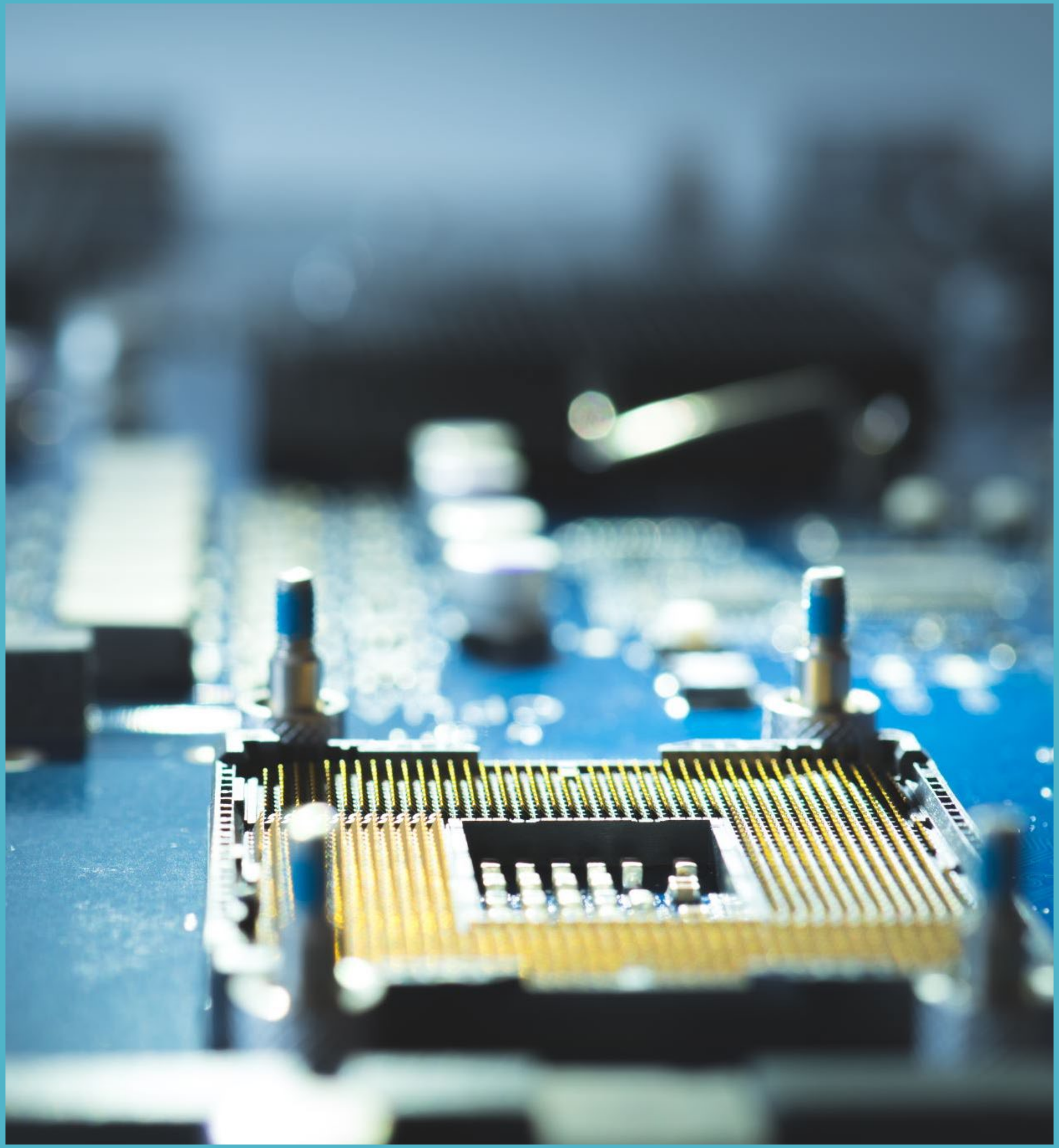
Z-Transform and Difference Equations

Digital Control

Dr. Ahmad Al-Mahasneh



In the previous lecture

1. Properties of z-transform
2. Inverse z-Transforms

Outline

1. Z-transform
2. Difference equations

Inverse Z-transform

Example 6.9

Find the inverse z-transform of the function

$$Y(z) = \frac{1}{(z-1)(z-2)}.$$

Inverse Z-transform

Solution

The above expression can be written as

$$\frac{Y(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}.$$

The values of A, B and C can be found by equating like powers in the numerator, i.e.

$$A(z-1)(z-2) + Bz(z-2) + Cz(z-1) \equiv 1$$

or

$$A(z^2 - 3z + 2) + Bz^2 - 2Bz + Cz^2 - Cz \equiv 1,$$

giving

$$\begin{aligned} A + \quad B + C &= 0, \\ -3A - 2B - C &= 0, \\ 2A \qquad\qquad &= 1. \end{aligned}$$

