Sampled Data Systems and the Z-Transform

Digital Control

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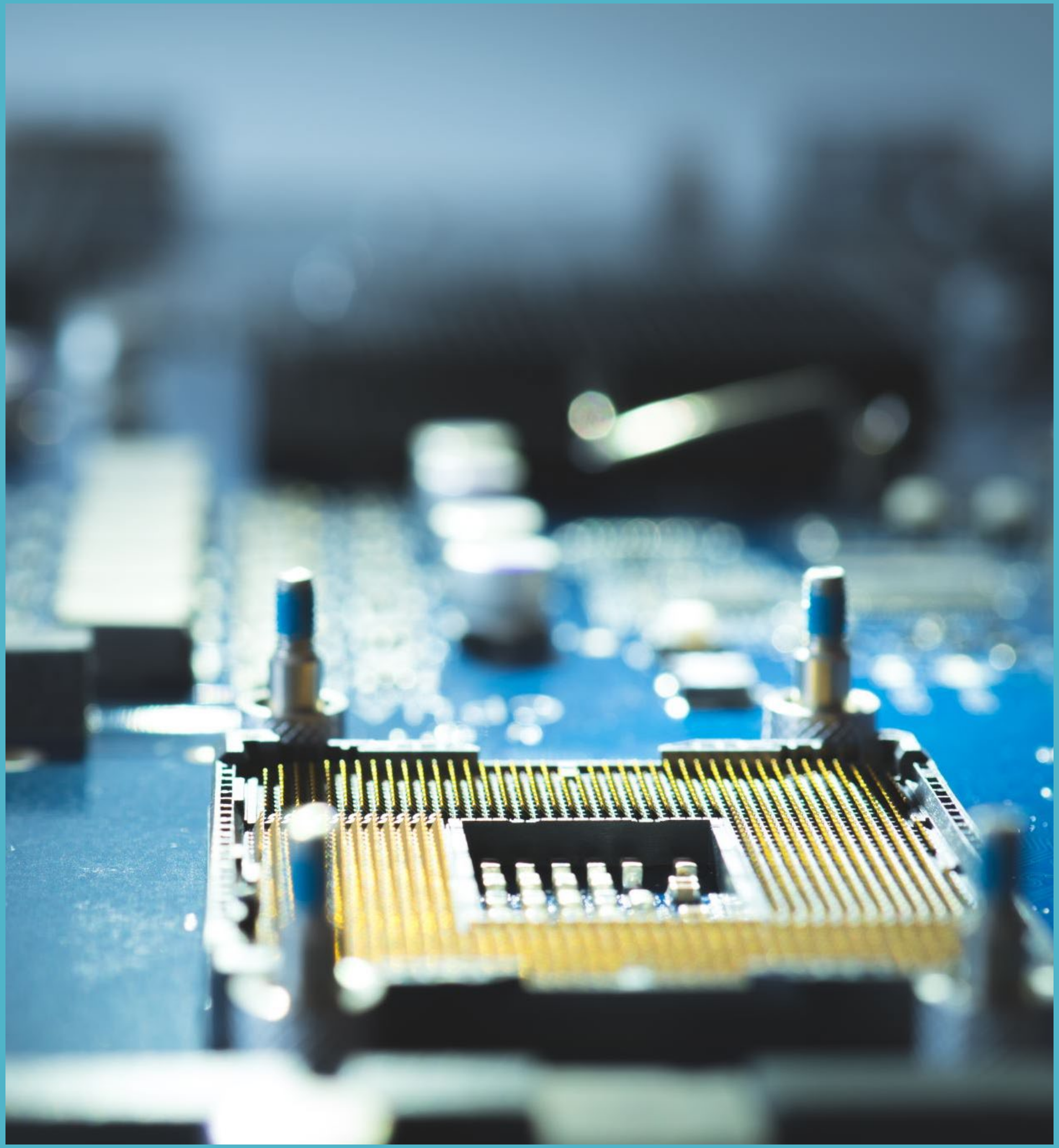


In the previous lecture

- Mechanical systems modelling
- Electrical systems modelling
- Electromechanical systems modelling

