Z-Transform

Digital Control

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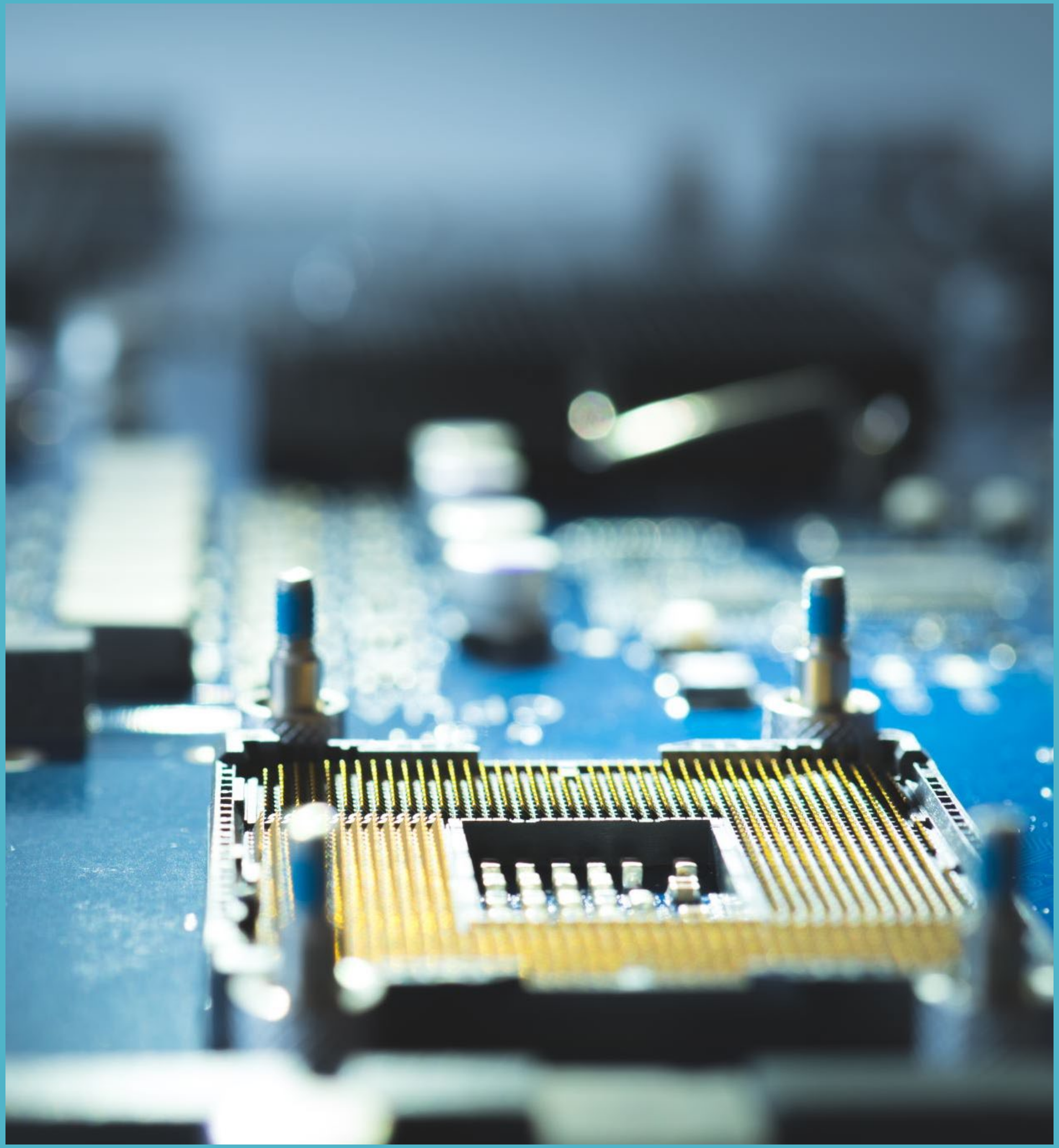

