Lecture: 7

Stability of Real-Time Systems

Prof. Kasim M. Al-Aubidy
Computer Engineering Department
Philadelphia University
Summer Semester, 2011
Course Objectives:
This unit is concerned with:
• stability analysis of microcontroller-based real-time systems.
• the various techniques available for the analysis of the stability.
• hardware and software design issues with stability requirements.
**Stability of Real-Time Systems:**
Suppose we have a closed-loop system transfer function:

\[
\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}
\]

where \( D(z) = 1 + GH(z) = 0 \) is known as the characteristic equation.

- The stability of the system depends on the location of the poles of the closed-loop transfer function, or the roots of the characteristic equation \( D(z) = 0 \).
- The left-hand side of the \( s \)-plane (a continuous system is stable) maps into the interior of the unit circle in the \( z \)-plane.
- We can say that a system in the \( z \)-plane will be stable if all the roots of the characteristic equation, \( D(z) = 0 \), lie inside the unit circle.
- There are several methods available to check for the stability of a discrete-time system:
  1. Factorize \( D(z) = 0 \) and find the positions of its roots, and hence the position of the closed loop poles.
  2. Determine the system stability without finding the poles of the closed-loop system, such as Jury’s test.
  3. Transform the problem into the \( s \)-plane and analyze the system stability using the well established \( s \)-plane techniques, such as frequency response analysis or the Routh–Hurwitz criterion.
  4. Use the root-locus graphical technique in the \( z \)-plane to determine the positions of the system poles.
The roots of the characteristic equation are:

\[ 1 + G(z) = 0, \quad 1 + \frac{1.729}{z - 0.135} = 0, \]

The solution of which is \( z = -1.594 \) which is outside the unit circle, i.e. the system is not stable.
**Example 2:** For the system given in Example 1, find the value of $T$ for which the system is stable.

\[ G(z) = \frac{2(1 - e^{-2T})}{z - e^{-2T}} \]

\begin{align*}
1 + G(z) &= 0, \text{ or } 1 + \frac{2(1 - e^{-2T})}{z - e^{-2T}} = 0 \\
&= z - e^{-2T} + 2(1 - e^{-2T}) = 0 \\
z &= 3e^{-2T} - 2 \\
|3e^{-2T} - 2| &< 1 \\
2T &< \ln \left( \frac{1}{3} \right) \quad \text{or} \quad T < 0.549
\end{align*}

- Thus, the system will be stable as long as the sampling time $T < 0.549$. 
**Jury’s Stability Test:**

- Jury’s stability test is similar to the Routh–Hurwitz stability criterion used for continuous time systems.
- Jury’s test can be applied to characteristic equations of any order, and its complexity increases for high-order systems.
- To describe Jury’s test, express the characteristic equation of a discrete-time system of order $n$ as

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0$$

where $a_n > 0$.

- We now form the array shown in the following table. The elements of this array are defined as follows:
- The elements of each of the even-numbered rows are the elements of the preceding row, in reverse order.
- The elements of the odd-numbered rows are defined as:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ n_{n-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}$$
**Table** Array for Jury’s stability tests

