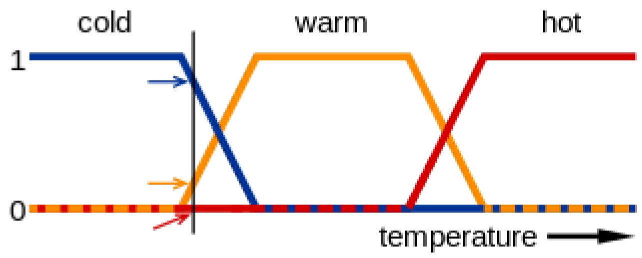
Machine intelligence

Introduction to fuzzy logic, fuzzy sets and membership functions



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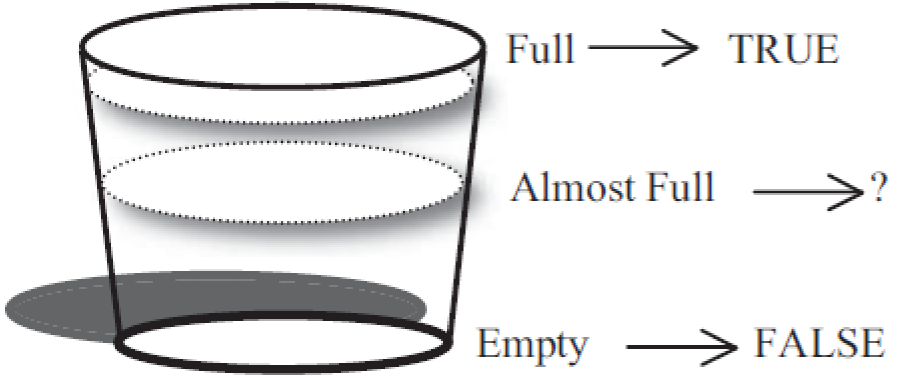
What is fuzzy logic ?

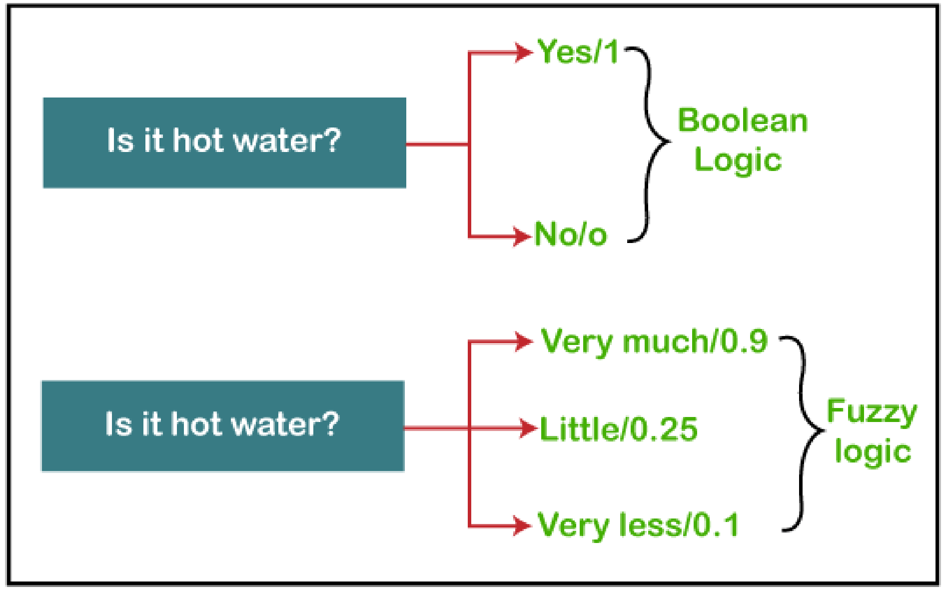
- Fuzzy logic:
  - A way to represent variation or imprecision in logic
  - A way to make use of natural language in logic
  - Approximate reasoning
  - A tool for Embedding Human Structured Knowledge (Experience, Expertise and Heuristic)
- Humans say things like "If it is sunny and warm today, I will drive fast"
- Linguistic variables:
  - Temp: {freezing, cool, warm, hot}
  - Cloud Cover: {overcast, partly cloudy, sunny}
  - Speed: {slow, fast}

Fuzzy logic example

- The glass is more than half full of water.
- The values true or false in classical two-valued logic cannot describe a situation like this.



