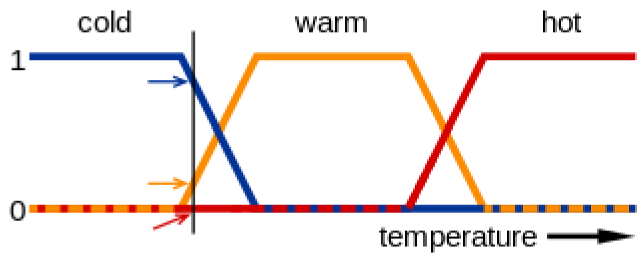
Machine intelligence

fuzzy logic part2



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Outline

- Operations on Fuzzy Sets
- Linguistic Variables
- Linguistic Hedges
- Fuzzy Relations
- Fuzzy If–Then Rules
- Fuzzification
- Defuzzification
- Inference Mechanism

Operations on fuzzy sets

Operations on Fuzzy Sets

The membership function is the main component defining the basic fuzzy set operations. Zadeh and other researchers have given additional and alternative definitions for set-theoretic operations.

α-cut of a fuzzy set: The α-cut of a fuzzy set A, denoted A_α, is a subset of X consisting of all the elements in X defined by

$$A_\alpha = \{x \mid \mu_{A_\alpha}(x) \geq \alpha \text{ and } x \in X\} \tag{2.11}$$

This means that the fuzzy set A_α contains all elements with a membership of $\alpha \in [0, 1]$ and higher, called the α-cut of the membership function. The α-cut of a fuzzy set A is shown in Figure 2.12. At a resolution level of α, it will have support of A_α. The higher the value of α, the higher the confidence in the parameter.

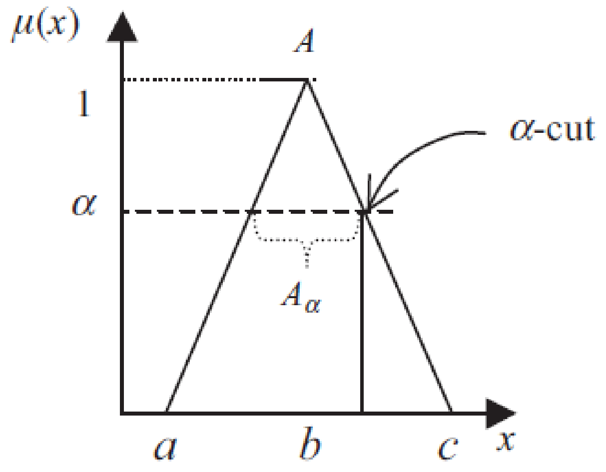

Figure 2.12 α-cut of the membership function

Operations on fuzzy sets

Example 2.3 Let A be a fuzzy set in the universe of discourse X and $(x_1, x_2, x_3, x_4) \in X$ defined as follows:

$$A = \{0.3/x_1, 1/x_2, 0.5/x_3, 0.9/x_4, 1/x_5\}$$

A_α for $\alpha > 0.5$ is

$$A_{\alpha>0.5} = \{1/x_2, 0.9/x_4, 1/x_5\}$$

Operations on fuzzy sets

Union of fuzzy sets: The union of two fuzzy sets A and B with membership functions μ_A and μ_B, respectively, is a fuzzy set C, denoted $C = A \cup B$, with the membership function μ_C. There are two definitions for the union operation: the max membership function and the product rule, as defined in Equations (2.12) and (2.13):

$$\mu_C(x) = \max\left[\mu_A(x), \mu_B(x)\right] \tag{2.12}$$

$$\mu_C(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x) \tag{2.13}$$

where x is an element in the universe of discourse X.

