DAMPING RATIO AND UNDAMPED NATURAL FREQUENCY IN THE Z-PLANE

Digital control

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Mapping S-plane to Z-plane

- The time responses of a sampled data system based on its pole positions in the z-plane are shown in Figure 7.8.
- It is clear from this figure that the system is stable if all the closed-loop poles are within the unit circle.

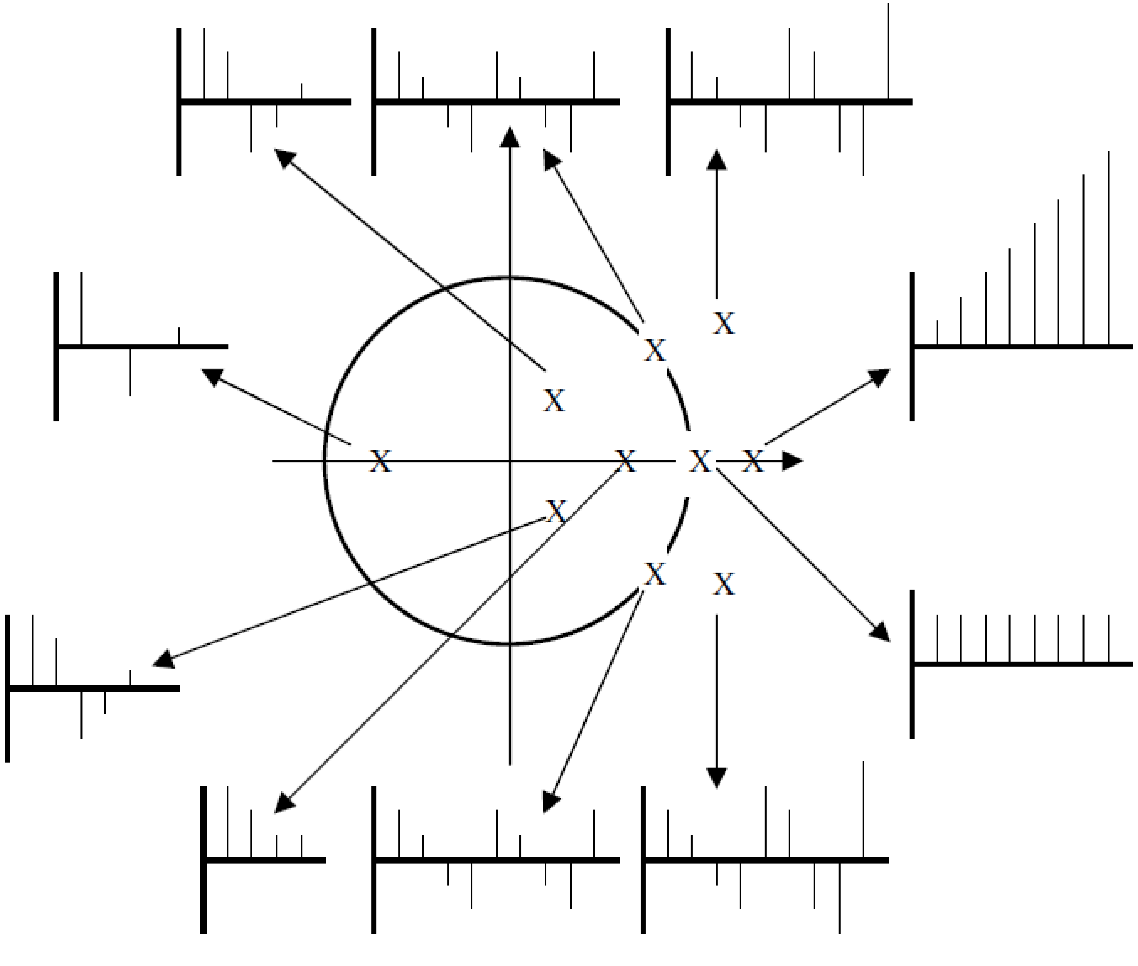

Figure 7.8 Time response of z-plane pole locations