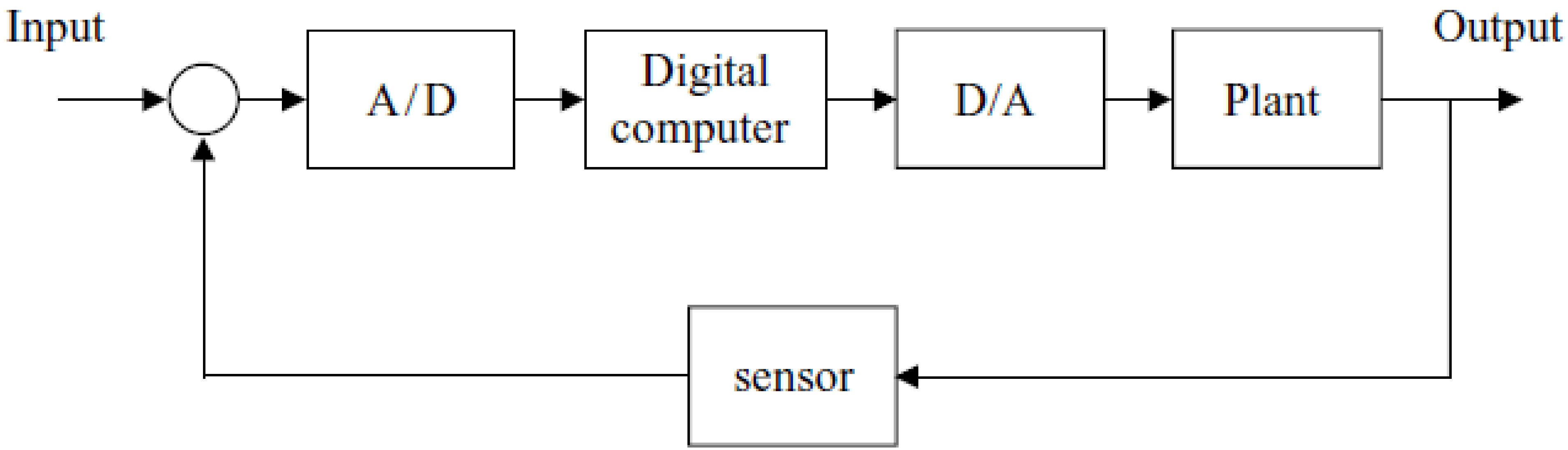
Outline

- Sampling
- Quantization
- Z-transform

Sampling

1. Sampled data system operates on discrete-time rather than continuous-time signals.
2. A **digital computer** is used as the controller in such a system.
3. A **D/A** converter is usually connected to the output of the computer to drive the plant.
4. We will assume that all the signals enter and leave the computer at the same fixed times, known as the **sampling times**.
5. The digital computer performs the controller or the compensation function within the system.
6. The **A/D** converter converts the error signal, which is a continuous signal, into digital form so that it can be processed by the computer.
7. At the computer output the D/A converter converts the digital output of the computer into a form which can be used to drive the plant.

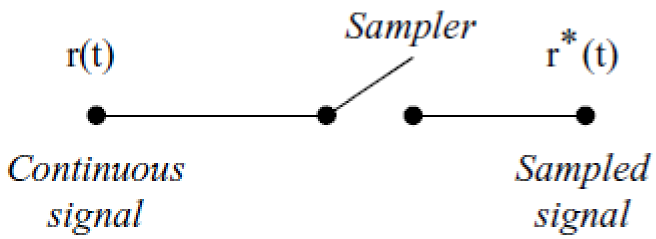
Sampling

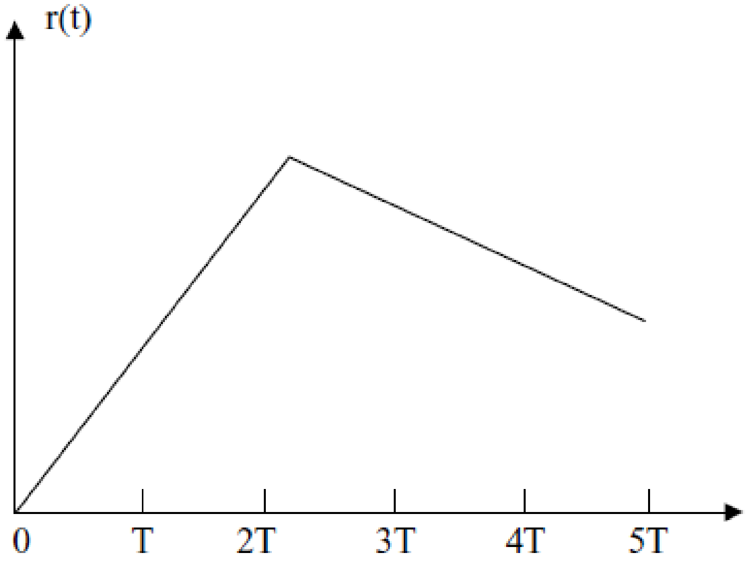


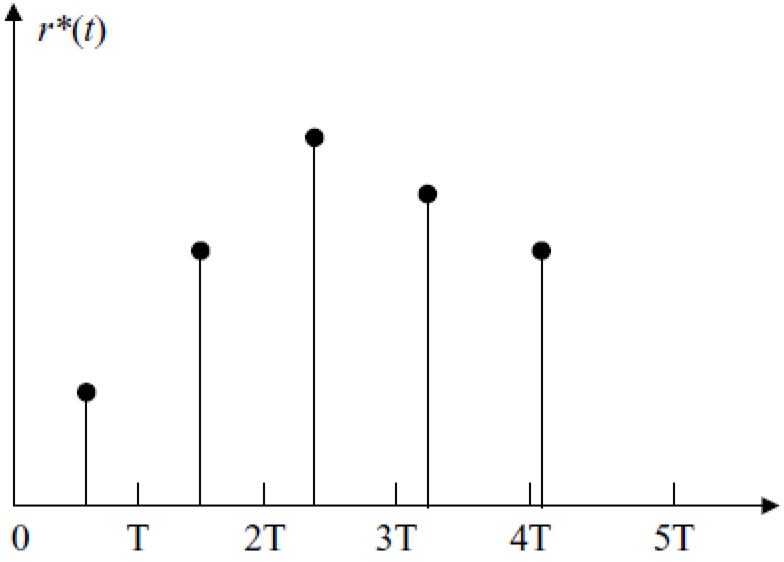
THE SAMPLING PROCESS

A sampler is basically a switch that closes every T seconds.