In the previous lecture

- Sampling
- Quantization
- Z-transform

Outline

1. Properties of z-transform
2. Inverse z-Transforms

Table 6.1 Some commonly used z-transforms

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\dfrac{z}{z-a}$
$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\dfrac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$

Laplace transform	Corresponding z-transform
$\frac{1}{s}$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s^3}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-a}{(s+a)(s+b)}$	$\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{(b-a)z^2-(be^{-aT}-ae^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\frac{z \sin aT}{z^2-2z \cos aT+1}$
$\frac{s}{s^2+a^2}$	$\frac{z^2-z \cos aT}{z^2-2z \cos aT+1}$
$\frac{s}{(s+a)^2}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$

Properties of Z-Transforms

Most of the properties of the z-transform are analogs of those of the Laplace transforms. Important z-transform properties are discussed in this section.

1. Linearity property

   Suppose that the z-transform of $f(nT)$ is $F(z)$ and the z-transform of $g(nT)$ is $G(z)$. Then

$$Z[f(nT) \pm g(nT)] = Z[f(nT)] \pm Z[g(nT)] = F(z) \pm G(z) \tag{6.20}$$

   and for any scalar a

$$Z[af(nT)] = aZ[f(nT)] = aF(z) \tag{6.21}$$

Properties of Z-Transforms

2. Left-shifting property
   Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT) = f(nT + mT)$. Then

$$Y(z) = z^m F(z) - \sum_{i=0}^{m-1} f(iT) z^{m-i}. \tag{6.22}$$

   If the initial conditions are all zero, i.e. $f(iT) = 0$, $i = 0, 1, 2, \ldots, m-1$, then,

$$Z[f(nT + mT)] = z^m F(z). \tag{6.23}$$

3. Right-shifting property
   Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT) = f(nT - mT)$. Then

$$Y(z) = z^{-m} F(z) + \sum_{i=0}^{m-1} f(iT - mT) z^{-i}. \tag{6.24}$$

   If $f(nT) = 0$ for $k < 0$, then the theorem simplifies to

$$Z[f(nT - mT)] = z^{-m} F(z). \tag{6.25}$$

Properties of Z-Transforms

4. Attenuation property
   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then,

$$Z[e^{-anT} f(nT)] = F[ze^{aT}]. \tag{6.26}$$

   This result states that if a function is multiplied by the exponential e^{-anT} then in the z-transform of this function z is replaced by ze^{aT}.

5. Initial value theorem
   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the initial value of the time response is given by

$$\lim_{n \to 0} f(nT) = \lim_{z \to \infty} F(z). \tag{6.27}$$

Properties of Z-Transforms

6. Final value theorem

   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the final value of the time response is given by

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(1 - z^{-1})F(z). \tag{6.28}$$

Note that this theorem is valid if the poles of $(1 - z^{-1})F(z)$ are inside the unit circle or at $z = 1$.

Properties of Z-Transforms: Examples

Example 6.3

The z-transform of a unit ramp function $r(nT)$ is

$$R(z) = \frac{Tz}{(z-1)^2}.$$

Find the z-transform of the function $5r(nT)$.

Solution

Using the linearity property of z-transforms,

$$Z[5r(nT)] = 5R(z) = \frac{5Tz}{(z-1)^2}.$$

Properties of Z-Transforms: Examples

Example 6.4

The z-transform of trigonometric function $r(nT) = \sin nwT$ is

$$R(z) = \frac{z \sin wT}{z^2 - 2z \cos wT + 1}.$$

find the z-transform of the function $y(nT) = e^{-2T} \sin nWT$.

Solution

Using property 4 of the z-transforms,

$$Z[y(nT)] = Z[e^{-2T} r(nT)] = R[ze^{2T}].$$

Thus,

$$Z[y(nT)] = \frac{ze^{2T} \sin wT}{(ze^{2T})^2 - 2ze^{2T} \cos wT + 1} = \frac{ze^{2T} \sin wT}{z^2 e^{4T} - 2ze^{2T} \cos wT + 1}$$

or, multiplying numerator and denominator by e^{-4T},

$$Z[y(nT)] = \frac{ze^{-2T} \sin wT}{z^2 - 2ze^{-2T} + e^{-4T}}.$$

Properties of Z-Transforms: Examples

Example 6.5

Given the function

$$G(z) = \frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)},$$

find the final value of $g(nT)$.