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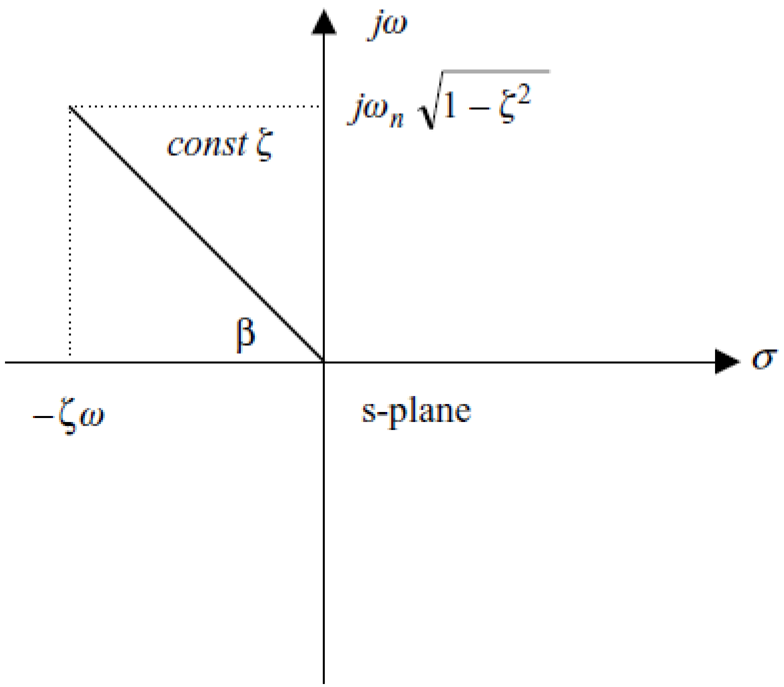


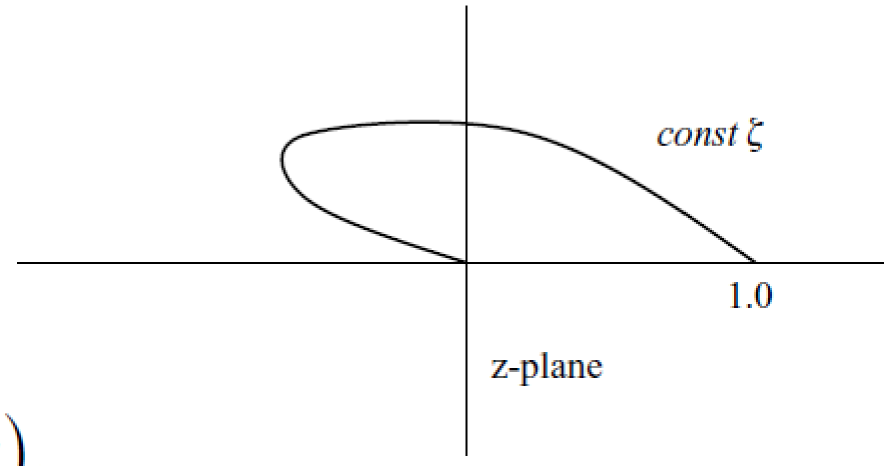
(a)

$$\zeta = \cos\alpha$$

$$z = e^{-\sigma\omega T}e^{j\omega T}.$$

$$\sigma = \tan\left(\frac{\pi}{2} - \cos^{-1}\zeta\right)$$

$$z = \exp\left[-\omega T \tan\left(\frac{\pi}{2} - \cos^{-1}\zeta\right)\right]e^{j\omega T}.$$



(b)

Figure 7.9 (a) Line of constant damping ratio in the s-plane, and (b) the corresponding locus in the z-plane