When a continuous signal $r(t)$ is sampled at regular intervals T, the resulting discrete-time signal







THE SAMPLING PROCESS

The ideal sampling process can be considered as the multiplication of a pulse train with a continuous signal, i.e.

$$r^*(t) = P(t)r(t),$$

where $P(t)$ is the delta pulse train as shown in Figure 6.6, expressed as

$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT);$$

thus,

$$r^*(t) = r(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

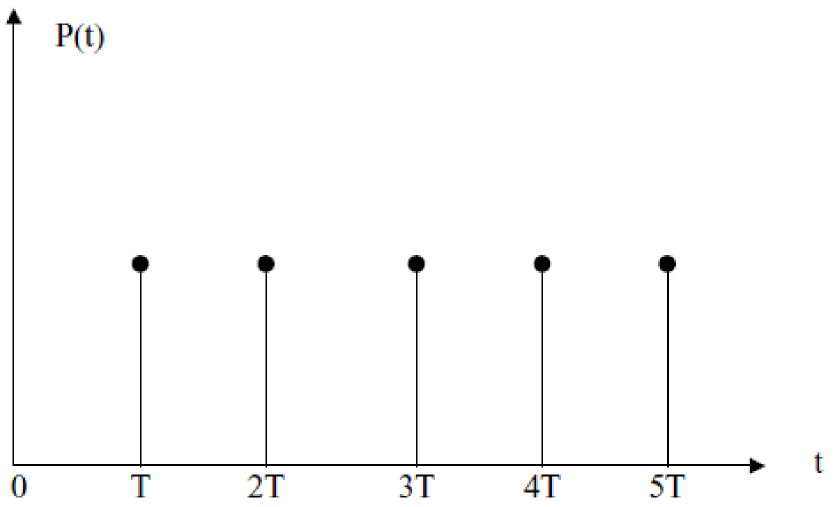


Figure 6.6 Delta pulse train

THE SAMPLING PROCESS

or

$$r^*(t) = \sum_{n=-\infty}^{\infty} r(nT)\delta(t - nT). \tag{6.4}$$

Now

$$r(t) = 0, \quad \text{for } t < 0, \tag{6.5}$$

and

$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t - nT). \tag{6.6}$$

Taking the Laplace transform of (6.6) gives

$$R^*(s) = \sum_{n=0}^{\infty} r(nT)e^{-snT}. \tag{6.7}$$

Equation (6.7) represents the Laplace transform of a sampled continuous signal $r(t)$.

Zero-order hold (ZOH)

A D/A converter converts the sampled signal $r^*(t)$ into a continuous signal $y(t)$. The D/A can be approximated by a zero-order hold (ZOH) circuit as shown in Figure 6.7. This circuit remembers the last information until a new sample is obtained, i.e. the zero-order hold takes the value $r(nT)$ and holds it constant for $nT \leq t < (n+1)T$, and the value $r(nT)$ is used during the sampling period.

The impulse response of a zero-order hold is shown in Figure 6.8. The transfer function of a zero-order hold is given by

$$G(t) = H(t) - H(t - T), \qquad (6.8)$$

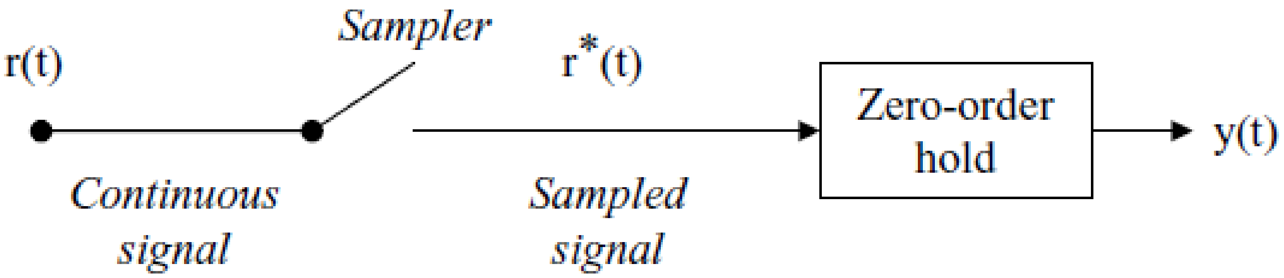


Figure 6.7 A sampler and zero-order hold

Zero-order hold (ZOH)

