Difference Equations

Digital Control

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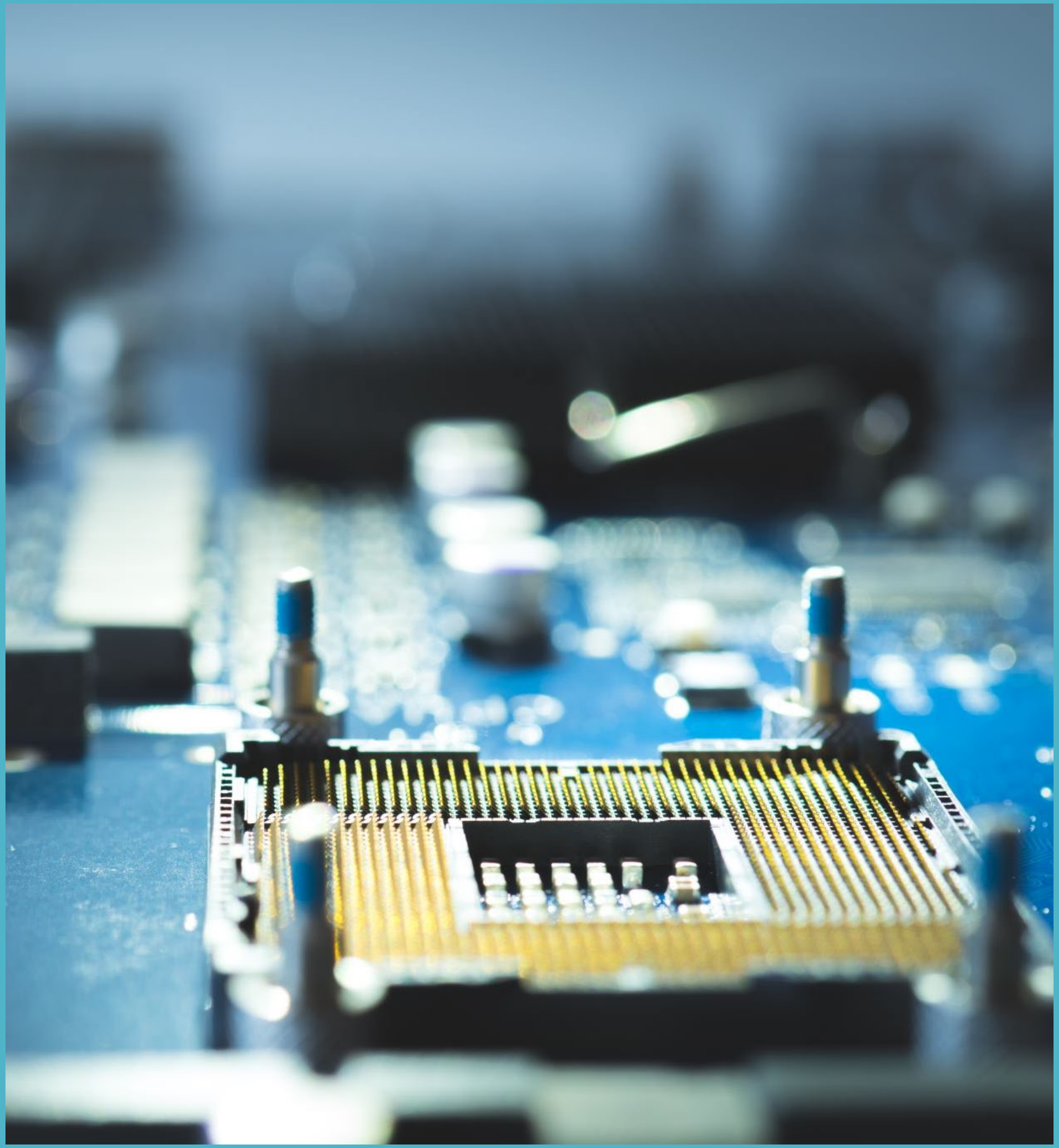

In the previous lecture

Inverse z-Transforms

Outline

Difference equations

Difference equations

Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values.

$$
\begin{aligned}
&y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) \\
&\quad = b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k)
\end{aligned}
$$

Difference equations

EXAMPLE 2.2

For each of the following difference equations, determine the order of the equation. Is the equation (a) linear, (b) time invariant, or (c) homogeneous?

