Mechanical Vibrations

Single Free System

Eng. Laith Batarseh
Single DoF free vibration system

Introduction
Examples

(a) Idealization of the tall structure

\[ k = \frac{3EI}{l^3} \]

(b) Equivalent spring-mass system
Examples

(a) Building frame

Rigid floor
(mass = $m$)

Elastic columns
(mass is negligible)

$\mathbf{x}(t)$

$\mathbf{x}(t)$

$k$

$m$

$\mathbf{x}(t)$
Single DoF free vibration system

- **Equation of motion**
  - Newton's 2nd law of motion
  - Other methods
    - D'Alembert's principle
    - Virtual displacement
    - Energy conservation
Single DoF free vibration system

- Newton's 2nd law of motion
  - Select a suitable coordinate
  - Determine the static equilibrium configuration of the system
  - Draw the free-body diagram
  - Apply Newton's second law of motion
  - The rate of change of momentum of a mass is equal to the force acting on it.
Newton's 2\textsuperscript{nd} law of motion

\[ \vec{F}(t) = \frac{d}{dt} \left( m \frac{d\vec{x}(t)}{dt} \right) = m \frac{d^2\vec{x}(t)}{dt^2} = m \ddot{x} \]

\[ \vec{F}(t) = -kx = m \ddot{x} \Rightarrow m \ddot{x} + kx = 0 \]

Energy conservation

Kinetic energy

\[ T = \frac{1}{2} m \dot{x}^2 \]

Potential energy

\[ U = \frac{1}{2} kx^2 \]

\[ T + U = \text{constant} \]

\[ \frac{d}{dt} (T + U) = 0 \]

\[ m \ddot{x} + kx = 0 \]
Single DoF free vibration system

- **Vertical system**

\[
\ddot{x} = -k(x + \delta_{st}) + mg ; \quad mg = k\delta_{st}
\]

\[\Rightarrow m\ddot{x} + kx = 0\]
Mathematical review

\[ A_1(t)\ddot{x}(t) + A_2(t)\dot{x}(t) + A_3(t)x(t) = F(t) \]

- \( F(t) = 0 \) → Homogenous
- \( F(t) \neq 0 \) → none homogenous

\[ x(t) = C_1x_1 + C_2x_2 \]

If \( A_1, A_2 \) and \( A_3 \) are constants:
- Characteristic equation:
  \[ A_1s^2 + A_2s + A_3 = 0 \]
- \( s = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1} \)

\[ x(t) = C_1x_1 + C_2x_2 + x_p \]

\( x_p \) form is the same type as \( F(t) \)
Single DoF free vibration system

\[ s = \frac{-A_2 \pm \sqrt{A_2^2 - 4A_1A_3}}{2A_1} \]

Only one real root:
\[ s_{1,2} \]
\[ x(t) = C_1e^{st} + C_2te^{st} \]

Two unequal real roots:
\[ s_{1,2} \]
\[ x(t) = C_1e^{s_1t} + C_2e^{s_2t} \]

Complex root:
\[ s_{1,2} = \alpha \pm \beta i \]
\[ x(t) = e^{\alpha t} \{ C_1\cos(\beta t) + C_1\sin(\beta t) \} \]
Solution

- \( m \ddot{x} + kx = 0 \) is 2nd order homogeneous differential equation

where: \( A_1 = m, A_2 = 0 \) and \( A_3 = k \)

\[
s = \frac{\pm \sqrt{-4mk}}{2m} = \pm i \sqrt{\frac{k}{m}} = \pm i \omega_n
\]

- the solution: \( x(t) = e^{\alpha t} \{ C_1 \cos(\beta t) + C_1 \sin(\beta t) \} \)

where \( \alpha = 0 \) and \( \beta = \omega_n \)

- \( x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) \)
Solution

$C_1$ and $C_2$ can be determined from initial conditions (I.Cs). For this case we need two I.Cs.