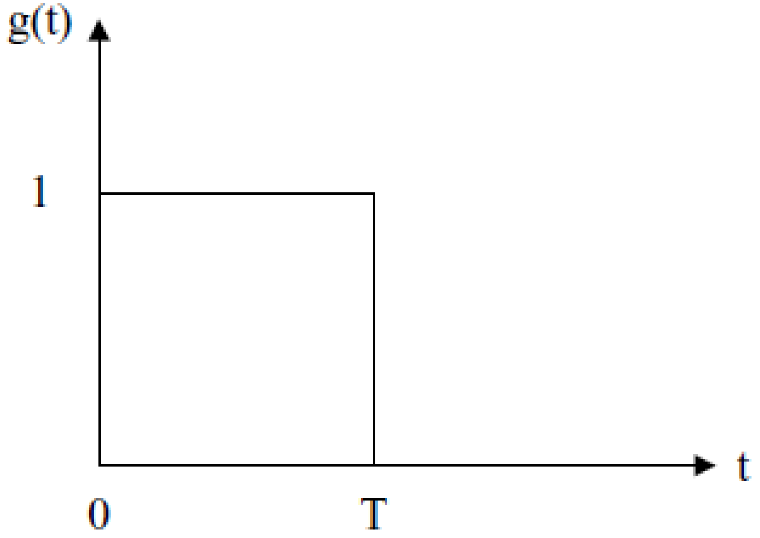


Figure 6.8 Impulse response of a zero-order hold

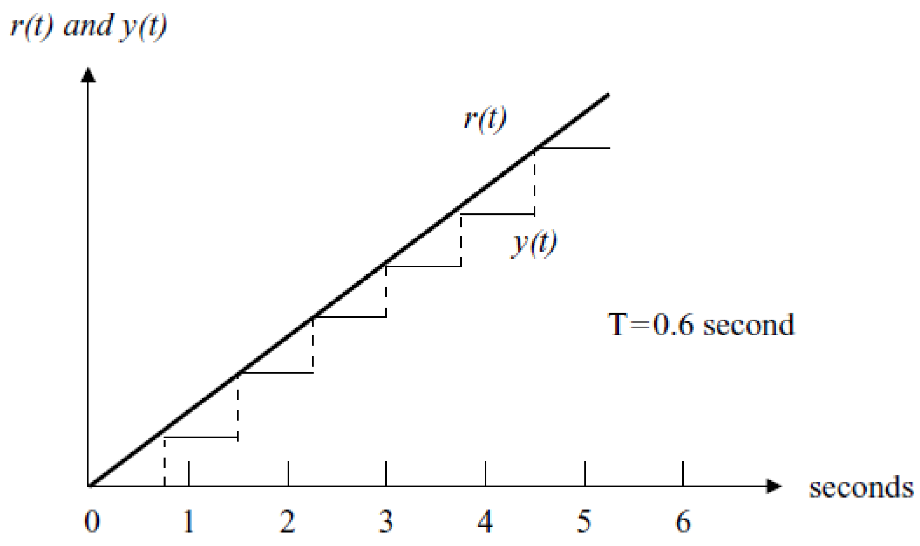
where $H(t)$ is the step function, and taking the Laplace transform yields

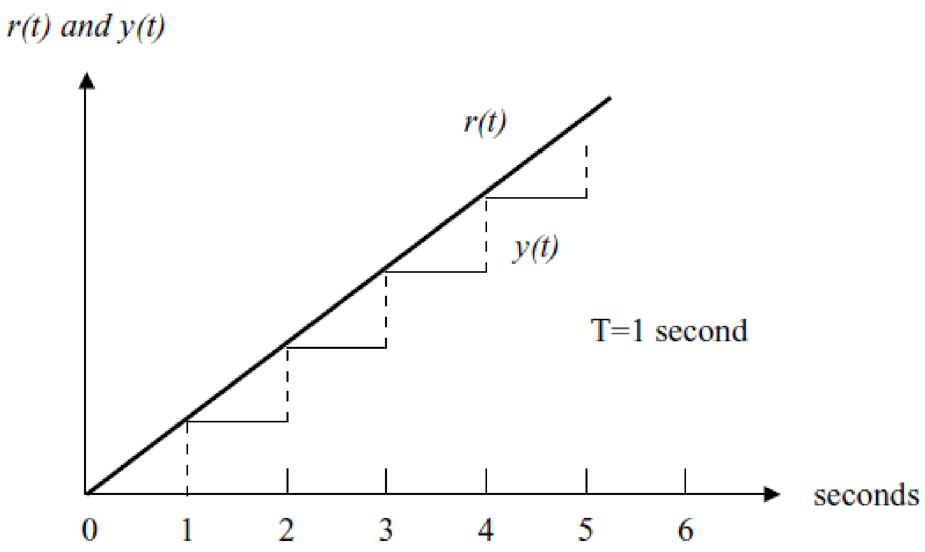
$$G(s) = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s}. \tag{6.9}$$

Zero-order hold (ZOH)

- A sampler and zero-order hold can accurately follow the input signal if the sampling time T is small compared to the transient changes in the signal.

- The response of a sampler and a zero-order hold to a ramp input is shown in Figure 6.9 for two different values of sampling period.





Example

Figure 6.10 shows an ideal sampler followed by a zero-order hold.

Assuming the input signal *r* (*t*) is as shown in the figure, show the waveforms after the sampler and also after the zero-order hold.