The values of the coefficients are found to be $A = 0.5$, $B = -1$ and $C = 0.5$. Thus,

$$\frac{Y(z)}{z} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

Inverse Z-transform

$$Y(z) = \frac{1}{2} - \frac{z}{z-1} + \frac{z}{2(z-2)}.$$

Using the inverse z-transform tables, we find

$$y(nT) = a - 1 + \frac{2^n}{2} = a - 1 + 2^{n-1}$$

where

$$a = \begin{cases} 1/2, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

the coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 0 \\ y(2T) &= 1 \\ y(3T) &= 3 \\ y(4T) &= 7 \\ y(5T) &= 15 \\ &\ldots, \end{aligned}$$

Inverse Z-transform

and the required sequence is

$$y(t) = \delta(t - 2T) + 3\delta(t - 3T) + 7\delta(t - 4T) + 15\delta(t - 5T) + \ldots.$$

Inverse Z-transform

The process of finding inverse z-transforms is aided by considering what form is taken by the roots of $Y(z)$. It is useful to distinguish the case of distinct real roots and that of multiple order roots.

Case I: Distinct real roots. When $Y(z)$ has distinct real roots in the form

$$Y(z) = \frac{N(z)}{(z - p_1)(z - p_2)(z - p_3)\ldots(z - p_n)},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \frac{A_3}{z - p_3} + \ldots + \frac{A_n}{z - p_n}$$

and the coefficients A_i can easily be found as

$$A_i = (z - p_i)\, Y(z)|_{z=p_i} \quad \text{for } i = 1,2,3,\ldots,n.$$

Inverse Z-transform

Example 6.10

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z}{(z-1)(z-2)}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2},$$

we find that

$$A = (z-1)\left.\frac{1}{(z-1)(z-2)}\right|_{z=1} = -1,$$

$$B = (z-2)\left.\frac{1}{(z-1)(z-2)}\right|_{z=2} = 1.$$

Thus,

$$Y(z) = \frac{z}{z-1} + \frac{z}{z-2}$$

and the inverse z-transform is obtained from the tables as

$$y(nT) = -1 + 2^n,$$

Inverse Z-transform

Example 6.11

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z^2 + z}{(z - 0.5)(z - 0.8)(z - 1)}.$$

Inverse Z-transform

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{A}{z-0.5} + \frac{B}{z-0.8} + \frac{C}{z-1}$$

we find that

$$A = (z-0.5)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=0.5} = 10,$$

$$B = (z-0.8)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=0.8} = -30,$$

$$C = (z-1)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=1} = 20.$$

Thus,

$$Y(z) = \frac{10z}{z-0.5} - \frac{30z}{z-0.8} + \frac{20z}{z-1}$$

The inverse transform is found from the tables as

$$y(nT) = 10(0.5)^n - 30(0.8)^n + 20$$

Inverse Z-transform

The coefficients of the power series are

$$
\begin{aligned}
y(0) &= 0 \\
y(T) &= 1 \\
y(2T) &= 3.3 \\
y(3T) &= 5.89 \\
&\cdots
\end{aligned}
$$

and the required sequence is

$$y(t) = \delta(t - T) + 3.3\delta(t - 2T) + 5.89\delta(t - 3T) + \ldots.$$

Inverse Z-transform

Case II: Multiple order roots. When $Y(z)$ has multiple order roots of the form

$$Y(z) = \frac{N(z)}{(z-p_1)(z-p_1)^2(z-p_1)^3 \ldots (z-p_1)^r},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{\lambda_1}{z-p_1} + \frac{\lambda_2}{(z-p_2)^2} + \frac{\lambda_3}{(z-p_1)^3} + \ldots + \frac{\lambda_r}{(z-p_1)^r}$$

and the coefficients λ_i can easily be found as

$$\lambda_{r-k} = \frac{1}{k!} \frac{d^k}{dz^k[(z-p_i)^r(X(z)/z)]}\bigg|_{z=p_i}. \qquad (6.29)$$

Inverse Z-transform

Example 6.12

Using (6.29), find the inverse z-transform of

$$Y(z) = \frac{z^2 + 3z - 2}{(z+5)(z-0.8)(z-2)^2}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2} = \frac{A}{z} + \frac{B}{z+5} + \frac{C}{z-0.8} + \frac{D}{(z-2)} + \frac{E}{(z-2)^2}$$

we obtain

$$A = z\left.\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\right|_{z=0} = \frac{-2}{5 \times (-0.8) \times 4} = 0.125,$$

$$B = (z+5)\left.\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\right|_{z=-5} = \frac{8}{-5 \times (-5.8) \times 49} = 0.0056,$$

$$C = (z-0.8)\left.\frac{z^2 + 3z - 2}{z(z+5)(z-0.8)(z-2)^2}\right|_{z=0.8} = \frac{1.04}{0.8 \times 5.8 \times 1.14} = 0.16,$$