Solution

Using the final value theorem,

$$\lim_{n\to\infty} g(nT) = \lim_{z\to 1}(1 - z^{-1})\frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}$$

$$= \lim_{z\to 1}\frac{0.792}{z^2 - 0.416z + 0.208}$$

$$= \frac{0.792}{1 - 0.416 + 0.208} = 1.$$

Inverse Z-transform

- The inverse z-transform is obtained in a similar way to the inverse Laplace transforms.
- Generally, the z-transforms are the ratios of polynomials in the complex variable z, with the numerator
- polynomial being of order no higher than the denominator.
- By finding the inverse z-transform we find the sequence associated with the given z-transform polynomial. As in the case of inverse Laplace transforms, we are interested in the output time response of a system.
- Therefore, we use an inverse transform to obtain $y(t)$ from $Y(z)$.

Inverse Z-transform

- There are several methods to find the inverse z-transform of a given function.

- The following methods will be described here:

1. Power series (long division)

2. Expanding Y (z) into partial fractions and using z-transform tables to find the inverse transforms.

3. Obtaining the inverse z-transform using an inversion integral.

Inverse Z-transform

Given a z-transform function $Y(z)$, we can find the coefficients of the associated sequence $y(nT)$at the sampling instants by using the inverse z-transform. The time function $y(t)$ is then determined as

$$y(t) = \sum_{n=0}^{\infty} y(nT)\delta(t - nT).$$

Method 1: Power series. This method involves dividing the denominator of $Y(z)$ into the numerator such that a power series of the form

$$Y(z) = y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3} + \dots$$

is obtained. Notice that the values of $y(n)$ are the coefficients in the power series.

Inverse Z-transform

Example 6.6

Find the inverse z-transform for the polynomial

$$Y(z) = \frac{z^2 + z}{z^2 - 3z + 4}.$$

Inverse Z-transform

Solution

Dividing the denominator into the numerator gives

$$
\begin{array}{r|l}
 & 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} \\
\hline
z^2 - 3z + 4 & z^2 + z \\
 & \underline{z^2 - 3z + 4} \\
 & \quad 4z - 4 \\
 & \quad \underline{4z - 12 + 16z^{-1}} \\
 & \qquad 8 - 16z^{-1} \\
 & \qquad \underline{8 - 24z^{-1} + 32z^{-2}} \\
 & \qquad\quad 8z^{-1} - 32z^{-2} \\
 & \qquad\quad 8z^{-1} - 24z^{-2} + 32z^{-3} \\
 & \qquad\quad \cdots
\end{array}
$$

and the coefficients of the power series are

$$
\begin{aligned}
y(0) &= 1, \\
y(T) &= 4, \\
y(2T) &= 8, \\
y(3T) &= 8,
\end{aligned}
$$

Inverse Z-transform

The required sequence is

$$y(t) = \delta(t) + 4\delta(t - T) + 8\delta(t - 2T) + 8\delta(t - 3T) + \ldots$$

Figure 6.15 shows the first few samples of the time sequence $y(nT)$.