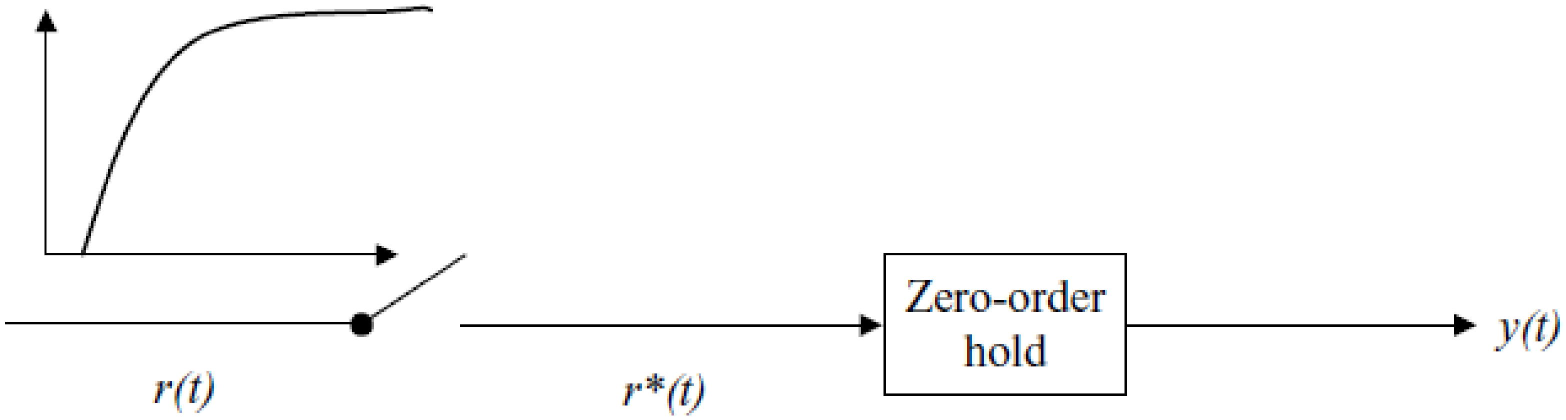


Figure 6.10 Ideal sampler and zero-order hold for Example 6.1

Example

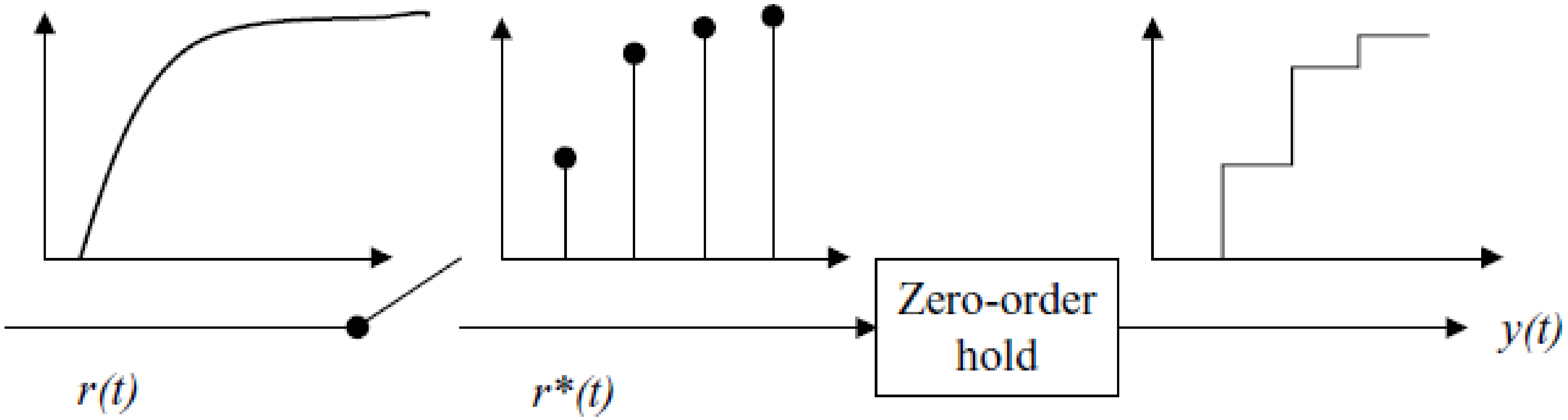


Figure 6.11 Solution for Example 6.1

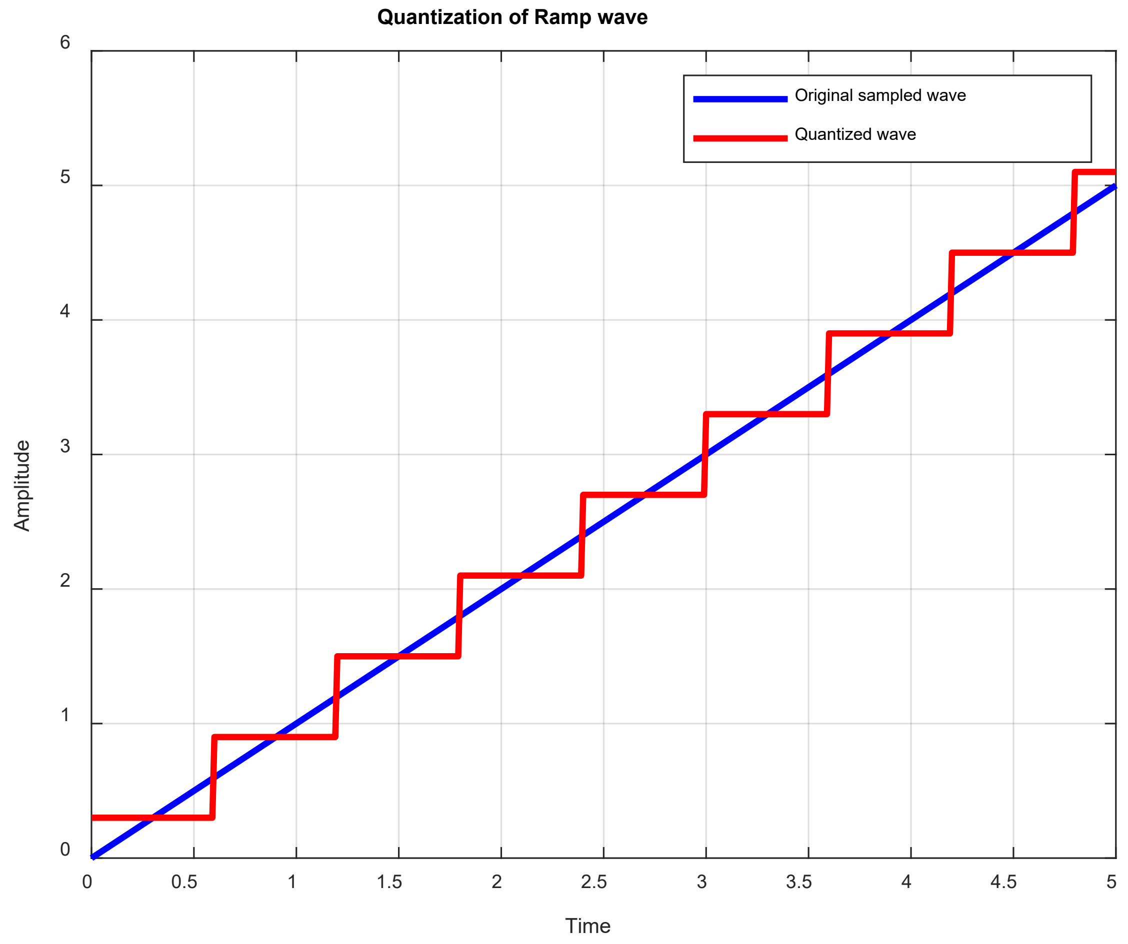
Quantization

- Quantization is the process of representing an analogue or continuous signal in discrete-states.
- Any A/D conversion involves **quantization error** which is due to the fact that analogue numbers should be rounded off to the nearest digital level.
- The analogue quantity is approximated by a finite digital number (digital word).
- Quantization Level depend on the Full-scale range (FSR) and the number of bits in the quantizer (n).
- Quantization level (Q)= FSR /(2^n)
- Quantization error ranges between [–Q/2 --- Q/2] = r(t)-y(t)

