } \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

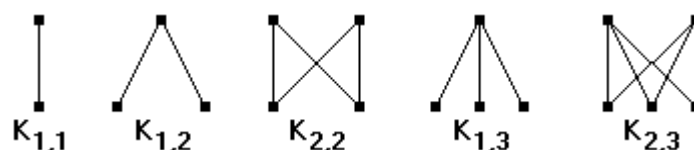
6.2 Convert these incidence matrices to adjacency matrices.

$$\text{a) } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{b) } \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{c) } \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A graph is **simple** if it has neither loops nor multiple edges. A simple graph in which every two vertices are connected by an edge is called a **complete** graph. Let  $K_n$  denote the complete graph with  $n$  vertices, as pictured below for  $n = 1, 2, 3, 4, 5$ .



A **complete bipartite** graph is a simple graph whose vertices can be partitioned into two subsets such that two vertices are connected if and only if they belong to different subsets. Let  $K_{m,n}$  denote the complete bipartite graph with the partition into  $m$  and  $n$  vertices:



6.3 Describe the adjacency matrices and the incidence matrices of  $K_n$  and  $K_{m,n}$ .

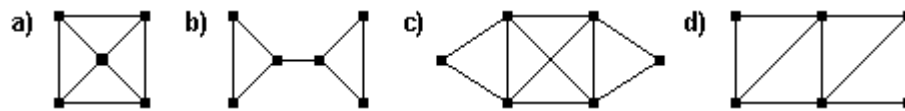
The **degree** of a vertex is the number of edges incidence on it, where a loop counts as two edges. The **degree** of a graph is the sum of all the degrees of its vertices. In the example  $G = \{a, b, c, d\} \cup \{ac, ac, ad, cc, cd\}$  given earlier, the degree of  $G = \deg(a) + \deg(b) + \deg(c) + \deg(d) = 3 + 0 + 5 + 2 = 10$ .

**Theorem:** The degree of any graph is twice its number of edges.

6.4 Find the numbers of edges and the degrees of  $K_n$  and  $K_{m,n}$ .

A **path** from vertex  $a$  to  $b$  is a continuous succession of distinct edges beginning at  $a$  and ending at  $b$ . If it happens that  $a = b$ , the path is a **circuit**. A graph is **connected** if every two vertices are connected by a path. A graph is an **Euler path/circuit** if all of its edges can be used to form a path/circuit. Clearly an Euler path must be connected.

6.5 Are these graphs Euler paths/circuits?



**Theorem:** A connected graph is an Euler circuit if and only if the degree of each vertex is even. Otherwise it is an Euler path if and only if exactly two vertices have odd degrees.

6.6 For which  $n$  is  $K_n$  an Euler path/circuit? How about  $K_{m,n}$ ?

A **Hamiltonian path** is one that goes through each vertex in the graph exactly once, except the first and the last vertices may be the same, in which case it is a Hamiltonian circuit.

6.7 Find a Hamiltonian path/circuit, if any, in each graph given in Problem 6.5.

6.8 Give an example of a connected graph which has no Hamiltonian path.

6.9 Draw an Euler path/circuit which has no Hamiltonian path/circuit.

A **bridge** is an edge or a vertex without which the graph would become disconnected.

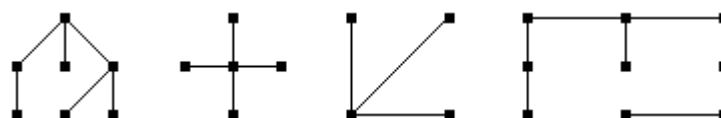
**Theorem:** If a connected graph has a bridge then it cannot have a Hamiltonian circuit. Also if it has 3 vertices of degree 1 then it cannot have a Hamiltonian path.

6.10 For which  $n$  does  $K_n$  have a Hamiltonian path/circuit? How about  $K_{m,n}$ ?

A **tree** is a connected graph whose edges are all bridges. Equivalently, a connected graph is a tree when any one of the following conditions is true.

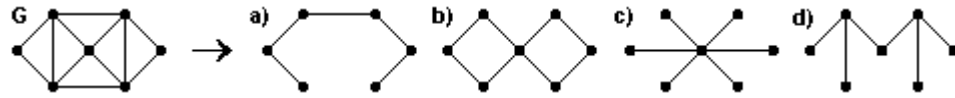
- 1) the graph contains no circuit.
- 2) the number of edges is one less than the number of vertices.
- 3) there is a unique path between any two vertices.