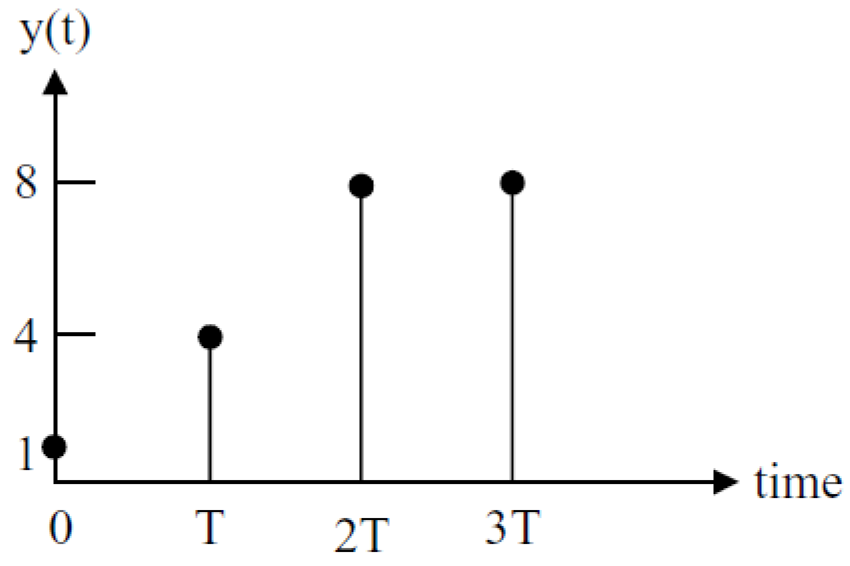

Figure 6.15 First few samples of $y(t)$

Inverse Z-transform

Example 6.7

Find the inverse z-transform for $Y(z)$ given by the polynomial

$$Y(z) = \frac{z}{z^2 - 3z + 2}.$$

Inverse Z-transform

Solution

Dividing the denominator into the numerator gives

$$\begin{array}{r|l} & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} \\ \hline z^2 - 3z + 2 & z \\ & \underline{z - 3 + 2z^{-1}} \\ & 3 - 2z^{-1} \\ & \underline{3 - 9z^{-1} + 6z^{-2}} \\ & 7z^{-1} - 6z^{-2} \\ & \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\ & 15z^{-2} - 14z^{-3} \\ & 15z^{-2} - 45z^{-3} + 30z^{-4} \\ & \cdots \end{array}$$

and the coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 1 \\ y(2T) &= 3 \\ y(3T) &= 7 \\ y(4T) &= 15 \end{aligned}$$

Inverse Z-transform

The required sequence is thus

$$y(t) = \delta(t - T) + 3\delta(t - 2T) + 7\delta(t - 3T) + 15\delta(t - 4T) + \dots.$$

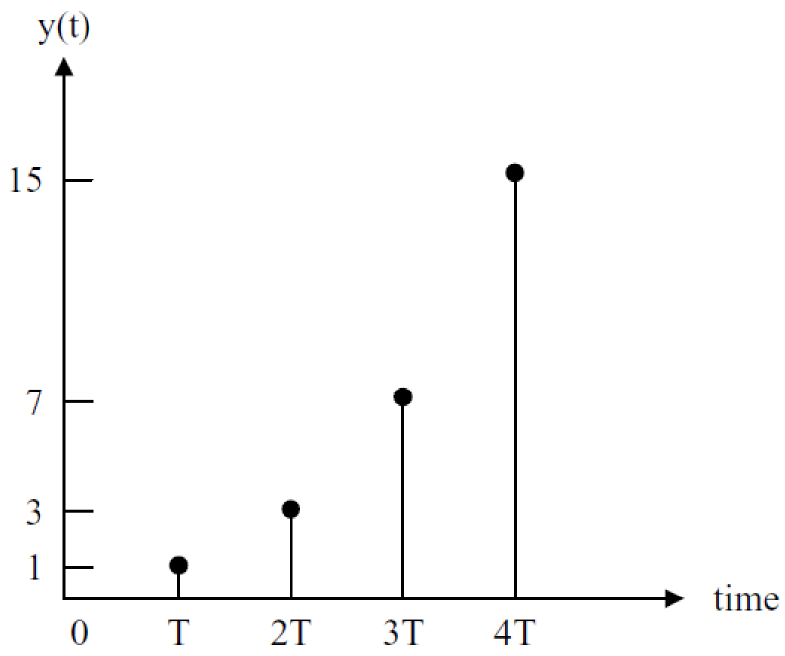

Figure 6.16 First few samples of $y(t)$

Inverse Z-transform

Method 2: Partial fractions. Similar to the inverse Laplace transform techniques, a partial fraction expansion of the function $Y(z)$ can be found, and then tables of known z-transforms can be used to determine the inverse z-transform.