Example Quantization

Find the Quantization error of the given signal at t = 0, 1, 2, 3, 4 and 5.

t	r(t)	y(t)	e(t)
0	0	0.3	-0.3
1	1	0.9	0.1
2	2	2.1	-0.1
3	3	2.8	0.2
4	4	3.9	0.1
5	5	5.1	-0.1



THE Z-TRANSFORM

The z-transformation is used in sampled data systems just as the Laplace transformation is used in continuous-time systems.

The z-transform is defined so that: $Z = e^{sT}$

the z-transform of the function $r(t)$ is $Z[r(t)] = R(z)$ which, from (6.7), is given by

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n}.$$

Notice that the z-transform consists of an infinite series in the complex variable z, and

$$R(z) = r(0) + r(T)z^{-1} + r(2T)z^{-2} + r(3T)z^{-3} + \ldots,$$

i.e. the $r(nT)$ are the coefficients of this power series at different sampling instants.

THE Z-TRANSFORM

- The response of a sampled data system can be determined easily by finding the z-transform of the output and then calculating the inverse z-transform.
- Just like the Laplace transform techniques used in continuous-time systems.

THE Z-TRANSFORM

6.2.1 Unit Step Function

Consider a unit step function as shown in Figure 6.12, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots$$

or

$$R(z) = \frac{z}{z-1}, \quad \text{for } |z| \ > 1.$$

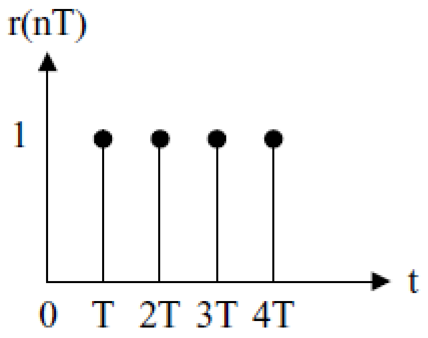


Figure 6.12 Unit step function

THE Z-TRANSFORM

6.2.2 Unit Ramp Function

Consider a unit ramp function as shown in Figure 6.13, defined by

$$r(nT) = \begin{cases} 0, & n < 0, \\ nT, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} nTz^{-n} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + 4Tz^{-4} + \ldots$$

or

$$R(z) = \frac{Tz}{(z-1)^2}, \qquad \text{for } |z| > 1.$$

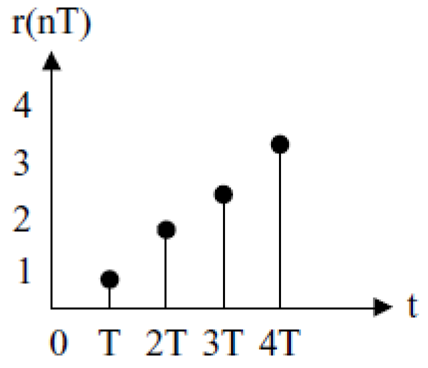


Figure 6.13 Unit ramp function

THE Z-TRANSFORM

6.2.3 Exponential Function

Consider the exponential function shown in Figure 6.14, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ e^{-anT}, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = 1 + e^{-aT} z^{-1} + e^{-2aT} z^{-2} + e^{-3aT} z^{-3} + \ldots$$

or

$$R(z) = \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}, \quad \text{for } |z| < e^{-aT}. \tag{6.12}$$

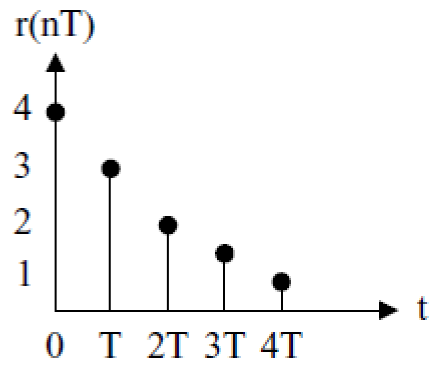


Figure 6.14 Exponential function

THE Z-TRANSFORM

6.2.4 General Exponential Function

Consider the general exponential function

$$r(n) = \begin{cases} 0, & n < 0, \\ p^n, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = 1 + pz^{-1} + p^2 z^{-2} + p^3 z^{-3} + \ldots$$

or

$$R(z) = \frac{z}{z - p}, \quad \text{for } |z| < |p|.$$

Similarly, we can show that

$$R(p^{-k}) = \frac{z}{z - p^{-1}}.$$

THE Z-TRANSFORM

6.2.5 Sine Function

Consider the sine function, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ \sin n\omega T. & n \geq 0. \end{cases}$$

Recall that

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j},$$

so that

$$r(nT) = \frac{e^{jn\omega T} - e^{-jn\omega T}}{2j} = \frac{e^{jn\omega T}}{2j} - \frac{e^{-jn\omega T}}{2j}. \tag{6.13}$$

