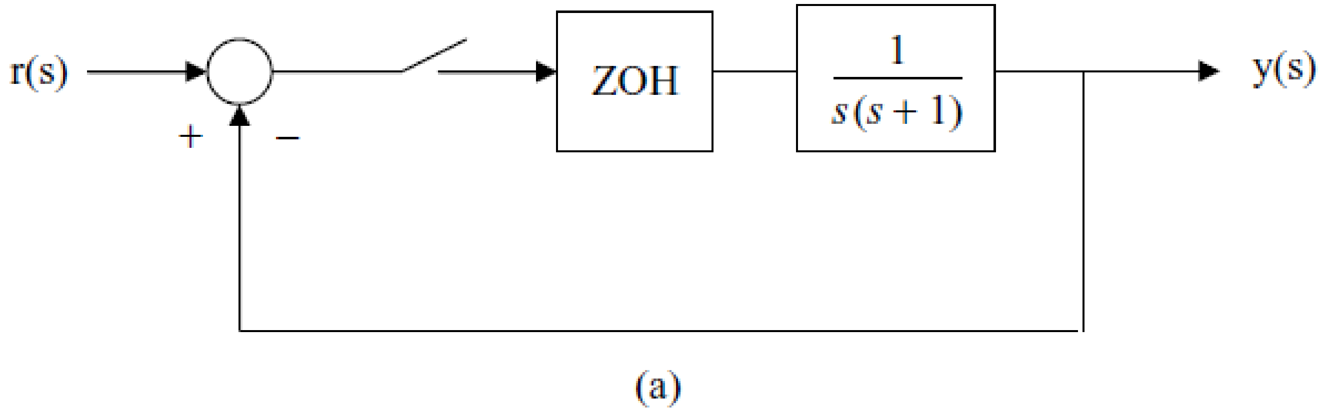
System Time Response Characteristics Digital control

Dr. Ahmad Al-mahasneh

TIME RESPONSE COMPARISON



$$\frac{y(z)}{r(z)} = \frac{G(z)}{1 + G(z)},$$

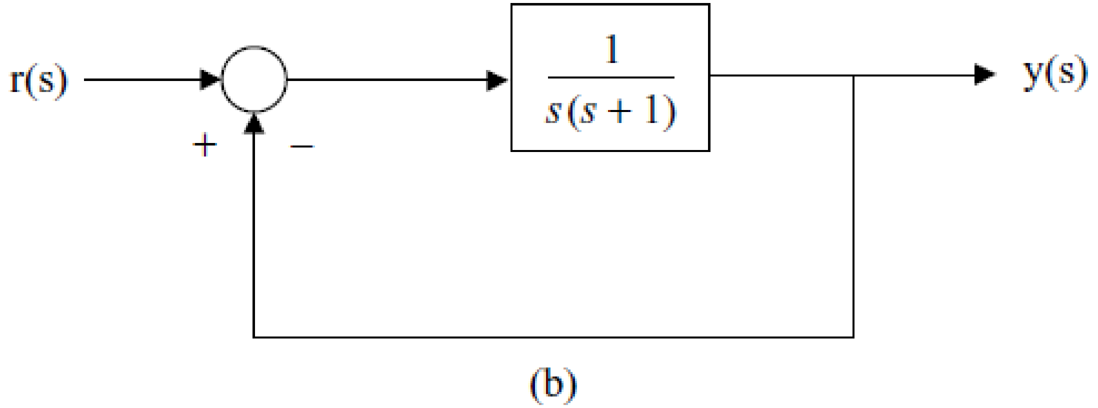


Figure 7.1 (a) Discrete system and (b) its continuous-time equivalent

TIME RESPONSE COMPARISON

$$\frac{y(z)}{r(z)} = \frac{G(z)}{1 + G(z)},$$

where

$$r(z) = \frac{z}{z - 1}$$

and the z-transform of the plant is given by

$$G(s) = \frac{1 - e^{-sT}}{s^2(s + 1)}.$$

Expanding by means of partial fractions, we obtain

$$G(s) = (1 - e^{-sT})\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s + 1}\right)$$