Looking at the z-transform tables, we see that there is usually a z term in the numerator. It is therefore more convenient to find the partial fractions of the function $Y(z)/z$ and then multiply the partial fractions by z to obtain a z term in the numerator.

Inverse Z-transform

Example 6.8

Find the inverse z-transform of the function

$$Y(z) = \frac{z}{(z-1)(z-2)}$$

Solution

The above expression can be written as

$$\frac{Y(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

The values of A and B can be found by equating like powers in the numerator, i.e.

$$A(z-2) + B(z-1) \equiv 1.$$

We find $A = -1$, $B = 1$, giving

$$\frac{Y(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

or

$$Y(z) = \frac{-z}{z-1} + \frac{z}{z-2}$$

Inverse Z-transform

From the z-transform tables we find that

$$y(nT) = -1 + 2^n$$

and the coefficients of the power series are

$$\begin{aligned} y(0) &= 0, \\ y(T) &= 1, \\ y(2T) &= 3, \\ y(3T) &= 7, \\ y(4T) &= 15, \\ &\dots \end{aligned}$$

so that the required sequence is

$$y(t) = \delta(t - T) + 3\delta(t - 2T) + 7\delta(t - 3T) + 15\delta(t - 4T) + \dots.$$

Inverse Z-transform

Example 6.9

Find the inverse z-transform of the function

$$Y(z) = \frac{1}{(z-1)(z-2)}.$$

Inverse Z-transform

Solution

The above expression can be written as

$$\frac{Y(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}.$$

The values of A, B and C can be found by equating like powers in the numerator, i.e.

$$A(z-1)(z-2) + Bz(z-2) + Cz(z-1) \equiv 1$$

or

$$A(z^2 - 3z + 2) + Bz^2 - 2Bz + Cz^2 - Cz \equiv 1,$$

giving

$$\begin{aligned} A + \quad B + C &= 0, \\ -3A - 2B - C &= 0, \\ 2A \qquad\qquad &= 1. \end{aligned}$$

The values of the coefficients are found to be $A = 0.5$, $B = -1$ and $C = 0.5$. Thus,

$$\frac{Y(z)}{z} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

Inverse Z-transform

$$Y(z) = \frac{1}{2} - \frac{z}{z-1} + \frac{z}{2(z-2)}.$$

Using the inverse z-transform tables, we find

$$y(nT) = a - 1 + \frac{2^n}{2} = a - 1 + 2^{n-1}$$

where

$$a = \begin{cases} 1/2, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

the coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 0 \\ y(2T) &= 1 \\ y(3T) &= 3 \\ y(4T) &= 7 \\ y(5T) &= 15 \\ &\ldots, \end{aligned}$$

Inverse Z-transform

and the required sequence is

$$y(t) = \delta(t - 2T) + 3\delta(t - 3T) + 7\delta(t - 4T) + 15\delta(t - 5T) + \ldots.$$

Inverse Z-transform

The process of finding inverse z-transforms is aided by considering what form is taken by the roots of $Y(z)$. It is useful to distinguish the case of distinct real roots and that of multiple order roots.

Case I: Distinct real roots. When $Y(z)$ has distinct real roots in the form

$$Y(z) = \frac{N(z)}{(z - p_1)(z - p_2)(z - p_3)\ldots(z - p_n)},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \frac{A_3}{z - p_3} + \ldots + \frac{A_n}{z - p_n}$$

and the coefficients A_i can easily be found as

$$A_i = (z - p_i)\, Y(z)\big|_{z=p_i} \quad \text{for } i = 1,2,3,\ldots,n.$$

Inverse Z-transform

Example 6.10

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z}{(z-1)(z-2)}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2},$$

we find that

$$A = (z-1)\left.\frac{1}{(z-1)(z-2)}\right|_{z=1} = -1,$$

$$B = (z-2)\left.\frac{1}{(z-1)(z-2)}\right|_{z=2} = 1.$$

Thus,

$$Y(z) = \frac{z}{z-1} + \frac{z}{z-2}$$

and the inverse z-transform is obtained from the tables as

$$y(nT) = -1 + 2^n,$$

Inverse Z-transform

Example 6.11

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z^2 + z}{(z - 0.5)(z - 0.8)(z - 1)}.$$

