Mathematics for Computing
Homework Collection 1

Use the integral test to determine convergent or divergent.

1. $\sum \frac{n}{n^2+1}$
   Answer: divergent

2. $\sum \frac{n}{e^{n^2}}$
   Solution: Consider $\int_0^\infty \frac{x}{e^{x^2}} dx$
   Let $u = x^2$ and $du = 2xdx$. The integral becomes $\int \frac{1}{2e^u} du = \frac{-1}{2} e^{-u} = \frac{-1}{2e^{x^2}}$
   Evaluate
   $$\int_0^\infty \frac{x}{e^{x^2}} dx = \left(\lim_{x\to\infty} \frac{-1}{2e^{x^2}} \right) - \left(\frac{-1}{2e^0} \right) = 0 - \left(\frac{-1}{2} \right) = \frac{1}{2}$$
   Answer: convergent

Use the ratio test to determine convergent or divergent.

1. $\sum \frac{(-10)^n}{n!}$
   Answer: $L = 0$, convergent

2. $\sum \frac{(-1)^n 2^{3n}}{7^n}$
   Solution:
   $$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} 2^{3n+3}}{7^{n+1}} \times \frac{7^n}{(-1)^n 2^{3n}} \right| = \frac{2^3}{7} > 1$$
   Answer: divergent

Find the interval of convergence.

1. $\sum \frac{nx^n}{2^n}$
   Answer: $(-2, 2)$

2. $\sum \frac{(2n)!\, x^n}{n!}$
   Answer: $\{0\}$

3. $\sum \frac{3^n x^n}{n^3}$ Answer: $[-\frac{1}{3}, \frac{1}{3}]$

4. $\sum \frac{(-1)^n x^n}{2^n (n+1)^2}$
   Solution: first ratio test,
   $$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(n+1)x^{n+1}}{2^{n+1}(n+2)^2} \times \frac{(-1)^n 2^n (n+1)^2}{(-1)^n n x^n} \right|$$
   $$= \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} \left| \frac{x}{2} \right| = \left| \frac{x}{2} \right|$$
   $L < 1 \to$ it is convergent for $-2 < x < 2$.
   $L = 1 \to x = \pm 2$
   $x = 2 \to \sum \frac{(-1)^n n}{(n+1)^2}$ convergent by the alternating series test.
   $x = -2 \to \sum \frac{n}{(n+1)^2}$ divergent by the integral test.
   Answer: $(-2, 2]$

Find the Taylor series at $x = 0$ and its interval of convergence.

1. $f(x) = \ln(1+x)$
   Answer: $\sum_1^\infty \frac{(-1)^{n+1}}{n} x^n$ on $(-1, 1]$

2. $f(x) = \sqrt{1+x}$
   Solution: Write $f(x) = (1+x)^{-1/2}$

$$\begin{aligned}
f &= (1+x)^{-1/2} && \to 1 \\
f' &= -\frac{1}{2}(1+x)^{-3/2} && \to -\frac{1}{2} \\
f^{(2)} &= \frac{1}{2}\cdot\frac{3}{2}(1+x)^{-5/2} && \to \frac{1}{2}\cdot\frac{3}{2} \\
f^{(3)} &= -\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1+x)^{-7/2} && \to -\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2} \\
f^{(4)} &= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}(1+x)^{-9/2} && \to \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}
\end{aligned}$$

   Answer: Sigma below, with I.C. by ratio test, $(-1, 1]$

$$\sum_{n=0}^{\infty}(-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\; n!}x^n$$

Use special function to derive the Taylor series at $x = 0$.

1. $\frac{\sin x}{x}$
   Answer: $\sum_0^\infty \frac{(-1)^n}{(2n+1)!} x^{2n}$

2. e^{-x^2}
   Answer: $\sum_0^\infty \frac{(-1)^n}{n!} x^{2n}$

3. $\sin x^2$
   Answer: $\sum_0^\infty \frac{(-1)^n}{(2n+1)!} x^{4n+2}$

4. $\frac{\ln(1+x)}{x}$
   Answer: $\sum_0^\infty \frac{(-1)^n}{n+1} x^n$

5. $\frac{1-\cos x}{x}$
   Answer: $\sum_1^\infty \frac{(-1)^{n+1}}{(2n)!} x^{2n-1}$

6. $\frac{1-e^{-x}}{x}$
   Solution:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\
e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots \\
1 - e^{-x} &= x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots \\
\frac{1 - e^{-x}}{x} &= 1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!} - \cdots
\end{aligned}$$

   Answer: $\sum_0^\infty \frac{(-1)^n}{(n+1)!} x^n$