TIME RESPONSE COMPARISON

and the z-transform is

$$G(z) = (1 - z^{-1})Z\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right\}.$$

From z-transform tables we obtain

$$G(z) = (1 - z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right].$$

Setting $T = 1$s and simplifying gives

$$G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}.$$

Substituting into (7.1), we obtain the transfer function

$$\frac{y(z)}{r(z)} = \frac{G(z)}{1 + G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632},$$

and then using (7.2) gives the output

$$y(z) = \frac{z(0.368z + 0.264)}{(z-1)(z^2 - z + 0.632)}.$$

TIME RESPONSE COMPARISON

The inverse z-transform can be found by long division: the first several terms are

$$y(z) = 0.368z^{-1} + z^{-2} + 1.4z^{-3} + 1.4z^{-4} + 1.15z^{-5} + 0.9z^{-6} + 0.8z^{-7} + 0.87z^{-8} + 0.99z^{-9} + \ldots$$

and the time response is given by

$$y(nT) = 0.368\delta(t-1) + \delta(t-2) + 1.4\delta(t-3) + 1.4\delta(t-4) + 1.15\delta(t-5) + 0.9\delta(t-6) + 0.8\delta(t-7) + 0.87\delta(t-8) + \ldots.$$

TIME RESPONSE COMPARISON

From Figure 7.1(b), the equivalent continuous-time system transfer function is

$$\frac{y(s)}{r(s)} = \frac{G(s)}{1 + G(s)} = \frac{1/(s(s+1))}{1 + (1/s(s+1))} = \frac{1}{s^2 + s + 1}.$$

Since $r(s) = 1/s$, the output becomes

$$y(s) = \frac{1}{s(s^2 + s + 1)}.$$

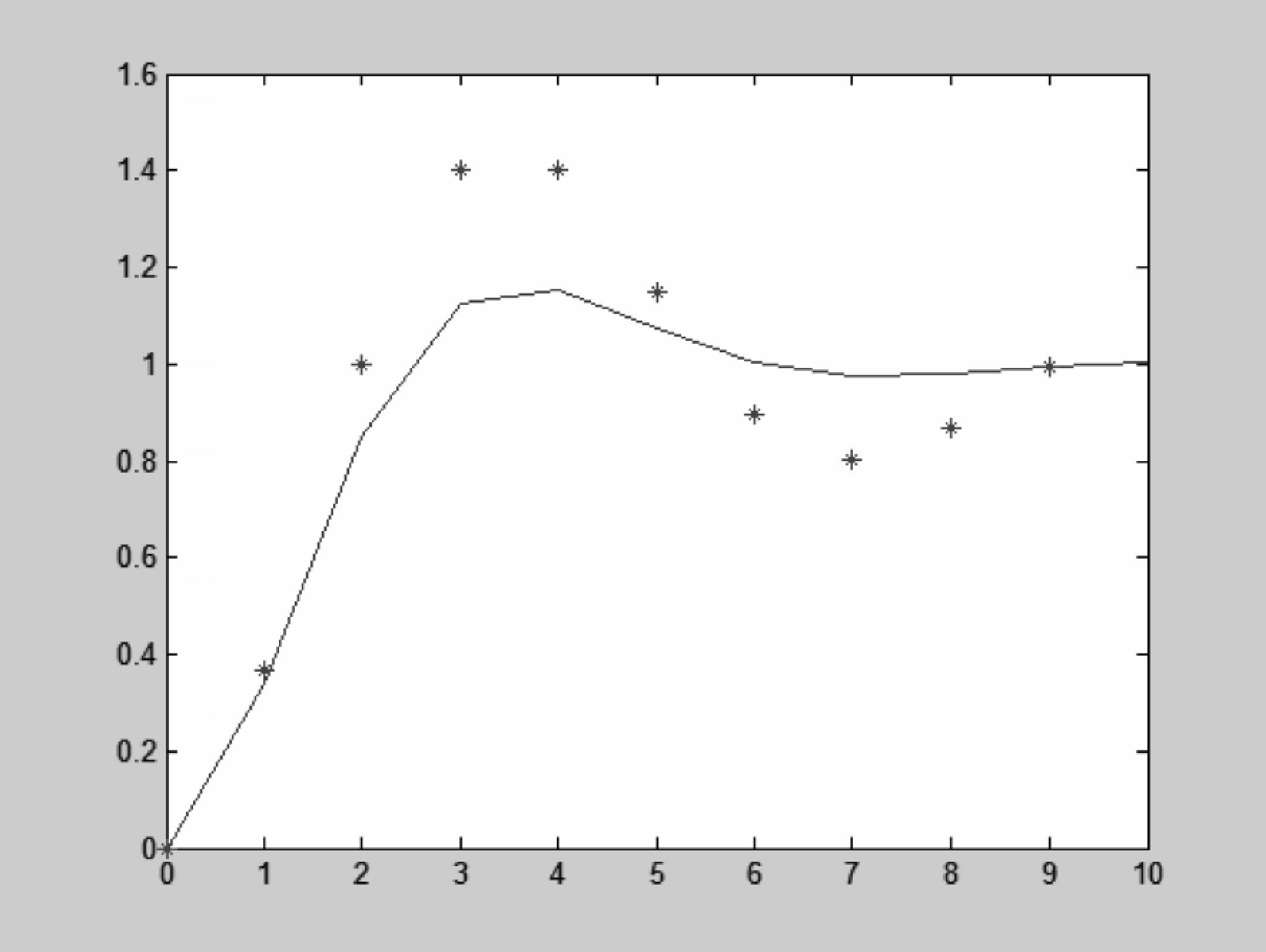
To find the inverse Laplace transform we can write