- Fuzzy logic is a transition from absolute truth to partial truth. That is, from a variable x (True or False) to a linguistic variable ‘Almost full’, ‘Very close to empty’, etc.



- From this perspective, fuzzy logic can be seen as a reasoning formalism of humans where all truths are partial or approximate and any falseness is represented by partial truth.

Fuzzy Sets

- A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ which associates with each point in x a real number in the interval [0,1], with the values of $\mu_A(x)$ at x representing the grade of membership of x in A.

- For example, a fuzzy set $A=\{x_1, x_2, x_3, x_4\}$ in X is characterized by the membership function $\mu_A(x)$ which maps each point x in X to real values 0.5, 1, 0.75 and 0.5.
- $\mu_A(x)$ represents the degree of membership of x in A and the mapping is only limited by $\mu_A(x) \in [0,1]$.
- In classical set theory, the membership function can take only two values: 0 and 1, i.e., either $\mu_A(x) = 1$ or $\mu_A(x) = 0$.
- In set-theoretic notation this is written as $\mu_A(x) \in \{0,1\}$.
- A fuzzy set is an extension of a classical set.
- If X is the universe of discourse and its elements are denoted by x, then a fuzzy set A in X is defined as a set of ordered pairs

$$A = \{x, \mu_A(x) \mid x \in X\}$$

Fuzzy Sets

In Figure 2.2, x_1, x_2, x_3 and x_4 have membership grades of 0.5, 1, 0.75 and 0.5, respectively, written as $\mu_A(x_1) = 0.5$, $\mu_A(x_2) = 1$, $\mu_A(x_3) = 0.75$ and $\mu_A(x_4) = 0.5$. A notational

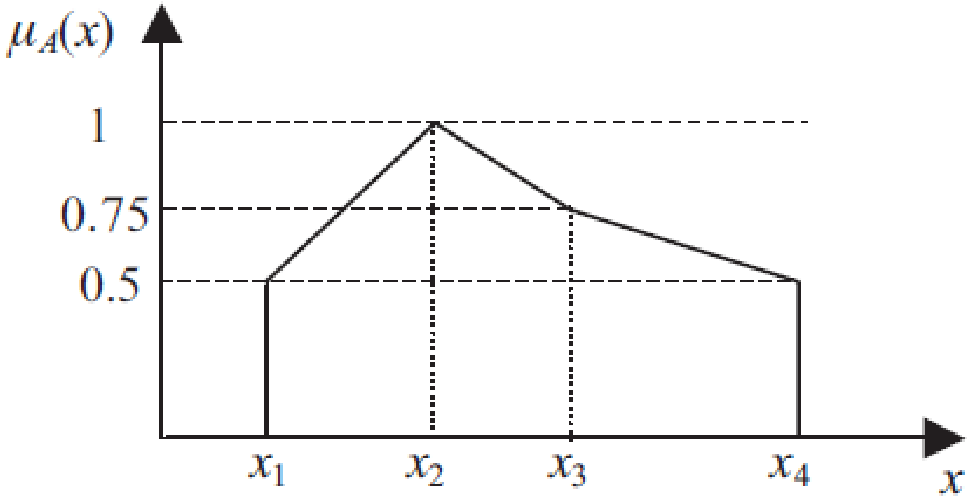


Figure 2.2 Fuzzy set

Fuzzy Sets

convention of fuzzy sets for a discrete and finite universe of discourse X in practice is written as

$$A = \{\mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n\} = \sum_{i=1}^{n} \mu_A(x_i)/x_i \tag{2.2}$$

where '+' does not mean arithmetic addition or logical OR.

Fuzzy sets

Example 2.1 Let $A = \{x_1, x_2, x_3, x_4\}$ in the universe of discourse X having membership values of 0.4, 1.0, 0.7 and 0.8, respectively. This fuzzy set can be written as

$$A = \{0.4/x_1 + 1.0/x_2 + 0.7/x_3 + 0.8/x_4\}$$

Example 2.2 Let $B_1 = \{x_1, x_2, x_3, x_4\}$ be a set of tall boxes and $B_2 = \{x_1, x_2, x_3, x_4\}$ be a set of very tall boxes in the universe of discourse X. The fuzzy sets for the tall and very tall boxes can be written as

$$B_1 = \{0.5/x_1 + 1.0/x_2 + 0.4/x_3 + 0/x_4\}$$
$$B_2 = \{0/x_1 + 0/x_2 + 0.6/x_3 + 0.9/x_4\}$$

The two fuzzy sets for tall and very tall boxes are shown graphically in Figure 2.3. It should be noted that box x_3 belongs to fuzzy set $B_1 = Tall$ with a grade of membership 0.4 and to $B_2 = very\ Tall$ with a grade of membership 0.6.