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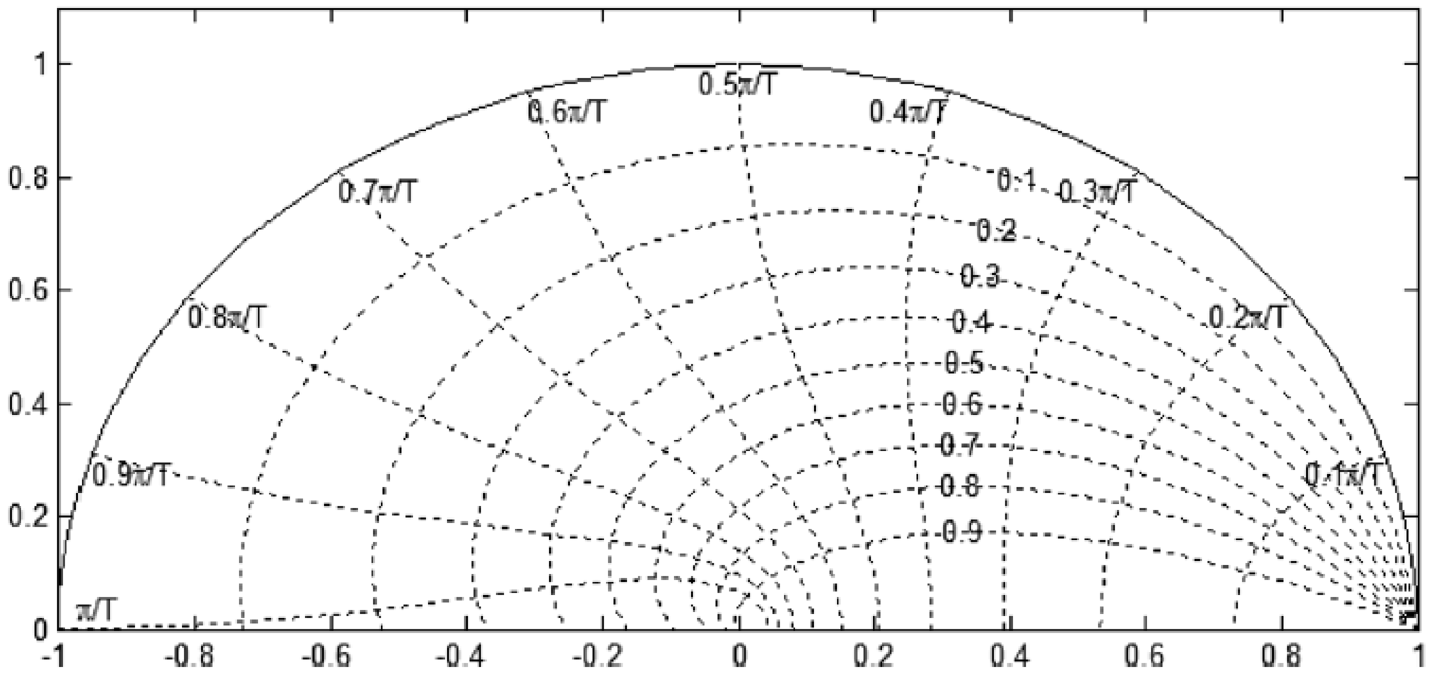

Figure 7.10 Lines of constant damping ratio for different ζ. The vertical lines are the lines of constant ω_n

The lines of constant damping ratio in the *z*-plane for various values of ζ .

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$$\omega^2 + \sigma^2 = \omega_n^2 \quad \text{or} \quad \sigma = \sqrt{\omega_n^2 - \omega^2}.$$

$$z = e^{-sT} = e^{-\sigma T} e^{-j\omega T} = \exp\left[-T\left(\sqrt{\omega_n^2 - \omega^2}\right)\right] e^{-j\omega T}$$