But we already know from (6.12) that the z-transform of an exponential function is

$$R(e^{-anT}) = R(z) = \frac{z}{z - e^{-aT}}.$$

Therefore, substituting in (6.13) gives

$$R(z) = \frac{1}{2j}\left(\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2j}\left(\frac{z(e^{j\omega T} - e^{-j\omega T}}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1}\right)$$

or

$$R(z) = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}.$$

6.2.6 Cosine Function

Consider the cosine function, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ \cos n\omega T, & n \geq 0. \end{cases}$$

Recall that

$$\cos x = \frac{e^{jx} + e^{-jx}}{2},$$

so that

$$r(nT) = \frac{e^{jn\omega T} + e^{-jn\omega T}}{2} = \frac{e^{jn\omega T}}{2} + \frac{e^{-jn\omega T}}{2}. \qquad (6.14)$$

But we already know from (6.12) that the z-transform of an exponential function is

$$R(e^{-anT}) = R(z) = \frac{z}{z - e^{-aT}}.$$

Therefore, substituting in (6.14) gives

$$R(z) = \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right)$$

or

$$R(z) = \frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}.$$

THE Z-TRANSFORM

6.2.7 Discrete Impulse Function

Consider the discrete impulse function defined as

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1.$$

THE Z-TRANSFORM

6.2.8 Delayed Discrete Impulse Function

The delayed discrete impulse function is defined as

$$\delta(n-k) = \begin{cases} 1, & n = k > 0, \\ 0, & n \neq k. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} z^{-n} = z^{-n}.$$

Tables of Z-Transforms

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
$(kT)^2$	$\dfrac{T^2 z(z+1)}{2(z-1)^3}$
$(kT)^3$	$\dfrac{T^3 z(z^2+4z+1)}{(z-1)^4}$
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\dfrac{z}{z-a}$
$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\dfrac{z \sin aT}{z^2 - 2z\cos aT + 1}$
$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$
$e^{-akT} \sin bkT$	$\dfrac{e^{-aT} z \sin bT}{z^2 - 2e^{-aT} z \cos bT + e^{-2aT}}$
$e^{-akT} \cos bkT$	$\dfrac{z^2 - e^{-aT} z \cos bT}{z^2 - 2e^{-aT} z \cos bT + e^{-2aT}}$

THE Z-TRANSFORM

6.2.10 The z-Transform of a Function Expressed as a Laplace Transform

It is important to realize that although we denote the z-transform equivalent of $G(s)$ by $G(z)$, $G(z)$ is *not* obtained by simply substituting z for s in $G(s)$. We can use one of the following methods to find the z-transform of a function expressed in Laplace transform format:

- Given $G(s)$, calculate the time response $g(t)$ by finding the inverse Laplace transform of $G(s)$. Then find the z-transform either from the first principles, or by looking at the z-transform tables.
- Given $G(s)$, find the z-tranform $G(z)$ by looking at the tables which give the Laplace transforms and their equivalent z-transforms (e.g. Table 6.1).
- Given the Laplace transform $G(s)$, express it in the form $G(s) = N(s)/D(s)$ and then use the following formula to find the z-transform $G(z)$:

$$G(z) = \sum_{n=1}^{p} \frac{N(x_n)}{D'(x_n)} \frac{1}{1 - e^{x_n T} z^{-1}}, \tag{6.15}$$

where $D' = \partial D/\partial s$ and the x_n, $n = 1, 2, \ldots, p$, are the roots of the equation $D(s) = 0$.

THE Z-TRANSFORM: Example

Example 6.2

Let

$$G(s) = \frac{1}{s^2 + 5s + 6}.$$

Determine $G(z)$ by the methods described above.

THE Z-TRANSFORM: Example

Solution

Method 1: By finding the inverse Laplace transform. We can express $G(s)$ as a sum of its partial fractions:

$$G(s) = \frac{1}{(s+3)(s+2)} = \frac{1}{s+2} - \frac{1}{s+3}. \tag{6.16}$$

The inverse Laplace transform of (6.16) is

$$g(t) = L^{-1}[G(s)] = e^{-2t} - e^{-3t}. \tag{6.17}$$

From the definition of the z-transforms we can write (6.17) as

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} (e^{-2nT} - e^{-3nT}) z^{-n} \\ &= (1 + e^{-2T} z^{-1} + e^{-4T} z^{-2} + \ldots) - (1 + e^{-3T} z^{-1} + e^{-6T} z^{-2} + \ldots) \\ &= \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-3T}} \end{aligned}$$

THE Z-TRANSFORM: Example

Method 2: By using the z-transform transform tables for the partial product. From Table 6.1, the z-transform of $1/(s+a)$ is $z/(z-e^{-aT})$. Therefore the z-transform of (6.16) is

$$G(z) = \frac{z}{z-e^{-2T}} - \frac{z}{z-e^{-3T}}$$

or

$$G(z) = \frac{z(e^{-2T}-e^{-3T})}{(z-e^{-2T})(z-e^{-3T})}.$$

THE Z-TRANSFORM: Example

Method 3: By using the z-transform tables for G(s). From Table 6.1, the z-transform of

$$G(s) = \frac{b-a}{(s+a)(s+b)} \tag{6.18}$$

is

$$G(z) = \frac{z(e^{-aT} - e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}. \tag{6.19}$$