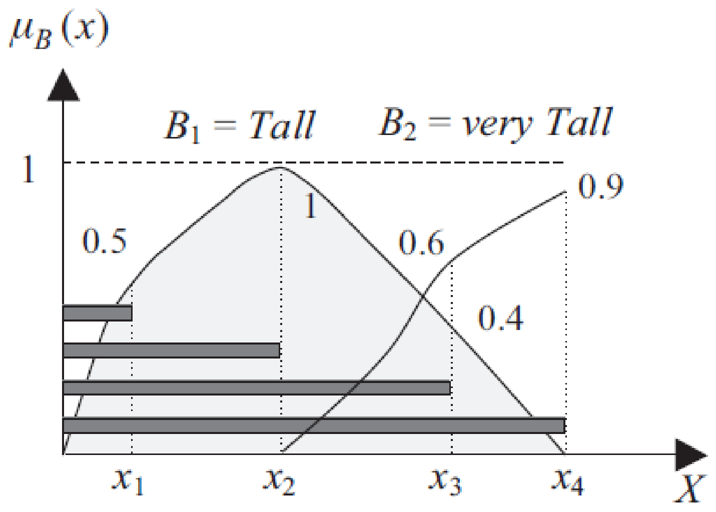


Figure 2.3 Set of tall boxes

Membership functions

- Very often, real-world situations are not certain and cannot be described precisely.
- For example, the uncertainty in Example 2.2 is belonging to Tall or very Tall.
- The uncertainties of expressions like ‘very nice’, ‘too small’, ‘high value’ are called fuzziness.
- The function that characterizes the fuzziness of a fuzzy set A in X, which associates each point in X with a real number in the interval [0, 1], is called a membership function (MF).
- There is no strict rule for defining a membership function.
- The choice of membership function is usually problem-dependent and often determined heuristically and subjectively.
- Most widely used MFs in the fuzzy logic literature are triangular, trapezoidal, Gaussian and bell-shaped functions.

Triangular MF

A triangular MF is specified by three parameters $\{a, b, c\}$, shown in Figure 2.4 and defined as

$$\mu(x) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right) \tag{2.3}$$

The parameters $\{a, b, c\}$ with $a < b < c$ determine the x coordinates of the three corners of the underlying triangular MF. Triangular MFs can be asymmetric, depending on the relations $a \leq b$ and $b \leq c$. Figure 2.4 shows a symmetric and an asymmetric triangular MF.

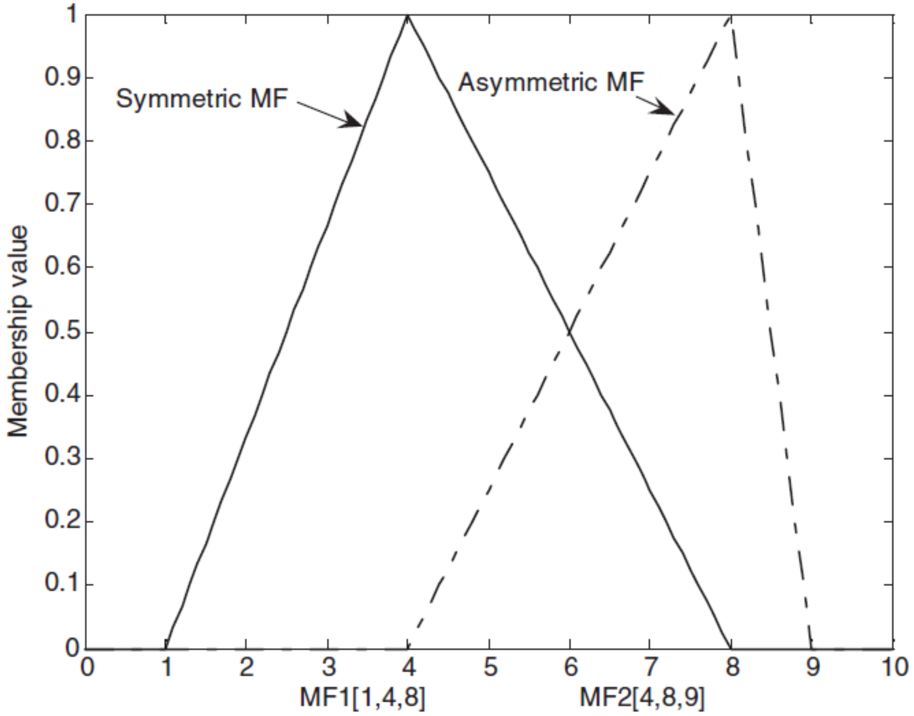


Figure 2.4 Triangular MF

Trapezoidal MF

A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$, shown in Figure 2.5 and defined as