$$x(t = 0) = C_1 = x_o$$

$$\dot{x}(t = 0) = C_2 \omega_n = \dot{x}_o$$

The I.Cs for this case would be:

So: $x(t) = x_o \cos(\omega_n t) + \frac{x_o}{\omega_n} \sin(\omega_n t) \quad \text{--- Eq.1}$
Harmonic motion

Introduce Eq.2 into Eq.1: \( x(t) = A \cos(\omega_n t - \phi) = A_o \sin(\omega_n t + \phi_o) \)
**Single DoF free vibration system**

### Harmonic motion

\[
x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)
\]

\[
x(t) = x_0 \cos(\omega_n t) + \frac{x_0}{\omega_n} \sin(\omega_n t)
\]

Assume:

\[C_1 = A \cos(\phi) \quad \text{--- Eq.2(a)}\]

\[C_2 = A \sin(\phi) \quad \text{--- Eq.2(b)}\]

**Harmonic functions in time.**

**Mass spring system is called harmonic oscillator.**

\[A = \sqrt{C_1^2 + C_2^2} = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \text{Amplitude}\]

\[A_o = A\]

\[\phi = \tan^{-1}\left(\frac{C_2}{C_1}\right) = \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n}\right) = \text{Phase angle}\]

\[\phi_o = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right)\]
Single DoF free vibration system

- Harmonic motion

\[ x(t) = A \cos (\omega_n t - \phi) \]

\[ A = \sqrt{x_0^2 + \left(\frac{x_0}{\omega_n}\right)^2} \]

Slope = \( \dot{x}_0 \)

Velocity maximum

Amplitude,
Natural Frequency (N.F)

A system property (i.e. depends on system parameters $m$ and $k$)

Unit: rad/sec

It is related to the periodic time ($\tau$): $\tau = \frac{2\pi}{\omega_n}$

Periodic time is the time taken to complete one cycle (i.e. 4 strokes)

The relation between the $\omega_n$ and $\tau$ is inverse relation
Example 2.1

The column of the water tank shown in Fig is 90m high and is made of reinforced concrete with a tubular cross section of inner diameter 2.4m and outer diameter 3m. The tank mass equal $3 \times 10^5$ kg when filled with water. By neglecting the mass of the column and assuming the Young’s modulus of reinforced concrete as 30 Gpa, determine the following:

- the natural frequency and the natural time period of transverse vibration of the water tank
- the vibration response of the water tank due to an initial transverse displacement of 0.3m.
- the maximum values of the velocity and acceleration experienced by the tank.
Example 2.1 solution:

Initial assumptions:

1. the water tank is a point mass
2. the column has a uniform cross section
3. the mass of the column is negligible
4. the initial velocity of the water tank equal zero
Example 2.1 solution:

a. Calculation of natural frequency:

1. Stiffness: \( k = \frac{3EI}{l^3} \) But: \( I = \frac{\pi}{64} (d_o^4 - d_i^4) = \frac{\pi}{64} (3^4 - 2.4^4) = 2.3475 \, m^4 \)

So: \( k = \frac{3 \times 30 \times 10^9 \times 2.3475}{90^3} = 289,812 \, N/m \)

2. Natural frequency: \( \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{289,812}{3 \times 10^5}} = 0.9829 \, rad/s \)
Example 2.1 solution:

b. Finding the response:

1. \( x(t) = A \sin(\omega_n t + \phi) \)

\[
A = \sqrt{x_o^2 + \left(\frac{x_o}{\omega_n}\right)^2} = x_o = 0.3m \\
\phi = \tan^{-1}\left(\frac{x_o \omega_n}{x_o \cdot \frac{\omega_n}{0}}\right) = \tan^{-1}\left(\frac{x_o \omega_n}{0}\right) = \frac{\pi}{2}
\]

So, \( x(t) = 0.3 \sin (0.9829 t + 0.5\pi) \)
Example 2.1 solution:

c. Finding the max velocity:

\[ \dot{x}(t) = 0.3(0.9829) \cos \left( 0.9829t + \frac{\pi}{2} \right) \Rightarrow \dot{x}_{\text{max}} = 0.3(0.9829) = 0.2949 \text{m/s} \]