$$y(s) = \frac{1}{s} - \frac{s+1}{s^2 + s + 1} = \frac{1}{s} - \frac{s + 0.5}{(s+0.5)^2 - 0.5^2} - \frac{0.5}{(s+0.5)^2 - 0.5^2}.$$

From inverse Laplace transform tables we find that the time response is

$$y(t) = 1 - e^{-0.5t}\left(\cos 0.5t + 0.577 \sin 0.5t\right).$$

TIME RESPONSE COMPARISON



TIME DOMAIN SPECIFICATIONS

- The performance of a control system is usually measured in terms of its response to a step input.
- The step input is used because it is easy to generate and gives the system a nonzero steady-state condition, which can be measured.

Most commonly used time domain performance measures refer to a second-order system with the transfer function:

$$\frac{y(s)}{r(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where ω_n is the undamped natural frequency of the system and ζ is the damping ratio of the system.

TIME DOMAIN SPECIFICATIONS

Based on this figure, the following performance parameters are usually defined: maximum overshoot; peak time; rise time; settling time; and steady-state error.

- The maximum overshoot, M_p, is the peak value of the response curve measured from unity.
- This parameter is usually quoted as a percentage.
- The amount of overshoot depends on the damping ratio and directly indicates the relative stability of the system.

$$M_p = e^{-(\zeta\pi/\sqrt{1-\zeta^2})},$$

$$T_p = \frac{\pi}{\omega_d},$$

where

$$\omega_d = \omega_n^2\sqrt{1-\zeta^2}$$

is the damped natural frequency.