$$\mu(x) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right) \tag{2.4}$$

The parameters $\{a, b, c, d\}$ with $a < b < c < d$ determine the x coordinates of the four corners of the underlying trapezoidal MF. Trapezoidal MFs can be asymmetric, depending on the relations $a \le b$ and $c \le d$. Both triangular and trapezoidal MFs can be symmetric or asymmetric, which is seen as an advantage for some applications. Owing to their simple formulae and computational efficiency, both triangular and trapezoidal MFs have been used extensively, especially in online applications.

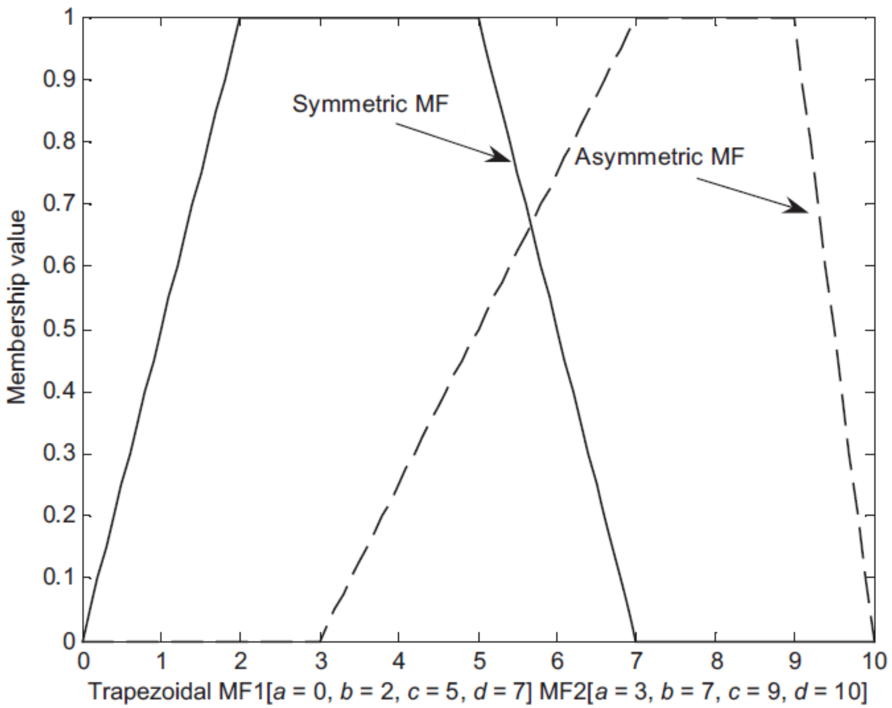


Figure 2.5 Trapezoidal MF

Gaussian MF

A Gaussian MF is specified by two parameters $\{m, \sigma\}$, shown in Figure 2.6 and defined as

$$\mu(x) = \exp\left[-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] \tag{2.5}$$

The parameters m and σ represent the centre and width of the Gaussian MF, respectively.