Examples:



6.11 For which  $n$  is  $K_n$  a tree? How about  $K_{m,n}$ ?

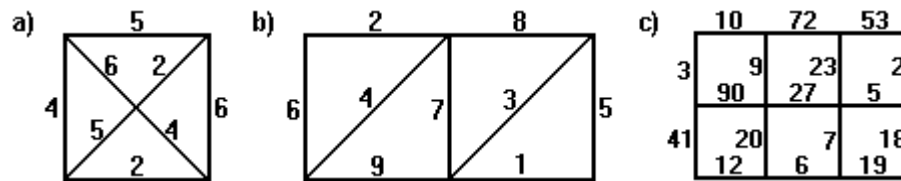
A **spanning tree** of a graph  $G$  is a tree subgraph of  $G$  containing all the vertices of  $G$ . In the example below (a) is not a spanning tree of  $G$  because it lacks one vertex of  $G$ , (b) is not because it is not a tree, (c) is not because it is not a subgraph of  $G$ , (d) is a spanning tree of  $G$ .



6.12 Find two spanning trees for each graph given in Problem 6.5.

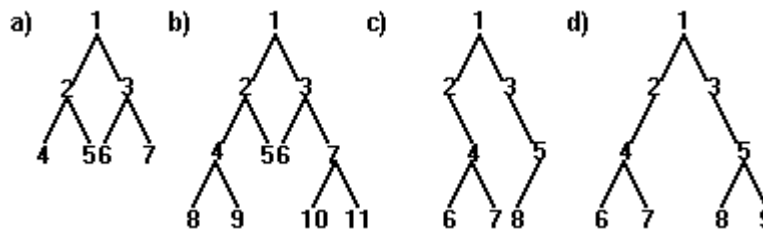
A graph is **weighted** if each edge is associated with a numerical value. A **minimal spanning tree** of a weighted graph is one with smallest possible total value. One way to obtain a minimal spanning tree from a weighted graph is by repeatedly removing the edge with largest value, provided that this action does not disconnect the graph, until what is left forms a tree.

6.13 Find a minimal spanning tree for each weighted graph.



A vertex in a tree can be selected as the **root**, which is placed topmost and from which every edge is directed downward. In this case a vertex one-edge down is called a **child** of the one above it. A **labeled binary tree** is a rooted tree in which every vertex has at most two children which are distinguished as a left and/or a right child, if any.

Examples:



There are 3 common algorithms for traversing the vertices of a labeled binary tree:

- 1) **pre-order** traversal: ROOT  $\rightarrow$  LEFT  $\rightarrow$  RIGHT
- 2) **post-order** traversal: LEFT  $\rightarrow$  RIGHT  $\rightarrow$  ROOT
- 3) **in-order** traversal: LEFT  $\rightarrow$  ROOT  $\rightarrow$  RIGHT

Example: Apply these algorithms to the labeled binary tree (a) in the above example.

pre-order: 1, 2, 4, 5, 3, 6, 7  
 post-order: 4, 5, 2, 6, 7, 3, 1  
 in-order: 4, 2, 5, 1, 6, 3, 7

6.14 Complete the example for (b), (c), and (d) using the 3 algorithms.

Labeled binary trees can be used to represent mathematical expressions in accordance with the in-order traversal. For example  $[5 \times (-3)] + [8 \div (9 - 7)]$  :

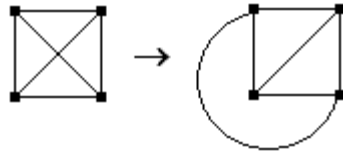


6.15 Represent these expressions using labeled binary trees.

- a)  $(x \times y) + [(y \div x) - (x + y)^3]$
- b)  $(A \cup B) \oplus [(A \cap C) \cup (B - C)]$
- c)  $(p \rightarrow \neg q) \leftrightarrow [\neg p \wedge (q \oplus r)]$

A graph is **planar** if it can be drawn without crossing any edge.

Example:  $K_4$  is planar.

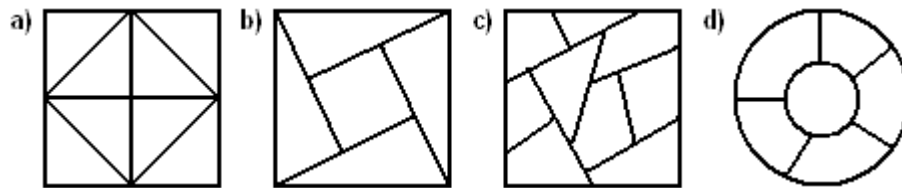


6.16 Are these graphs planar ?

- a)  $K_5$
- b)  $K_{2,2}$
- c)  $K_{2,3}$
- d)  $K_{3,3}$

This particular drawing of a planar graph is called a **map**, and it partitions the plane into a number of **regions**. For example the map of  $K_4$  partitions the plane into 3 interior regions. The **chromatic number** of a map is the minimum number of colors needed to color the interior regions of the map such that regions which share an edge are of different colors.