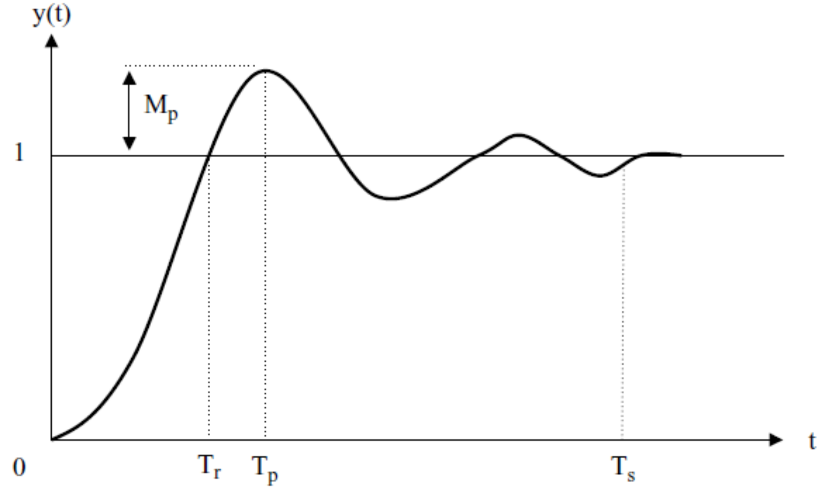


Figure 7.3 Second-order system unit step response

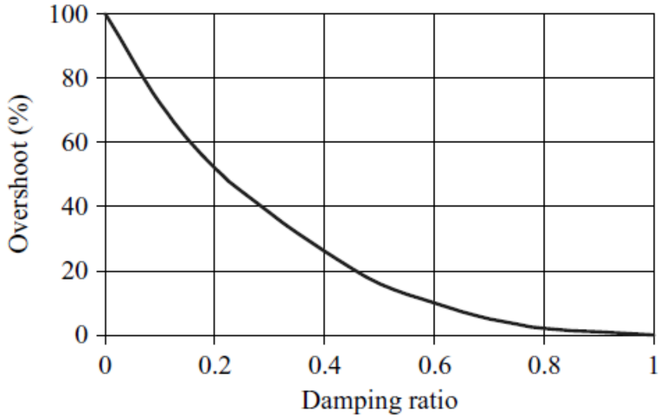


Figure 7.4 Variation of overshoot with damping ratio

TIME DOMAIN SPECIFICATIONS

- The peak time, T_p, is defined as the time required for the response to reach the first peak of the overshoot.
- The system is more responsive when the peak time is smaller, but this gives rise to a higher overshoot.

- The rise time, T_r , is the time required for the response to go from 0 % to 100 % of its final value.

- It is a measure of the responsiveness of a system, and smaller rise times make the system more responsive.

$$T_r = \frac{\pi - \beta}{\omega_d},$$

where

$$\beta = \tan^{-1} \frac{w_d}{\zeta \omega_n}.$$

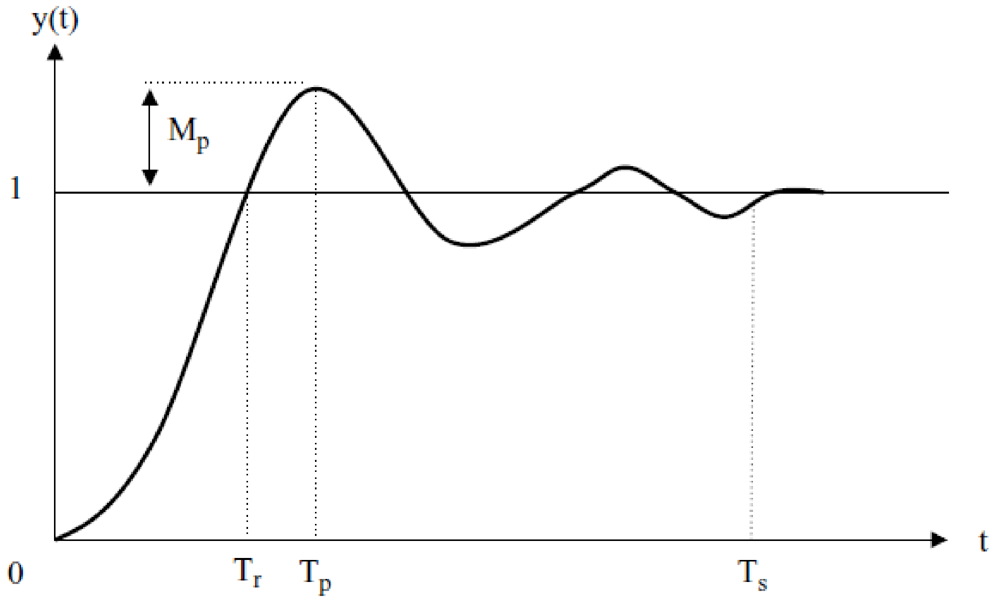


Figure 7.3 Second-order system unit step response

TIME DOMAIN SPECIFICATIONS

- The settling time, T_s , is the time required for the response curve to reach and stay within a range about the final value. A value of 2–5% is usually used in performance specifications.