Operations on fuzzy sets

Example 2.4 Let A and B be two fuzzy sets in the universe of discourse X and $(x_1, x_2, x_3, x_4) \in X$ defined as follows:

$$A = \{0/x_1 + 1/x_2 + 0.7/x_3 + 0.4/x_4 + 0.2/x_5 + 0/x_6\}$$
$$B = \{0/x_1 + 0.4/x_2 + 0.7/x_3 + 0.8/x_4 + 1/x_5 + 0/x_6\}$$

The union of fuzzy sets A and B using the max membership function is

$$C_{\max} = A \cup B = \{0/x_1 + 1/x_2 + 0.7/x_3 + 0.8/x_4 + 1/x_5 + 0/x_6\}$$

where $\mu_C(x_i)$ is calculated from $\max[\mu_A(x_i), \mu_B(x_i)]$ for $i = 1, 2, 3, \ldots, 6$. Alternatively, using the product rule it is

$$C_{\text{prod}} = A \cup B = \{0/x_1 + 1/x_2 + 0.91/x_3 + 0.88/x_4 + 1/x_5 + 0/x_6\}$$

where $\mu_C(x_i)$ is calculated using $[\mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i)^* \mu_B(x_i)]$ for $i = 1, 2, 3, \ldots, 6$.

The union operation of fuzzy sets A and B is shown in Figure 2.13.

Operations on fuzzy sets

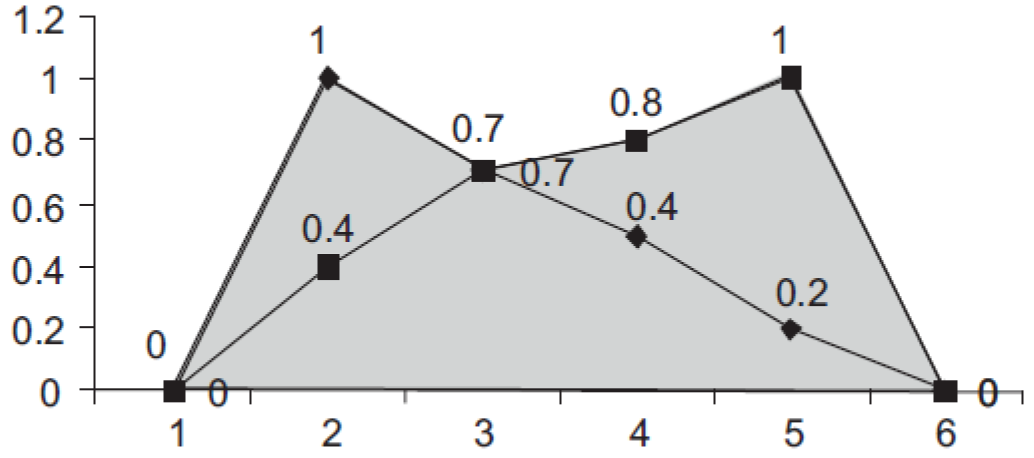

Figure 2.13 Union of fuzzy sets *A* and *B* using max operation

Operations on fuzzy sets

Intersection of fuzzy sets: The intersection of two fuzzy sets A and B with membership functions μ_A and μ_B, respectively, is a fuzzy set C, denoted $C = A \cap B$, with membership function μ_C defined using the min membership function or the product rule as

$$\mu_C(x) = \min[\mu_A(x), \mu_B(x)] \tag{2.14}$$

$$\mu_C(x) = \mu_A(x)^* \mu_B(x) \tag{2.15}$$

Example 2.5 Let A and B be two fuzzy sets in the universe of discourse X and $(x_1, x_2, x_3, x_4) \in X$ defined as in the previous example.

The intersection of fuzzy sets A and B using the min membership function is

$$C_{\min} = A \cap B = \{0/x_1 + 0.4/x_2 + 0.7/x_3 + 0.4/x_4 + 0.2/x_5 + 0/x_6\}$$

where $\mu_C(x_i)$ is calculated from $\mu_C(x) = \min[\mu_A(x), \mu_B(x)]$ for $i = 1, 2, 3, \ldots, 6$. Alternatively, using the product rule it is

$$C_{\mathrm{prod}} = A \cap B = \{0/x_1 + 0.4/x_2 + 0.49/x_3 + 0.32/x_4 + 0.2/x_5 + 0/x_6\}$$

where $\mu_C(x_i)$ is calculated from $\mu_C(x) = \mu_A(x)^* \mu_B(x)$ for $i = 1, 2, 3, \ldots, 6$.