6.17 Find the chromatic numbers of these maps.



**Theorem:** The chromatic number of any map is at most 4. (The Four-Color Theorem)

6.18 Draw a map with 4 interior regions and with chromatic number equals 4.

The **dual graph**  $G$  of a map  $M$  is defined as follow.

- 1) The vertices of  $G$  are the interior regions of  $M$
- 2) The edges of  $G$  are the boundaries between two regions of  $M$

Example: The dual graph of  $K_4$  is  $K_3$  (Verify it!)

6.19 Draw the dual graphs for the maps given in Problem 6.17.

6.20 The coloring of the map corresponds to the coloring of the vertices of its dual graph such that those connected by an edge have different colors. Now do Problem 6.17 again by coloring the dual graphs.

## Appendix 1 Personalized Projects

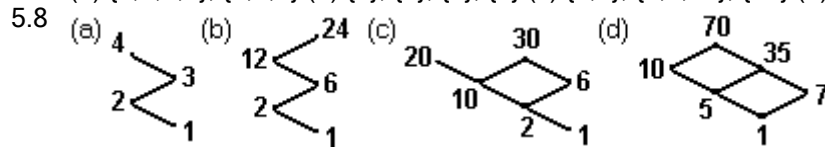
- Convert your university number to (a) binary (b) hexadecimal and (c) octal.
- Use the Euclidean algorithm to compute GCD (m, n) where m is your university number and n is the same number with the digits reversed from right to left.
- How many different permutations can be formed using all the digits of your university number?
- The set A consists of the digits of your university number and  $R = \{(a, b) \mid a \bmod 3 = b \bmod 3\}$ . Show that R is an equivalence relation and then find the equivalence classes.
- The set A consists of the digits of your university number and  $R = \{(a, b) \mid a \text{ divides } b\}$ . Show that  $R \cup \{(0, 0)\}$  is a partial order relation and then draw the Hasse diagram.
- Write your university number in binary and then enter the digits into a 5x5 zero-one matrix, starting from the upper left corner. Ignore any leftover digits.
  - Find the elements of R and draw its digraph.
  - Is R reflexive, symmetric, anti-symmetric, or transitive?
  - Find the matrix for the transitive closure of R.

## Appendix 2 Selected Answers

- 1.1 (a) Amman is in Jordan and  $2 + 2 \neq 5$  (b) T (c) Amman is not in Jordan or  $2 + 2 \neq 5$  (d) T  
 1.2 (a)  $q \leftrightarrow \neg r$  (b)  $p \oplus \neg p$  (c)  $\neg r \rightarrow \neg q$  (d)  $\neg(p \vee r)$  (e)  $r \rightarrow (q \vee p)$   
 1.3 (a) F T T T (b) T T F F (c) T T F T (d) T F T T T F T F (e) F F T F F F F F  
 1.6 (a) If I do not get good mark then I do not study hard (b) If it is hot then it does not rain  
 1.7 (a) tautology (b) tautology (c) contingency (d) tautology (e) contradiction  
 1.9 (a) valid (b) invalid  
 1.11 (a) F (b) T (c) F (d) T  
 1.12 (a)  $(p \vee \neg q) \wedge (p \vee q) \equiv (p \wedge q) \vee (p \wedge \neg q)$  (b)  $(p \vee \neg q) \equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$   
 1.13 (a)  $(\neg p \vee q) \wedge (p \vee q)$  (b)  $(p \wedge \neg q)$  (c)  $(\neg p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee \neg r)$   
 1.14 (a) F (b) F (c)  $0 < x < 1$   
 1.15 (a) T (b) T (c) F (d)  $(x, 0) \vee (0, y)$   
 1.16 (a) T (b) F  
 1.17 (a) T (b) F (c) T (d) T (e) T
- 3.1 (a) 42 (b) 328 (c) 183 (d) 65  
 3.2 (a) 100101 (b) 1100011 (c) 111110100 (d) 1111100111  
 3.3 (a) 1534 (b) 3021 (c) 41120 (d) 69905  
 3.4 (a) 25 (b) 63 (c) 1F4 (d) 3E7  
 3.5 (a) 1011111110 (b) 101111001101 (c) 1010000010100000 (d) 10001000100010001  
 3.6 (a) 2A (b) 148 (c) B7 (d) 41  
 3.7 (a) 45 (b) 143 (c) 764 (d) 1747  
 3.8 (a) 0.8125 (b) 0.015625 (c) 7.875 (d) 0.65625 (e) 273.06665039062  
 3.9 (a)  $0.00001 = 0.08$  (b)  $0.110001 = 0.C4$  (c)  $0.101 = 0.A$  (d)  $0.010101\dots = 0.555\dots$   
 3.10 (a) 0 (b) 5 (c) 3 (d) 11  
 3.13 (a) 5 (b) 3 (c) 24 (d) 25  
 3.14 (a) 6325 (b) 40959 (c) 2736 (d) 29000  
 3.15 (a)  $f(n) = f(n-1) + f(n-2) + f(n-3)$  (b)  $f(n-1) + 10$  (c)  $n \times f(n-1)$  (d)  $f(n-1) + \lfloor \frac{1}{2} f(n-2) \rfloor$   
 3.16 (a)  $f(n) = 3(2^n) - 2(-1)^n$  (b)  $f(n) = 2^n + n(2^{n-1})$  (c)  $a_n = \frac{1}{4} [3^{n+1} + (-1)^n]$  (d)  $a_n = 3n + 1$   
 3.17  $f(n) = \{(1+\sqrt{5})/2\}^n / \sqrt{5} - \{(1-\sqrt{5})/2\}^n / \sqrt{5}$