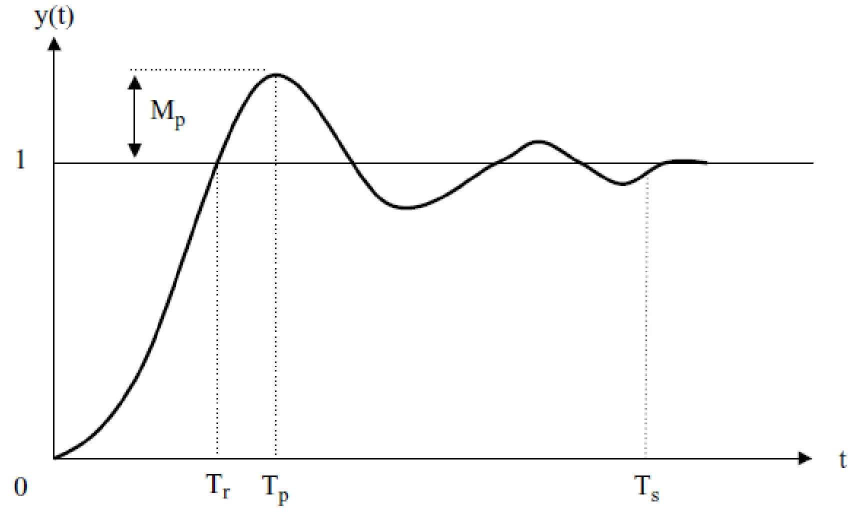


Figure 7.3 Second-order system unit step response

The settling time is usually specified for a 2 % or 5 % tolerance band, and is given by

$$T_s = \frac{4}{\zeta \omega_n} \quad \text{(for 2\% settling time)},$$

$$T_s = \frac{3}{\zeta \omega_n} \quad \text{(for 5\% settling time)}.$$

TIME DOMAIN SPECIFICATIONS

- The steady-state error, E_{ss} , is the error between the system response and the reference input value (unity) when the system reaches its steady-state value.
- A small steady-state error is a requirement in most control systems.
- In some control systems, such as position control, it is one of the requirements to have no steady-state error.

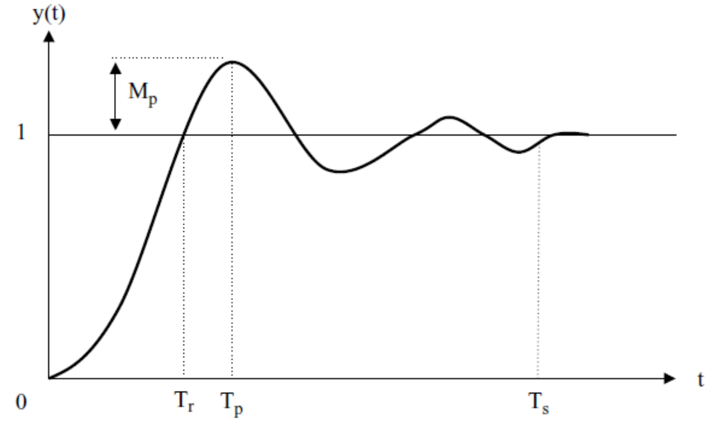


Figure 7.3 Second-order system unit step response

The steady-state error can be found by using the final value theorem, i.e. if the Laplace transform of the output response is $y(s)$, then the final value (steady-state value) is given by

$$\lim_{s \to 0} s y(s),$$

and the steady-state error when a unit step input is applied can be found from

$$E_{ss} = 1 - \lim_{s \to 0} s\, y(s).$$

TIME DOMAIN SPECIFICATIONS: Example

Determine the performance parameters of the system given

$$\frac{y(s)}{r(s)} = \frac{1}{s^2 + s + 1}.$$

Solution

Comparing this system with the standard second-order system transfer function

$$\frac{y(s)}{r(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

we find that $\zeta = 0.5$ and $\omega_n = 1$ rad/s. Thus, the damped natural frequency is