Operations on fuzzy sets

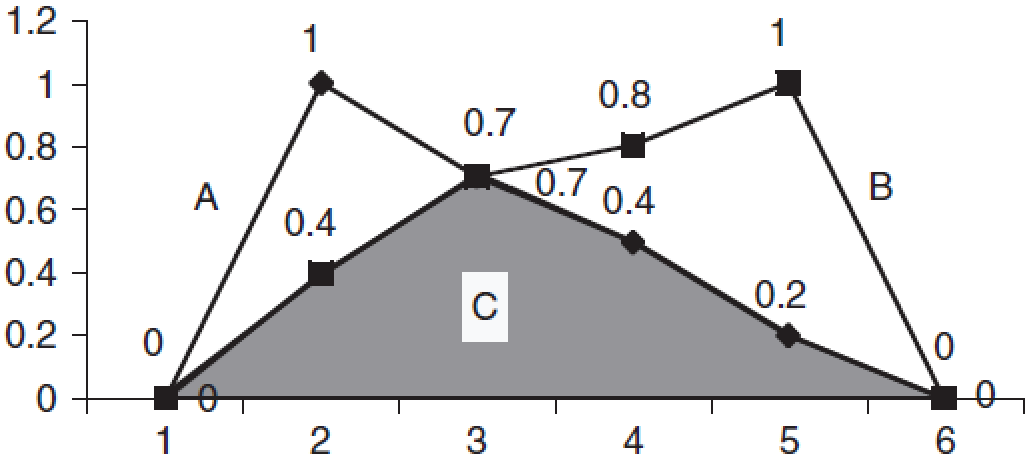

Figure 2.14 Intersection of fuzzy sets *A* and *B* using the min operation

Operations on fuzzy sets

Complement of fuzzy set: The complement of a fuzzy set A with membership function μ_A is a fuzzy set, denoted $\sim A$, with membership function $\mu_{\sim A}$ defined as

$$\mu_{\sim A}(x) = 1 - \mu_A(x) \tag{2.16}$$

Example 2.6 Let A be a fuzzy set in the universe of discourse X and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in X$ defined as follows:

$$\mathrm{A} = \{1/x_1 + 1/x_2 + 0.9/x_3 + 0.8/x_4 + 0.7/x_5 + 0.3/x_6 + 0.1/x_7 + 0/x_8\}$$

The complement of fuzzy set A is $\sim A$:

$$\sim A = \{0/x_1 + 0/x_2 + 0.1/x_3 + 0.2/x_4 + 0.3/x_5 + 0.7/x_6 + 0.9/x_7 + 1/x_8\}$$

where $\mu_{\sim A}(x)$ is calculated from $[1 - \mu_A(x)]$ for $i = 1, 2, 3, \ldots, 8$.

The complement operation of fuzzy set A is shown in Figure 2.15.

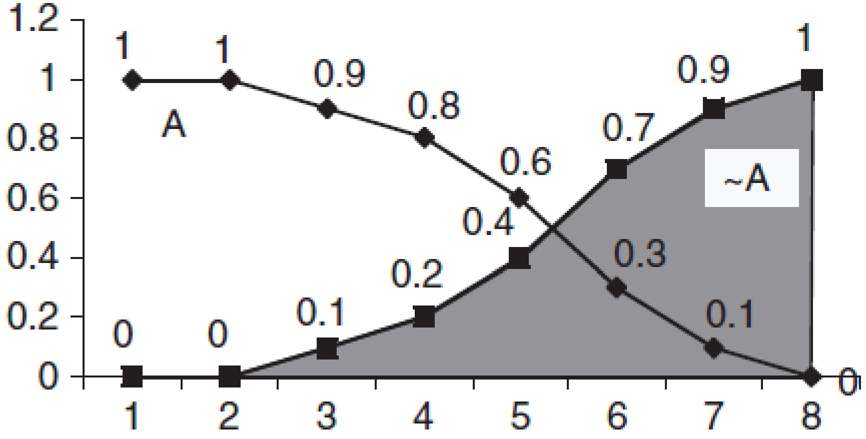

Figure 2.15 Complement of fuzzy set A

Operations on fuzzy sets

Fuzzy subsets or containment: Let A and B be two fuzzy sets with membership functions μ_A and μ_B, respectively. A is a subset of B (or A is contained in B), written $A \subset B$, if and only if