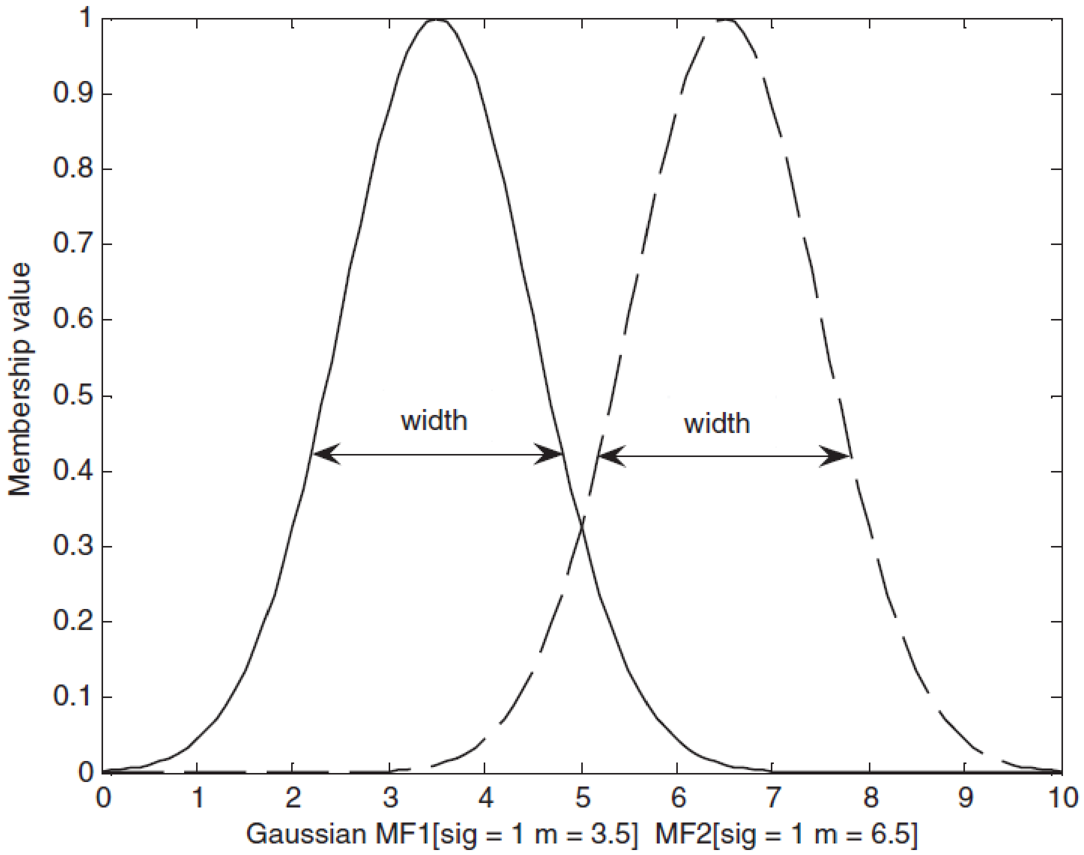


Figure 2.6 Gaussian MF

Bell-shaped MF

A bell-shaped MF is specified by three parameters $\{m, \sigma, a\}$, shown in Figure 2.7 and defined as

$$\mu(x) = \frac{1}{1 + \left|\dfrac{x - m}{\sigma}\right|^{2a}} \tag{2.6}$$

The parameters m and σ represent the centre and width of the bell-shaped MF, respectively. Parameter a, usually positive, controls the slope of the MF as shown in Figure 2.7. The MF is narrower with increasing value of a.