$$\omega_d = \omega_n^2\sqrt{1 - \zeta^2} = 0.866\text{rad/s}.$$

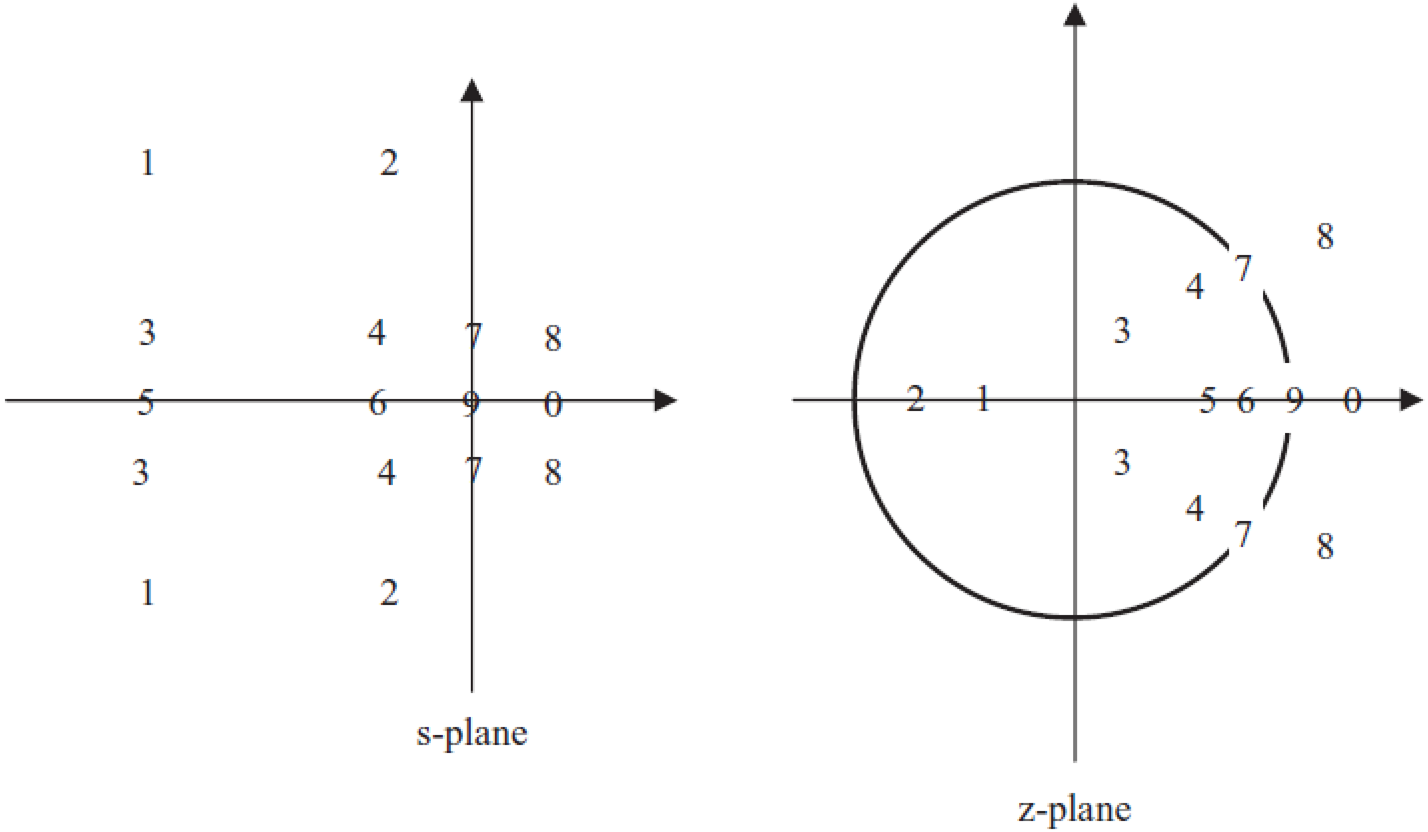
The peak overshoot is

$$M_p = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} = 0.16$$

or 16 %. The peak time is

$$T_p = \frac{\pi}{\omega_d} = 3.627\,\text{s}$$

TIME DOMAIN SPECIFICATIONS: Example



$$T_s = \frac{\quad}{\zeta \omega_n} = 8 \text{ s.}$$

Finally, the steady state error is

$$E_{ss} = 1 - \lim_{s \to 0} s\, y(s) = 1 - \lim_{s \to 0} s \frac{1}{s(s^2 + s + 1)} = 0.$$

MAPPING THE s-PLANE INTO THE z-PLANE

- The pole locations of a closed-loop continuous-time system in the s-plane determine the behaviour and stability of the system, and we can shape the response of a system by positioning its poles in the s-plane. It is desirable to do the same for the sampled data systems.