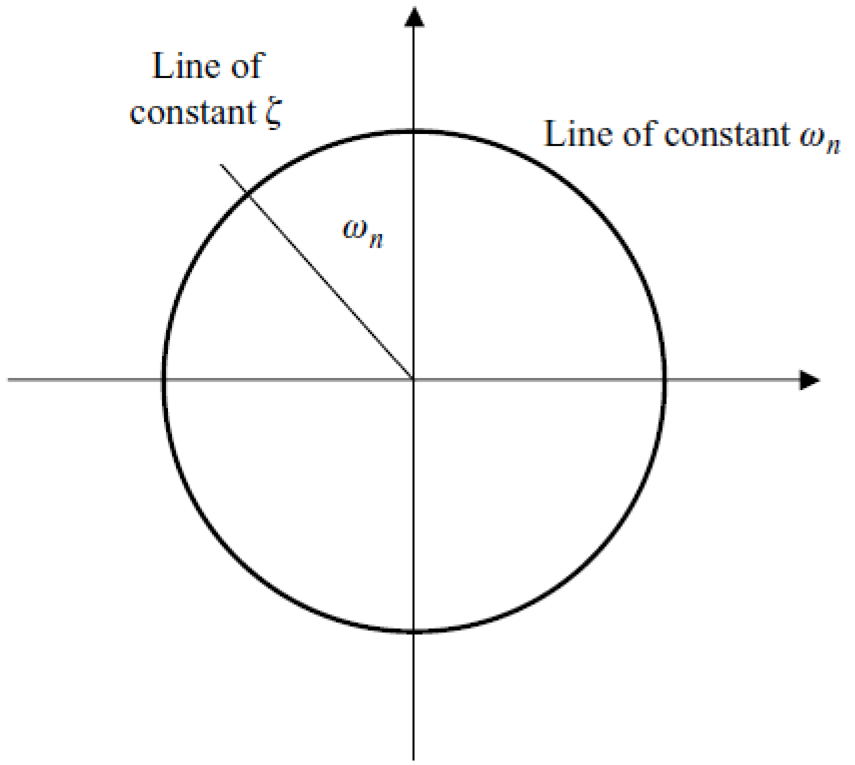

Figure 7.11 Locus of constant ω_n in the s-plane

Figure 7.11, the locus of constant undamped natural frequency in the s-plane is a circle with radius ω_n

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$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The poles of this system are at

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$$

We can now find the equivalent z-plane poles by making the substitution $z = e^{sT}$, i.e.

$$z = e^{sT} = e^{-\zeta\omega_n T} \angle \pm \omega_n T\sqrt{1-\zeta^2},$$

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which we can write as

$$z = r\angle \pm \theta, \tag{7.11}$$

where

$$r = e^{-\zeta\omega_n T} \quad \text{or} \quad \zeta\omega_n T = -\ln r \tag{7.12}$$

and

$$\theta = \omega_n T\sqrt{1-\zeta^2}. \tag{7.13}$$

From (7.12) and (7.13) we obtain

$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{-\ln r}{\theta}$$

or

$$\zeta = \frac{-\ln r}{\sqrt{(\ln r)^2 + \theta^2}}, \tag{7.14}$$

and from (7.12) and (7.14) we obtain

$$\omega_n = \frac{1}{T}\sqrt{(\ln r)^2 + \theta^2}. \tag{7.15}$$

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Example 7.2

Consider the system described in Section 7.1 with closed-loop transfer function

$$\frac{y(z)}{r(z)} = \frac{G(z)}{1 + G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632}.$$

Find the damping ratio and the undamped natural frequency. Assume that $T = 1$ s.

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Solution

We need to find the poles of the closed-loop transfer function. The system characteristic equation is $1 + G(z) = 0$,
i.e.

$$z^2 - z + 0.632 = (z - 0.5 - j0.618)(z - 0.5 + j0.618) = 0,$$

which can be written in polar form as

$$z_{1,2} = 0.5 \pm j0.618 = 0.795\angle \pm 0.890 = r\angle \pm \theta$$

(see (7.11)). The damping ratio is then calculated using (7.14) as

$$\zeta = \frac{-\ln r}{\sqrt{(\ln r)^2 + \theta^2}} = \frac{-\ln 0.795}{\sqrt{(\ln 0.795)^2 + 0.890^2}} = 0.25,$$

and from (7.15) the undamped natural frequency is, taking $T = 1$,

$$\omega_n = \frac{1}{T}\sqrt{(\ln r)^2 + \theta^2} = \sqrt{(\ln 0.795)^2 + 0.890^2} = 0.92.$$

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Example 7.3

Find the damping ratio and the undamped natural frequency for Example 7.2 using the graphical method.