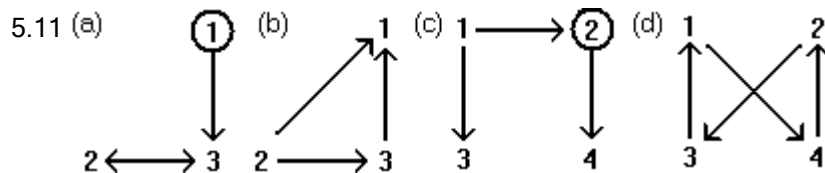
- 4.1 (a) {2,4} (b) {0,6} (c) {0,1,3,5,6} (d) {1,2,3,4,5}
- 4.2 (a) T (b) T (c) F (d) T (e) T
- 4.3 (a)  $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$  (c)  $\{\emptyset\}$  (d)  $\{\emptyset, \{\emptyset\}\}$  (e)  $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
- 4.4 (a) F (b) T (c) F
- 4.5 (a) 97 (b) 13 (c) 49 (d) 1465
- 4.6 (a) 93 (b) 67 (c) 94 (d) 176
- 4.7 (a) 734 (b) 333 (c) 667 (d) 816
- 4.8 (a)  $24! + 23! - 21!$  (b)  $25! + 24! - 22! - 21! + 20!$  (c)  $26! - 25! - 24! + 23!$
- 4.9 (a) 12 (b) 20160 (c) 4989600 (d) 840
- 4.10 (a) {000, 001, 010, 011, 100, 101, 110, 111} (b) {aa, ab, ac, ba, bb, bc, ca, cb, cc}
- 4.11 (a) 35 (b) 35 (c) 1 (d) 8 (e) 127
- 4.12 (a) 78 (b) 45 (c) 21 (d) 6 (e) 9

- 5.1 (a)  $(R^{-1})^{-1} = R$  (b)  $R^3 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (2,4), (3,3), (4,1), (4,3), (4,4)\}$
- 5.2 (c)  $R \circ R^{-1} \neq R^{-1} \circ R$  (d) both =  $\{(1,1), (1,2), (2,2), (2,4), (3,2), (3,3), (3,4), (4,1)\}$
- 5.3 (a) and (b) reflexive, anti-symmetric, transitive (c) anti-symmetric (d) symmetric
- 5.4 (a) {0,2,4,6}, {1,3,5} (b) {1}, {2}, {3}, {4} (c) {0,9}, {5,8,11}, {10} (d) {1,6,11}, {2,7}, {3}



- 5.9 (a) and (b)

5.10 (a)  $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$  (c)  $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  (d)  $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$



- 5.12 (a)  $\{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$  (b)  $R \cup (2,2)$  (c)  $R \cup (2,4)$  (d)  $A \times A$

5.13 (a)  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$  (c)  $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$  (d)  $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

6.1 (a)  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$  (c)  $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

6.2 (a)  $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$  (c)  $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

- 6.3 (a)  $\deg Kn = n(n-1)$  (b)  $\deg Km, n = 2mn$
- 6.4 (a) no (b) Euler path (c) Euler circuit (d) Euler path
- 6.5 (a) path and circuit (b) path only (c) path and circuit (d) path and circuit
- 6.6 (a) 12 (b) 17 (c) 115
- 6.7 (a) pre (1 2 4 8 9 5 3 6 7 10 11) post (8 9 4 5 2 6 10 11 7 3 1) in (8 4 9 2 5 1 6 3 10 7 11)
- 6.8 (a) (12467358), (67428531), (26471385) (d) (124673589), (674289531), (647213859)
- 6.9 (a) no (b) yes (c) yes (d) no
- 6.10 (a) 2 (b) 3 (c) 4 (d) 4