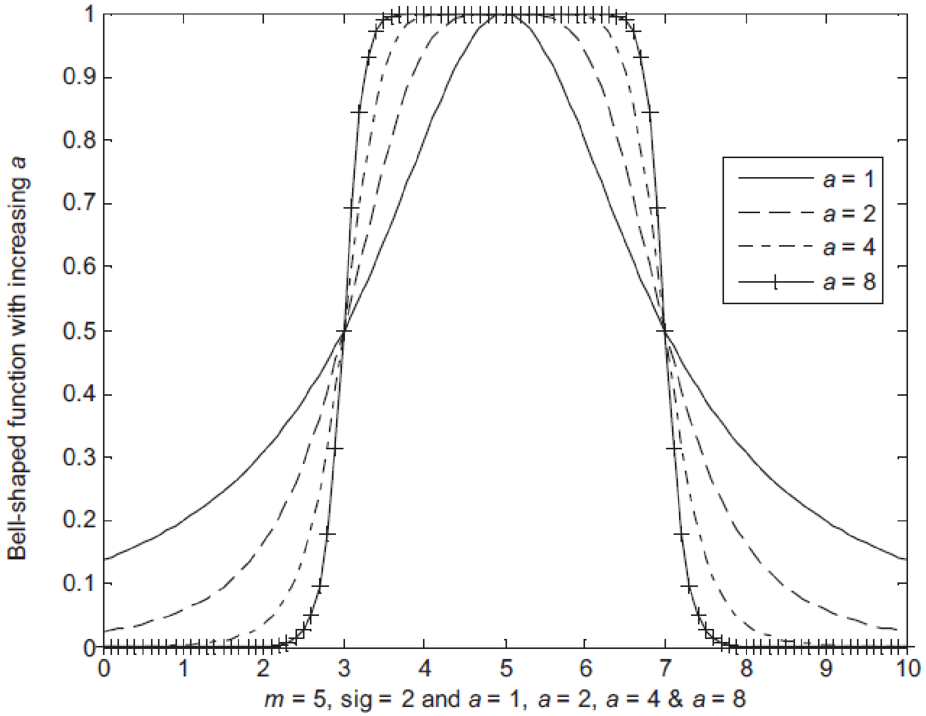


Figure 2.7 Bell-shaped MF with increasing value of a

Sigmoidal MF

$$\mu(x) = \frac{1}{1 + \exp\left[-a\left(x - c\right)\right]} \tag{2.7}$$

The parameter a controls the slope of the MF at the cross-point $x = c$. Two sigmoidal MFs are shown in Figure 2.8. One is right open and the other is left open. The sign of the parameter a determines the open-end direction of the sigmoidal MF. If a is positive, the MF will open to the right and if a is negative, the MF will open to the left. This property of the sigmoidal MF helps to define extreme positive or extreme negative MFs.

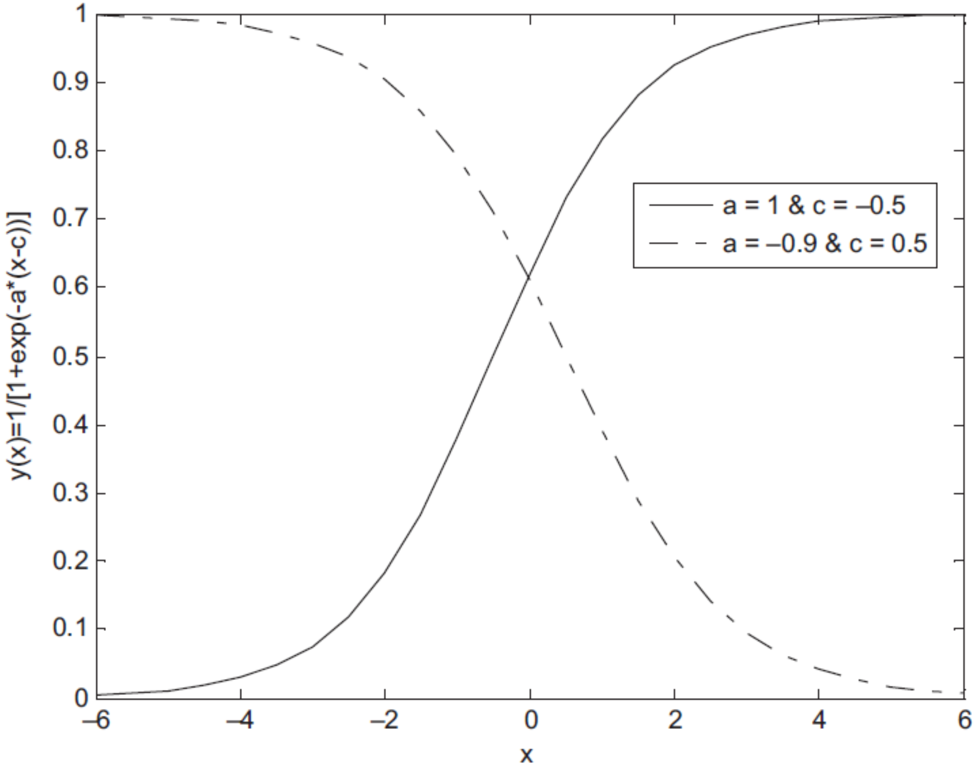


Figure 2.8 Two sigmoidal MFs

Selecting the MF

- There is no general rule for choosing the type of MFs for a particular problem or application.

- It is rather application-dependent; the shape of MF depending on the parameters of the MF used, which greatly influences the performance of a fuzzy system.

- There are different approaches to construct membership functions, such as heuristic selection (the most widely used), clustering approach, C-means clustering approach, adaptive vector quantization and self-organizing map.