1. $y(k+2) + 0.8y(k+1) + 0.07y(k)u(k)$
2. $y(k+4) + \sin(0.4k)y(k+1) + 0.3y(k) = 0$
3. $y(k+1) = -0.1y^2(k)$

Difference equations

Solution

1. The equation is second order. All terms enter the equation linearly and have constant coefficients. The equation is therefore LTI. A forcing function appears in the equation, so it is nonhomogeneous.
2. The equation is fourth order. The second coefficient is time dependent, but all the terms are linear and there is no forcing function. The equation is therefore linear time varying and homogeneous.
3. The equation is first order. The right-hand side (RHS) is a nonlinear function of $y(k)$, but does not include a forcing function or terms that depend on time explicitly. The equation is therefore nonlinear, time invariant, and homogeneous.

Converting from difference equations to Z-transform

$$y_{n+1} - 3y_n = 4 \qquad n = 0, 1, 2, \ldots$$

with initial condition $y_0 = 1$.
We multiply both sides of (1) by z^{-n} and sum each side over all positive integer values of n and zero. We obtain

$$\sum_{n=0}^{\infty} (y_{n+1} - 3y_n) z^{-n} = \sum_{n=0}^{\infty} 4z^{-n}$$

or

$$\sum_{n=0}^{\infty} y_{n+1} z^{-n} - 3 \sum_{n=0}^{\infty} y_n z^{-n} = 4 \sum_{n=0}^{\infty} z^{-n} \qquad (2)$$

Converting from difference equations to Z-transform

The right-hand side is the z-transform of the constant sequence $\{4,\ 4, \ldots\}$ which is $\dfrac{4z}{z-1}$.

If $Y(z) = \sum_{n=0}^{\infty} y_n z^{-n}$ denotes the z-transform of the sequence $\{y_n\}$ that we are seeking then $\sum_{n=0}^{\infty} y_{n+1} z^{-n} = z\,Y(z) - zy_0$ (by the left shift theorem).

Consequently (2) can be written

$$z\,Y(z) - zy_0 - 3\,Y(z) = \frac{4z}{z-1} \tag{3}$$

Converting from difference equations to Z-transform

$$(z-3)Y(z)-z = \frac{4z}{(z-1)}$$

$$(z-3)Y(z) = \frac{4z}{z-1}+z=\frac{z^2+3z}{z-1}$$

so $$Y(z)=\frac{z^2+3z}{(z-1)(z-3)}$$

Converting from z-transform to difference equation

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{2(1-z^{-1})}$$

$$2(1-z^{-1})Y(z) = (1+z^{-1})X(z)$$

$$Y(z) - Y(z)z^{-1} = \frac{1}{2}X(z) + \frac{1}{2}X(z)z^{-1}$$

$$y[n] - y[n-1] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

$$y[n] = y[n-1] + \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

Table 6.1 Some commonly used z-transforms

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\frac{z}{z-1}$
kT	$\frac{Tz}{(z-1)^2}$
e^{-akT}	$\frac{z}{z-e^{-aT}}$
kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\frac{z}{z-a}$
$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\frac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$

Laplace transform	Corresponding z-transform
$\frac{1}{s}$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s^3}$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-a}{(s+a)(s+b)}$	$\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{(b-a)s}{(s+a)(s+b)}$	$\frac{(b-a)z^2-(be^{-aT}-ae^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\frac{z \sin aT}{z^2-2z \cos aT+1}$
$\frac{s}{s^2+a^2}$	$\frac{z^2-z \cos aT}{z^2-2z \cos aT+1}$
$\frac{s}{(s+a)^2}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$

End

Thanks