Inverse Z-transform

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{A}{z-0.5} + \frac{B}{z-0.8} + \frac{C}{z-1}$$

we find that

$$A = (z-0.5)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=0.5} = 10,$$

$$B = (z-0.8)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=0.8} = -30,$$

$$C = (z-1)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=1} = 20.$$

Thus,

$$Y(z) = \frac{10z}{z-0.5} - \frac{30z}{z-0.8} + \frac{20z}{z-1}$$

The inverse transform is found from the tables as

$$y(nT) = 10(0.5)^n - 30(0.8)^n + 20$$

Inverse Z-transform

The coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 1 \\ y(2T) &= 3.3 \\ y(3T) &= 5.89 \\ &\cdots \end{aligned}$$

and the required sequence is

$$y(t) = \delta(t - T) + 3.3\delta(t - 2T) + 5.89\delta(t - 3T) + \ldots.$$

Inverse Z-transform

Case II: Multiple order roots. When $Y(z)$ has multiple order roots of the form

$$Y(z) = \frac{N(z)}{(z-p_1)(z-p_1)^2(z-p_1)^3\ldots(z-p_1)^r},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{\lambda_1}{z-p_1} + \frac{\lambda_2}{(z-p_2)^2} + \frac{\lambda_3}{(z-p_1)^3} + \ldots + \frac{\lambda_r}{(z-p_1)^r}$$

and the coefficients λ_i can easily be found as

$$\lambda_{r-k} = \frac{1}{k!}\left.\frac{d^k}{dz^k[(z-p_i)^r(X(z)/z)]}\right|_{z=p_i}. \qquad (6.29)$$

Inverse Z-transform

Example 6.12

Using (6.29), find the inverse z-transform of

$$Y(z) = \frac{z^2 + 3z - 2}{(z+5)(z-0.8)(z-2)^2}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2} = \frac{A}{z} + \frac{B}{z+5} + \frac{C}{z-0.8} + \frac{D}{(z-2)} + \frac{E}{(z-2)^2}$$

we obtain

$$A = z\,\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0} = \frac{-2}{5 \times (-0.8) \times 4} = 0.125,$$

$$B = (z+5)\,\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=-5} = \frac{8}{-5 \times (-5.8) \times 49} = 0.0056,$$

$$C = (z-0.8)\,\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0.8} = \frac{1.04}{0.8 \times 5.8 \times 1.14} = 0.16,$$

Inverse Z-transform

$$E = (z-2)^2 \left. \frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2} \right|_{z=2} = \frac{8}{2\times 7\times 1.2} = 0.48,$$

$$D = \frac{d}{dz}\left[\frac{z^2+3z-2}{z(z+5)(z-0.8)}\right]\Bigg|_{z=2}$$

$$= \left. \frac{[z(z+5)(z-0.8(2z+3)-(z^2+3z-2)(3z^2+8.4z-4)]}{(z^3+4.2z^2-4z)^2} \right|_{z=2} = -0.29.$$

We can now write $Y(z)$ as

$$Y(z) = 0.125 + \frac{0.0056z}{z+5} + \frac{0.016z}{z-0.8} - \frac{0.29z}{(z-2)} + \frac{0.48z}{(z-2)^2}$$

The inverse transform is found from the tables as

$$y(nT) = 0.125a + 0.0056(-5)^n + 0.016(0.8)^n - 0.29(2)^n + 0.24n(2)^n,$$

where

$$a = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

End

Thanks