Comparing (6.18) with (6.16) we have, $a = 2$, $b = 3$. Thus, in (6.19) we get

$$G(z) = \frac{z(e^{-2T} - e^{-3T})}{(z-e^{-2T})(z-e^{-3T})}.$$

Laplace transform	Corresponding z-transform
$\frac{1}{s}$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s^3}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-a}{(s+a)(s+b)}$	$\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{(b-a)z^2-(be^{-aT}-ae^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\frac{z \sin aT}{z^2-2z \cos aT+1}$
$\frac{s}{s^2+a^2}$	$\frac{z^2-z \cos aT}{z^2-2z \cos aT+1}$
$\frac{s}{(s+a)^2}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$

THE Z-TRANSFORM: Example

Method 4: By using equation (6.15). Comparing our expression

$$G(s) = \frac{1}{s^2 + 5s + 6}$$

with (6.15), we have $N(s) = 1$, $D(s) = s^2 + 5s + 6$ and $D'(s) = 2s + 5$, and the roots of $D(s) = 0$ are $x_1 = -2$ and $x_2 = -3$. Using (6.15),

$$G(z) = \sum_{n=1}^{2} \frac{N(x_n)}{D'(x_n)} \frac{1}{1 - e^{x_n T} z^{-1}}$$

or, when $x_1 = -2$,

$$G_1(z) = \frac{1}{1} \frac{1}{1 - e^{-2T} z^{-1}}$$

and when $x_1 = -3$,

$$G_2(z) = \frac{1}{-1} \frac{1}{1 - e^{-3T} z^{-1}}.$$

Thus,

$$G(z) = \frac{1}{1 - e^{-2T} z^{-1}} - \frac{1}{1 - e^{-3T} z^{-1}} = \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-3T}}$$

Table 6.1 Some commonly used z-transforms

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\dfrac{z}{z-a}$
$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\dfrac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$

Properties of Z-Transforms

Most of the properties of the z-transform are analogs of those of the Laplace transforms. Important z-transform properties are discussed in this section.

1. Linearity property

   Suppose that the z-transform of $f(nT)$ is $F(z)$ and the z-transform of $g(nT)$ is $G(z)$. Then

$$Z[f(nT) \pm g(nT)] = Z[f(nT)] \pm Z[g(nT)] = F(z) \pm G(z) \tag{6.20}$$

   and for any scalar a

$$Z[af(nT)] = aZ[f(nT)] = aF(z) \tag{6.21}$$

Properties of Z-Transforms

2. Left-shifting property
   Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT) = f(nT + mT)$. Then

$$Y(z) = z^m F(z) - \sum_{i=0}^{m-1} f(iT) z^{m-i}. \tag{6.22}$$

   If the initial conditions are all zero, i.e. $f(iT) = 0$, $i = 0, 1, 2, \ldots, m-1$, then,

$$Z[f(nT + mT)] = z^m F(z). \tag{6.23}$$

3. Right-shifting property
   Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT) = f(nT - mT)$. Then

$$Y(z) = z^{-m} F(z) + \sum_{i=0}^{m-1} f(iT - mT) z^{-i}. \tag{6.24}$$

   If $f(nT) = 0$ for $k < 0$, then the theorem simplifies to

$$Z[f(nT - mT)] = z^{-m} F(z). \tag{6.25}$$

Properties of Z-Transforms

4. Attenuation property
   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then,

$$Z[e^{-anT} f(nT)] = F[ze^{aT}]. \tag{6.26}$$

   This result states that if a function is multiplied by the exponential e^{-anT} then in the z-transform of this function z is replaced by ze^{aT}.

5. Initial value theorem
   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the initial value of the time response is given by

$$\lim_{n\to 0} f(nT) = \lim_{z\to\infty} F(z). \tag{6.27}$$

Properties of Z-Transforms

6. Final value theorem

   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the final value of the time response is given by

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(1 - z^{-1})F(z). \tag{6.28}$$

Note that this theorem is valid if the poles of $(1 - z^{-1})F(z)$ are inside the unit circle or at $z = 1$.

Properties of Z-Transforms: Examples

Example 6.3

The z-transform of a unit ramp function $r(nT)$ is

$$R(z) = \frac{Tz}{(z-1)^2}.$$

Find the z-transform of the function $5r(nT)$.

Solution

Using the linearity property of z-transforms,

$$Z[5r(nT)] = 5R(z) = \frac{5Tz}{(z-1)^2}.$$

Properties of Z-Transforms: Examples

Example 6.4

The z-transform of trigonometric function $r(nT) = \sin nwT$ is

$$R(z) = \frac{z \sin wT}{z^2 - 2z \cos wT + 1}.$$

find the z-transform of the function $y(nT) = e^{-2T} \sin nWT$.

Solution

Using property 4 of the z-transforms,

$$Z[y(nT)] = Z[e^{-2T} r(nT)] = R[ze^{2T}].$$

Thus,

$$Z[y(nT)] = \frac{ze^{2T} \sin wT}{(ze^{2T})^2 - 2ze^{2T} \cos wT + 1} = \frac{ze^{2T} \sin wT}{z^2 e^{4T} - 2ze^{2T} \cos wT + 1}$$

or, multiplying numerator and denominator by e^{-4T},

$$Z[y(nT)] = \frac{ze^{-2T} \sin wT}{z^2 - 2ze^{-2T} + e^{-4T}}.$$

Properties of Z-Transforms: Examples

Example 6.5

Given the function

$$G(z) = \frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)},$$

find the final value of $g(nT)$.

Solution

Using the final value theorem,

$$\lim_{n\to\infty} g(nT) = \lim_{z\to 1}(1 - z^{-1})\frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}$$

$$= \lim_{z\to 1} \frac{0.792}{z^2 - 0.416z + 0.208}$$

$$= \frac{0.792}{1 - 0.416 + 0.208} = 1.$$

End

Thanks