Finding the max acceleration:

\[ \ddot{x}(t) = -0.3(0.9829)^2 \sin \left( 0.9829t + \frac{\pi}{2} \right) \Rightarrow \ddot{x}_{\text{max}} = 0.3(0.9829)^2 = 0.2898 \text{m/s}^2 \]
Example 2: Q2.13
Find the natural frequency of the pulley system shown in Fig. by neglecting the friction and the masses of the pulleys.
Example 2: Q2.13

Solution:
1. Free body diagram

2. \( x = 2x_1 + 2x_2 \) ---- Eq.1
Example 2: Q2.13
Solution:

3. Equilibrium for pulley_1: \(2P = k_1 x_1 = 2k x_1 \) ---- Eq.2

4. Equilibrium for pulley_2: \(2P = k_2 x_2 = 2k x_2 \) ---- Eq.3

5. Substitute Eqs 2 and 3 in Eq.1: \(x = 2 \left( \frac{2P}{k_1} \right) + 2 \left( \frac{2P}{k_2} \right) = 4P \left( \frac{1}{2k} + \frac{1}{2k} \right) = \frac{4P}{k} \)

6. Let \(k_{eq}\) is the equivalent spring constant for the system: \(k_{eq} = \frac{P}{x} = \frac{k}{4} \)

7. Mathematical model: \(m x'' + k x = 0 \)

8. Natural frequency: \(\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}} \)
Rotational system

Governing equation:

\[ \sum M_o = J_o \alpha = J_o \ddot{\theta} \]

\[ J_o \ddot{\theta} + mgl \sin(\theta) = 0 \]

Assume \( \theta \) is very small

\[ \sin(\theta) \approx \theta \]

\[ J_o \ddot{\theta} + (mgl)\theta = 0 \]

Natural frequency \( (\omega_n) \)

\[ \omega_n = \sqrt{\frac{mgl}{J_o}} \implies \tau = 2\pi \sqrt{\frac{J_o}{mgl}} \]
## Torsional system

### Governing equation:

\[
\sum M_O = J_o \alpha = J_o \ddot{\theta}
\]

\[
J_o \ddot{\theta} + k_T \theta = 0
\]

### Natural frequency \((\omega_n)\)

\[
\omega_n = \sqrt{\frac{k_T}{J_o}} \Rightarrow \tau = 2\pi \sqrt{\frac{J_o}{k_T}}
\]

\[
J_o = \frac{\rho h \pi D^4}{32} = \frac{WD^2}{8g}
\]
Single DoF free vibration system

Solution

\[ \theta(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t) \]

\[ \theta(t = 0) = A_1 = \theta_o \]

\[ \dot{\theta}(t = 0) = A_2 \omega_n = \dot{\theta}_o \]

\[ \theta(t) = \theta_o \cos(\omega_n t) + \frac{\dot{\theta}_o}{\omega_n} \sin(\omega_n t) \]
Example 2.3

Any rigid body pivoted at a point other than its center of mass will oscillate about the pivot point under its own gravitational force. Such a system is known as a compound pendulum (see the Fig). Find the natural frequency of such a system.
**Solution**

The governing equation is found as:

\[ J_o \ddot{\theta} + Wd \sin(\theta) = 0 \]

Assume small angle of vibration:

\[ J_o \ddot{\theta} + (Wd)\theta = 0 \]

So:

\[ \omega_n = \sqrt{\frac{Wd}{J_o}} = \sqrt{\frac{mgd}{J_o}} \]
Example 2.4: Q2.12

Find the natural frequency of the system shown in Fig. with the springs $k_1$ and $k_2$ in the end of the elastic beam.
Example 2.4: Q2.12

Solution: F.B,D
Example 2.4: Q2.12

Solution:

- $k_{eq}$ is equivalent stiffness for the combination of $k_1$, $k_2$ and $k_{beam}$

$$k_{beam} = \frac{3EI}{l^3}$$

- $k_1$ and $k_2$ equivalent: apply energy concept

$$\frac{1}{2} k_{eq,1,2} x^2 = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2 \Rightarrow k_{eq} = k_1 \left( \frac{x_1}{x} \right)^2 + k_2 \left( \frac{x_2}{x} \right)^2$$
Example 2.4: Q2.12

Solution:

Finding $k_{eq}$

\[
\frac{1}{k_{eq}} = \frac{1}{k_{eq,1,2}} + \frac{1}{k_{beam}} \implies k_{eq} = \frac{k_{eq,1,2}k_{beam}}{k_{eq,1,2} + k_{beam}}
\]
Example 2.4: Q2.12

Solution:

Finding natural frequency

\[ \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq,1,2}k_{beam}}{m(k_{eq,1,2} + k_{beam})}} \]

\[ \omega_n = \sqrt{\frac{\left( k_1 \left( \frac{x_1}{x} \right)^2 + k_2 \left( \frac{x_2}{x} \right)^2 \right) k_{beam}}{m \left( k_1 \left( \frac{x_1}{x} \right)^2 + k_2 \left( \frac{x_2}{x} \right)^2 + k_{beam} \right)}} \]

x\textsubscript{1}, x\textsubscript{2} and x can be found from strength relation
Example 2.5: Q2.7

Three springs and a mass are attached to a rigid, weightless bar PQ as shown in Fig. Find the natural frequency of vibration of the system.
Example 2.5: Q2.7

Solution:
Assume small angular motion \( \sin(\theta) \approx \theta \)

\[
\frac{1}{2} k_{eq,1,2} (\theta_3)^2 = \frac{1}{2} k_1 (\theta_1)^2 + \frac{1}{2} k_2 (\theta_2)^2 \implies k_{eq,1,2} = \frac{k_1 l_1^2 + k_2 l_2^2}{l_3^2}
\]

Let \( k_{eq} \) is the equivalent stiffness for the whole system

\[
\frac{1}{k_{eq}} = \frac{1}{k_{eq,1,2}} + \frac{1}{k_3} \implies k_{eq} = \frac{k_{eq,1,2} k_3}{k_3 + k_{eq,1,2}}
\]
Example 2.5: Q2.7

Solution:
Now find the natural frequency

\[ \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}} \]
Example 2.5: Q2.45

Draw the free-body diagram and derive the equation of motion using Newton's second law of motion for each of the systems shown in Fig.
Example 2.5: Q2.45

Solution F.B.D

![Diagram of a single DoF free vibration system](image-url)
Example 2.5: Q2.45

Equation of motion:

The distance: \( x = 4r(\theta + \theta_o) \)

For mass \( m \): \( mg - T = m\ddot{x} \) \quad \text{--- (1)}

For pulley \( J_o \): \( J_o \ddot{\theta} = Tr - 4rk(\theta + \theta_o)(4r) \) \quad \text{--- (2)}

According to static equilibrium: \( mgr = k(4r)(4r)\theta_o \Rightarrow \theta_o = \frac{mg}{16rk} \) \quad \text{--- (3)}
Example 2.5: Q2.45

Equation of motion [cont]:
Substitute equations 1 and 3 into equation 1:

\[ J_o \ddot{\theta} = \left( mg - m x \right) r - 16kr^2 \left( \theta + \frac{mg}{16rk} \right) \]

\[ J_o \ddot{\theta} - \ddot{m}gr + m x r + 16kr^2 \theta + \ddot{m}gr = 0 \Rightarrow J_o \ddot{\theta} + m x r + 16kr^2 \theta = 0 \]

Use the relation \( x = r \theta \Rightarrow \ddot{x} = r \ddot{\theta} \) to relate the translational motion with the rotational one:

\[ (J_o + mr^2) \ddot{\theta} + (16kr^2) \theta = 0 \]
End of chapter 2 – part I