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Solution

The characteristic equation of the system is found to be

$$z^2 - z + 0.632 = (z - 0.5 - j0.618)(z - 0.5 + j0.618) = 0$$

and the poles of the closed-loop system are at

$$z_{1,2} = 0.5 \pm j0.618.$$

Figure 7.12 shows the loci of the constant damping ratio and the loci of the undamped natural frequency with the poles of the closed-loop system marked with an $\times$ on the graph. From the graph we can read the damping ratio as 0.25 and the undamped natural frequency as

$$\omega_n = \frac{0.29\pi}{T} = 0.91.$$

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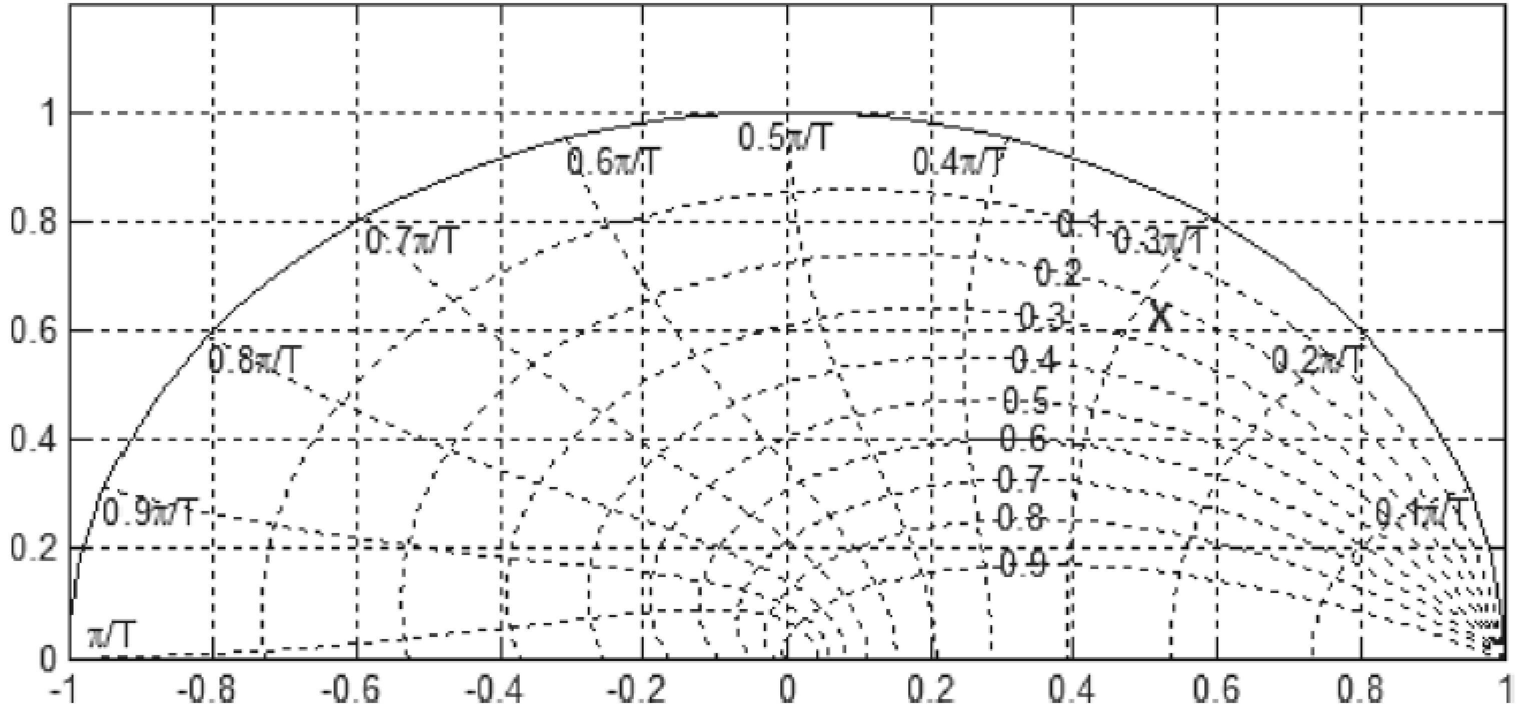

Figure 7.12 Finding ζ and ω_n graphically