MAPPING THE s-PLANE INTO THE z-PLANE

First of all, consider the mapping of the left-hand side of the s-plane into the z-plane. Let $s = \sigma + j\omega$ describe a point in the s-plane. Then, along the $j\omega$ axis,

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}.$$

But $\sigma = 0$ so we have

$$z = e^{j\omega T} = \cos \omega T + j \sin \omega T = 1\angle\omega T.$$

Hence, the pole locations on the imaginary axis in the s-plane are mapped onto the unit circle in the z-plane. As ω changes along the imaginary axis in the s-plane, the angle of the poles on the unit circle in the z-plane changes.

MAPPING THE s-PLANE INTO THE z-PLANE

If ω is kept constant and σ is increased in the left-hand s-plane, the pole locations in the z-plane move towards the origin, away from the unit circle. Similarly, if σ is decreased in the left-hand s-plane, the pole locations in the z-plane move away from the origin in the z-plane. Hence, the entire left-hand s-plane is mapped into the interior of the unit circle in the z-plane. Similarly, the right-hand s-plane is mapped into the exterior of the unit circle in the z-plane. As far as the system stability is concerned, a sampled data system will be stable if the closed-loop poles (or the zeros of the characteristic equation) lie within the unit circle.

MAPPING THE s-PLANE INTO THE z-PLANE

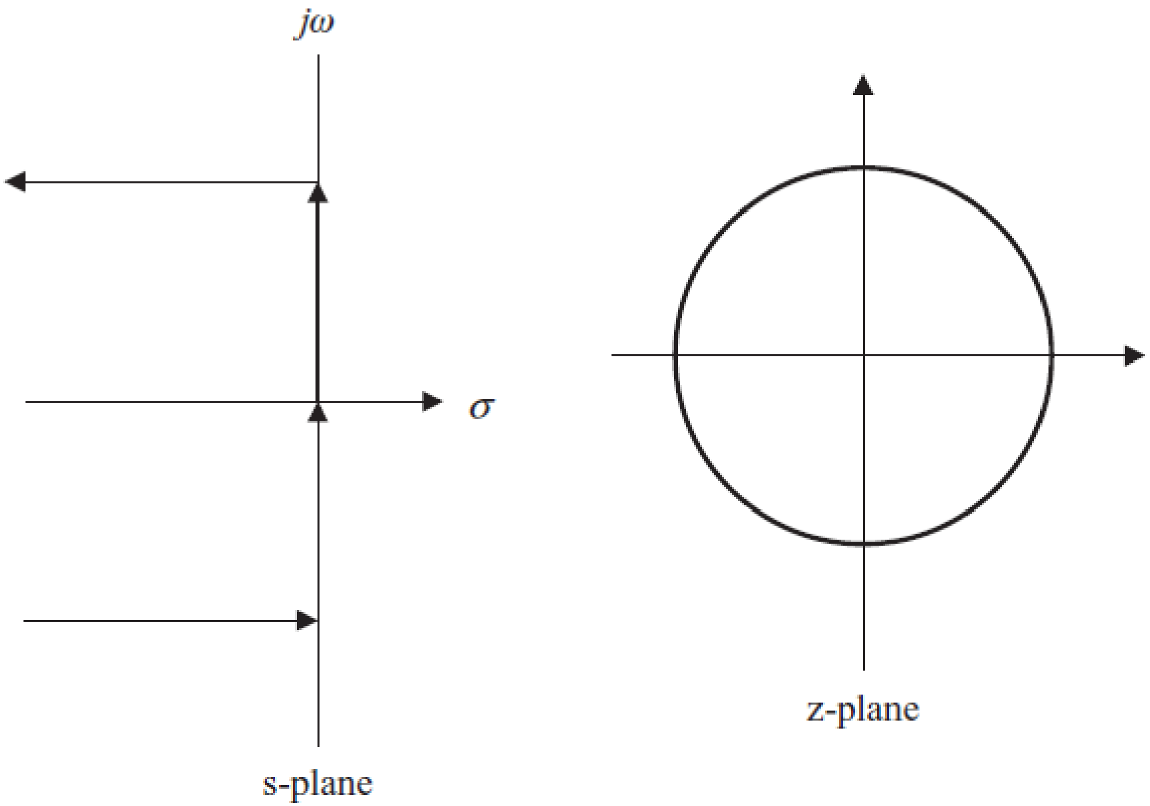


Figure 7.5 Mapping the left-hand *s*-plane into the *z*-plane

MAPPING THE s-PLANE INTO THE z-PLANE

As shown in Figure 7.6, lines of constant σ in the s-plane are mapped into circles in the z-plane with radius $e^{\sigma T}$. If the line is on the left-hand side of the s-plane then the radius of the circle in the z-plane is less than 1. If on the other hand the line is on the right-hand side of the s-plane then the radius of the circle in the z-plane is greater than 1. Figure 7.7 shows the corresponding pole locations between the s-plane and the z-plane.

MAPPING THE s-PLANE INTO THE z-PLANE

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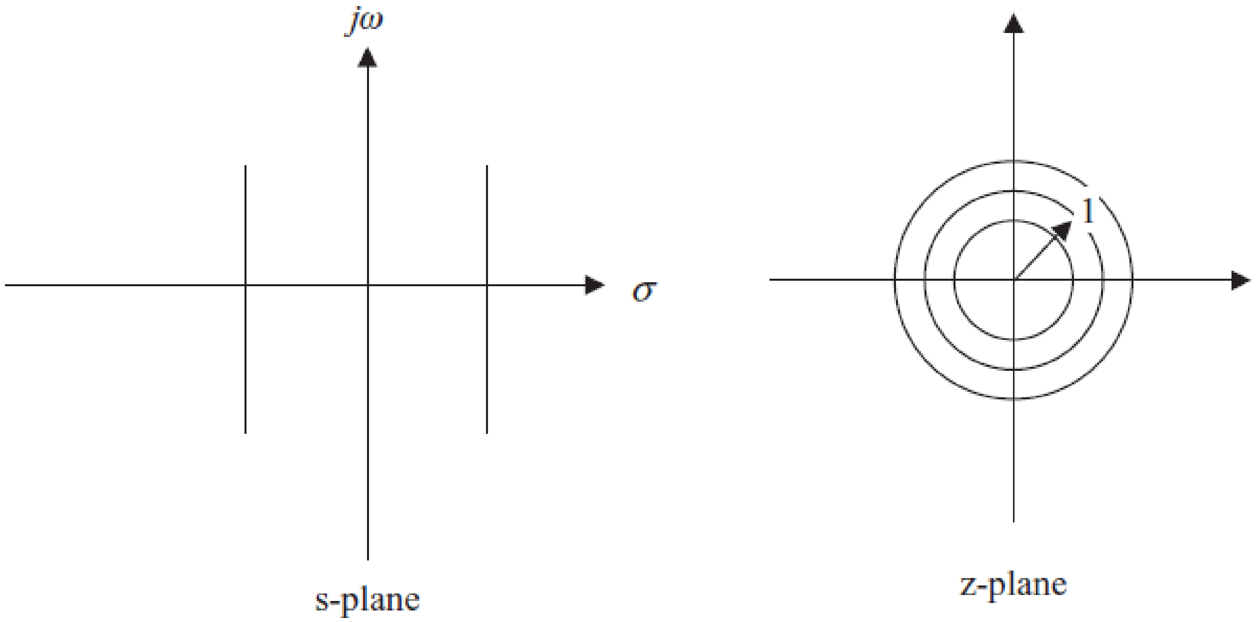

Figure 7.6 Mapping the lines of constant σ

MAPPING THE s-PLANE INTO THE z-PLANE