<table>
<thead>
<tr>
<th>$z^0$</th>
<th>$z^1$</th>
<th>$z^2$</th>
<th>...</th>
<th>$z^{n-k}$</th>
<th>...</th>
<th>$z^{n-1}$</th>
<th>$z^n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0$</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>...</td>
<td>$a_{n-k}$</td>
<td>...</td>
<td>$a_{n-1}$</td>
<td>$a_n$</td>
</tr>
<tr>
<td>$a_n$</td>
<td>$a_{n-1}$</td>
<td>$a_{n-2}$</td>
<td>...</td>
<td>$a_k$</td>
<td>...</td>
<td>$a_1$</td>
<td>$a_0$</td>
</tr>
<tr>
<td>$b_0$</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>...</td>
<td>$b_{n-k}$</td>
<td>...</td>
<td>$b_{n-1}$</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$b_{n-1}$</td>
<td>$b_{n-2}$</td>
<td>$b_{n-3}$</td>
<td>...</td>
<td>$b_{k-1}$</td>
<td>...</td>
<td>$b_0$</td>
<td>$b_0$</td>
</tr>
<tr>
<td>$c_0$</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>...</td>
<td>$c_{n-k}$</td>
<td>...</td>
<td>$c_{n-k}$</td>
<td>$c_{n-2}$</td>
</tr>
<tr>
<td>$c_{n-2}$</td>
<td>$c_{n-3}$</td>
<td>$c_{n-4}$</td>
<td>...</td>
<td>$c_{k-2}$</td>
<td>...</td>
<td>$...$</td>
<td>$...$</td>
</tr>
<tr>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>...</td>
<td>$...$</td>
<td>...</td>
<td>$...$</td>
<td>$...$</td>
</tr>
<tr>
<td>$l_0$</td>
<td>$l_1$</td>
<td>$l_2$</td>
<td>$l_3$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
</tr>
<tr>
<td>$l_3$</td>
<td>$l_2$</td>
<td>$l_1$</td>
<td>$l_0$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
</tr>
<tr>
<td>$m_0$</td>
<td>$m_1$</td>
<td>$m_2$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
<td>$...$</td>
</tr>
</tbody>
</table>
Jury’s Stability Test Conditions:

- The necessary and sufficient conditions for the characteristic equation to have roots inside the unit circle are given as:

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$F(1) &gt; 0$,</td>
</tr>
<tr>
<td>2.</td>
<td>$(-1)^n F(-1) &gt; 0$,</td>
</tr>
<tr>
<td>3.</td>
<td>$</td>
</tr>
<tr>
<td>4.</td>
<td>$</td>
</tr>
</tbody>
</table>

- Jury’s test is then applied as follows:
- Check the first three conditions and stop if any of these conditions is not satisfied.
- Construct the array given in the Table and check the fourth conditions given above. Stop if any condition is not satisfied.
Example 3:
Check stability of a system has an open loop transfer function:

\[ G(z) = \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2} \]

Solution
The characteristic equation is

\[ 1 + G(z) = 1 + \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2} = 0 \]

\[ z^2 - z + 0.7 = 0. \]

Applying Jury’s test,

\[ F(1) = 0.7 > 0, \quad F(-1) = 2.7 > 0, \quad 0.7 < 1 \]

All the conditions are satisfied and the system is stable.
Example 4
The characteristic equation of a system is given by

\[ 1 + G(z) = 1 + \frac{K(0.2z + 0.5)}{z^2 - 1.2z + 0.2} = 0 \]

Determine the value of \( K \) for which the system is stable.

Solution
The characteristic equation is

\[ z^2 + z(0.2K - 1.2) + 0.5K = 0, \quad \text{where} \ K > 0. \]

Applying Jury's test,

\[ F(1) = 0.7K - 0.2 > 0, \]
\[ F(-1) = 0.3K + 2.2 > 0, \quad 0.5K < 1 \]

Thus, the system is stable for \[ 0.285 < K < 2. \]
Example 5
The characteristic equation of a system is given by
\[ F(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0 \]
Determine the stability of the system.

Solution
Applying Jury’s test, \( a_3 = 1, a_2 = -2, a_1 = 1.4, a_0 = -0.1 \) and
\[ F(1) = 0.3 > 0, \quad F(-1) = -4.5 < 0, \quad 0.1 < 1 \]
The first conditions are satisfied. Applying the other condition,
\[
\begin{vmatrix}
-0.1 & 1 \\
1 & -0.1
\end{vmatrix} = -0.99 \quad \text{and} \quad \begin{vmatrix}
-0.1 & 1.4 \\
1 & -2
\end{vmatrix} = -1.2
\]
since \(|0.99| < |-1.2|\), the system is not stable.
Hardware and Software Design and Stability Requirements:

- This will be discussed with examples during lecture.