Inverse Z-transform

$$E = (z-2)^2 \left.\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\right|_{z=2} = \frac{8}{2\times 7\times 1.2} = 0.48,$$

$$D = \frac{d}{dz}\left[\frac{z^2+3z-2}{z(z+5)(z-0.8)}\right]\Bigg|_{z=2}$$

$$= \left.\frac{[z(z+5)(z-0.8(2z+3)-(z^2+3z-2)(3z^2+8.4z-4)]}{(z^3+4.2z^2-4z)^2}\right|_{z=2} = -0.29.$$

We can now write $Y(z)$ as

$$Y(z) = 0.125 + \frac{0.0056z}{z+5} + \frac{0.016z}{z-0.8} - \frac{0.29z}{(z-2)} + \frac{0.48z}{(z-2)^2}$$

The inverse transform is found from the tables as

$$y(nT) = 0.125a + 0.0056(-5)^n + 0.016(0.8)^n - 0.29(2)^n + 0.24n(2)^n,$$

where

$$a = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Difference equations

Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values.

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)$$
$$= b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)$$

Difference equations

EXAMPLE 2.2

For each of the following difference equations, determine the order of the equation. Is the equation (a) linear, (b) time invariant, or (c) homogeneous?

1. $y(k+2) + 0.8y(k+1) + 0.07y(k)u(k)$
2. $y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$
3. $y(k+1) = -0.1y^2(k)$

Difference equations

Solution

1. The equation is second order. All terms enter the equation linearly and have constant coefficients. The equation is therefore LTI. A forcing function appears in the equation, so it is nonhomogeneous.
2. The equation is fourth order. The second coefficient is time dependent, but all the terms are linear and there is no forcing function. The equation is therefore linear time varying and homogeneous.
3. The equation is first order. The right-hand side (RHS) is a nonlinear function of $y(k)$, but does not include a forcing function or terms that depend on time explicitly. The equation is therefore nonlinear, time invariant, and homogeneous.

Converting from difference equations to Z-transform

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots$$

with initial condition $y_0 = 1$.
We multiply both sides of (1) by z^{-n} and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty}(y_{n+1} - 3y_n)z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$

or

$$\sum_{n=0}^{\infty} y_{n+1}z^{-n} - 3 \sum_{n=0}^{\infty} y_n z^{-n} = 4 \sum_{n=0}^{\infty} z^{-n} \tag{2}$$

Converting from difference equations to Z-transform

The right-hand side is the z-transform of the constant sequence $\{4,\ 4, \ldots\}$ which is $\dfrac{4z}{z-1}$.

If $Y(z) = \displaystyle\sum_{n=0}^{\infty} y_n z^{-n}$ denotes the z-transform of the sequence $\{y_n\}$ that we are seeking then $\displaystyle\sum_{n=0}^{\infty} y_{n+1} z^{-n} = z\,Y(z) - z y_0$ (by the left shift theorem).

Consequently (2) can be written

$$z\,Y(z) - z y_0 - 3\,Y(z) = \frac{4z}{z-1} \tag{3}$$

Converting from difference equations to Z-transform

$$\begin{aligned}(z-3)Y(z)-z &= \frac{4z}{(z-1)} \\ (z-3)Y(z) &= \frac{4z}{z-1}+z=\frac{z^2+3z}{z-1}\end{aligned}$$

so $$Y(z)=\frac{z^2+3z}{(z-1)(z-3)}$$

Converting from z-transform to difference equation

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{2(1-z^{-1})}$$

$$2(1-z^{-1})Y(z) = (1+z^{-1})X(z)$$

$$Y(z) - Y(z)z^{-1} = \frac{1}{2}X(z) + \frac{1}{2}X(z)z^{-1}$$

$$y[n] - y[n-1] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

$$y[n] = y[n-1] + \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

Table 6.1 Some commonly used z-transforms

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\dfrac{z}{z-a}$
$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\dfrac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$

Laplace transform	Corresponding z-transform
$\frac{1}{s}$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s^3}$	$\frac{T^2z(z+1)}{2(z-1)^3}$
$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-a}{(s+a)(s+b)}$	$\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{(b-a)z^2-(be^{-aT}-ae^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\frac{z \sin aT}{z^2-2z\cos aT+1}$
$\frac{s}{s^2+a^2}$	$\frac{z^2-z\cos aT}{z^2-2z\cos aT+1}$
$\frac{s}{(s+a)^2}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$

End

Thanks