Features of MFs

- Support

The support of a fuzzy set A is the set of all points $x \in X$ at which $\mu_A(x) > 0$. Assume A is a fuzzy subset of X. The support of A, denoted $Support(A)$, is the crisp subset of X whose elements have nonzero membership grades in A, defined as

$$Support(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\} \tag{2.8}$$

- Core

The core of a fuzzy set A, denoted $Core(A)$, is the crisp subset of X consisting of all elements with membership grade 1. This is defined as

$$Core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\} \tag{2.9}$$

Features of MFs

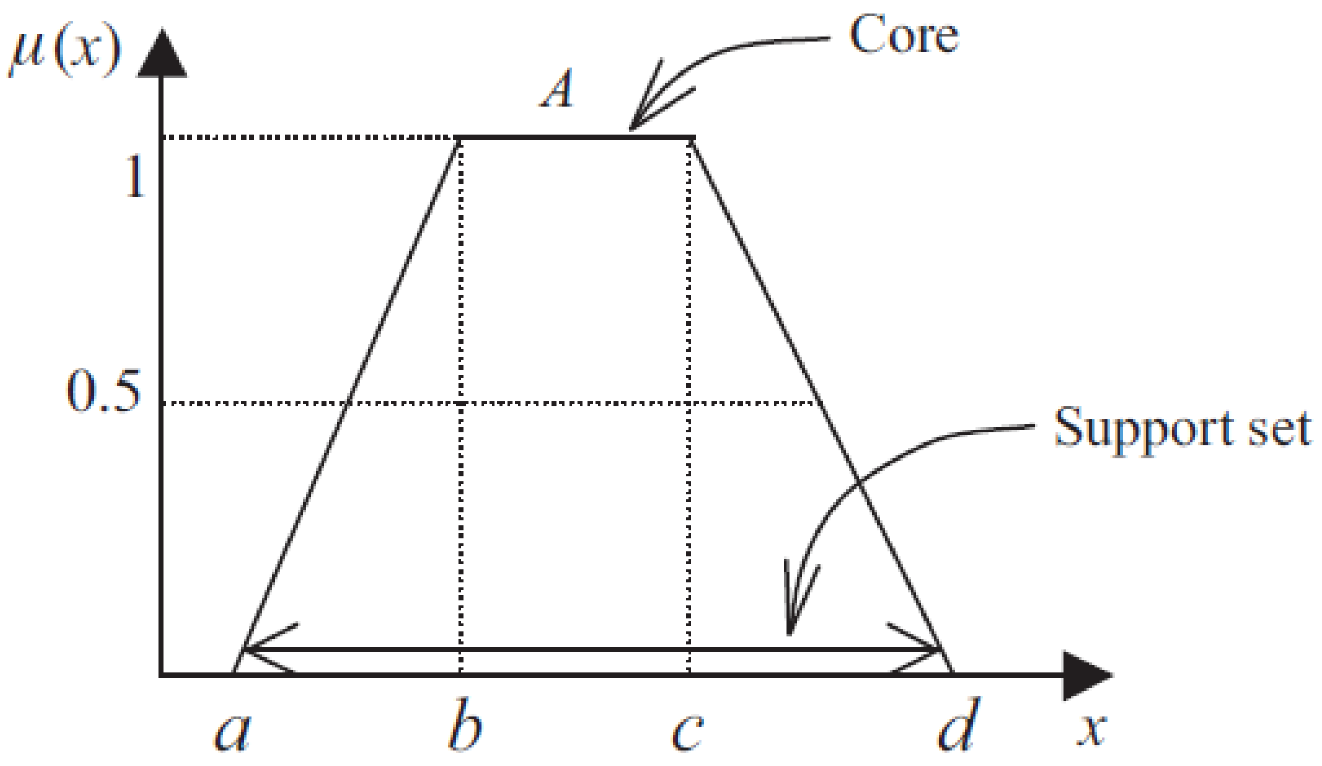


Figure 2.9 Support set and core of MF *A*

Features of MFs: Fuzzy Singleton

A fuzzy set whose support is a single point in X at which $\mu_A(x) = 1$ is called a singleton. A singleton is shown in Figure 2.10.

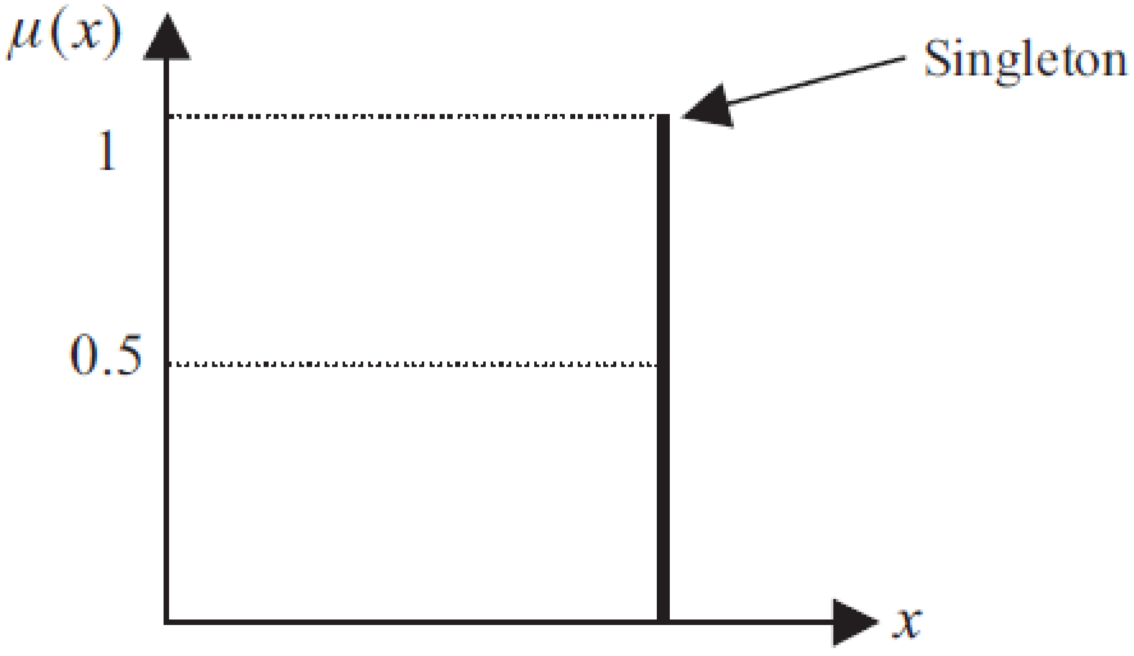


Figure 2.10 Singleton

Features of MFs: crossover point

A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = \alpha$ with $\alpha \in [0, 1]$:

$$Crossover(A) = \{x \mid \mu_A(x) = 0.5\} \quad (2.10)$$

- At the peak point, point b of MF A and point c of MF B in Figure 2.11, the membership value is 1.
- The left width is the distance of the left anchor point from the peak point and the right width is the distance of the right anchor point from the peak point.
- The left and right widths for the MF B are shown in Figure 2.11.

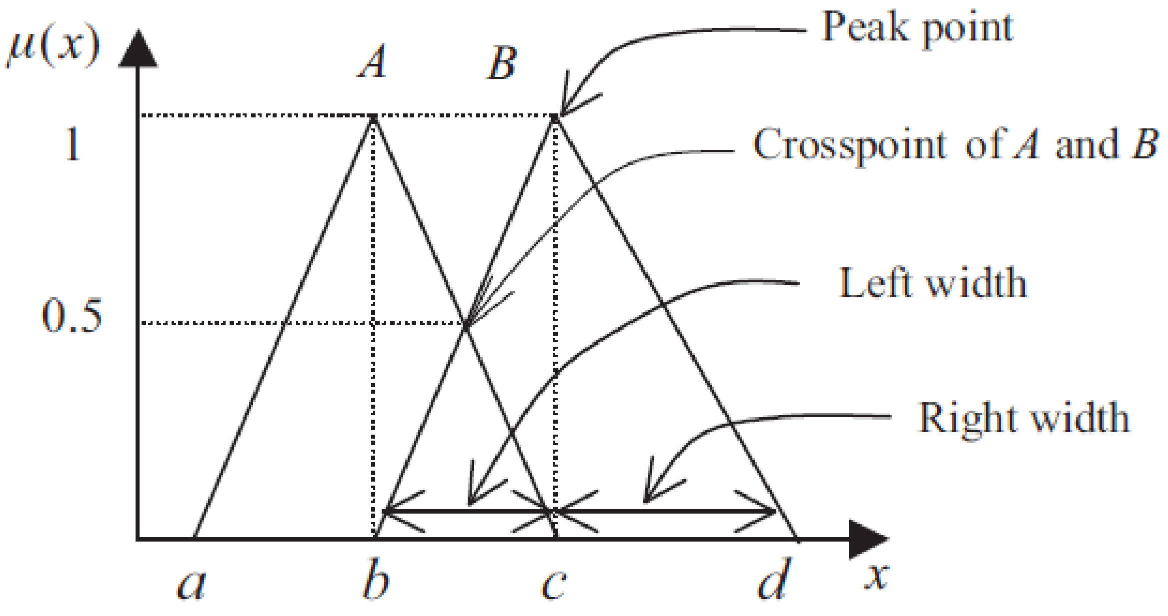


Figure 2.11 Crossover point, left and right width of MF

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