$$\mu_A \leq \mu_B \; \forall x, x \in X \tag{2.17}$$

Equality of fuzzy sets: Two fuzzy sets A and B with membership functions μ_A and μ_B, respectively are equal, written $A = B$, if and only if

$$\mu_A = \mu_B \; \forall x, x \in X \tag{2.18}$$

Example 2.7 Let A and B be two fuzzy sets in the universe of discourse X and $(x_1, x_2, x_3, x_4) \in X$ defined as follows:

$$A = \{0/x_1 + 1/x_2 + 0.7/x_3 + 0.4/x_4 + 0.2/x_5 + 0/x_6\}$$
$$B = \{0/x_1 + 1/x_2 + 0.7/x_3 + 0.4/x_4 + 0.2/x_5 + 0/x_6\}$$

All the membership values of A are equal to those of B, i.e., $\mu_A = \mu_B$, therefore $A = B$.

Null (or empty) fuzzy set: A fuzzy set is null (or empty, denoted $\emptyset$) if and only if its membership function $\forall x \in X$ (for all elements in X) is identically zero on X. This is defined as

$$\mu_{\emptyset}(x) = \{x \mid \mu_{\emptyset}(x) = 0 \text{ and } \forall x \in X\} \tag{2.19}$$

Operations on fuzzy sets

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Operations on fuzzy sets

Properties of fuzzy sets: Assume A, B and C are fuzzy sets of X. The following properties hold for union, intersection and fuzzy subsets.

(i) **Commutativity**

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

(ii) **Idempotency**

$$A \cup A = A$$

$$A \cap B = B \cap A$$

(iii) **Associativity**

$$A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$$

(iv) **Distributivity**

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup \emptyset = A$$

$$A \cap \emptyset = \emptyset$$

$$A \cup X = X$$

$$A \cap X = A$$

Operations on fuzzy sets

(v) **Transitivity**

If $A \subset B$ then $B = A \cup B$ and $A = A \cap B$

If $A \subset B$ and $B \subset C$ then $A \subset C$

The following properties hold for complements of fuzzy subsets.

(vi) **De Morgan's Law**

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

A significant feature of fuzzy set that distinguishes them from classical sets is that $\bar{A} \cap \emptyset \neq \emptyset$ and $\bar{A} \cup A \neq X$.

Linguistic variables

A linguistic variable is a variable whose values are words or sentences, used as labels of fuzzy subsets (Zadeh, 1975a,b, 1976a). Such linguistic variables serve as a means of approximate characterization of systems which cannot be described precisely by numerical values or other traditional quantitative terms. For example, speed is a linguistic variable if its values are slow, medium, fast, not slow, very fast, not very slow, etc. In this case, fast is a linguistic value of speed and is imprecise compared with an exact numeric value such as 'speed is 77 mph'. The relation between a numerical variable $s = 77$ and the linguistic variables slow, medium and fast is illustrated graphically in Figure 2.16.

In general, a linguistic variable is characterized by a quintuple $\{X, T, U, G, M\}$, where X is the name of the variable (e.g., Speed), T denotes the term set of X (i.e., the set of names of linguistic labels of X over a universe of discourse U: slow, medium, fast, etc.), G is the syntactic rule or grammar for generating names and M is the semantic rule for associating with each X its meaning, $M(X) \subseteq U$ (Zadeh, 1975a).

Linguistic variables

$$T(Speed) = \{slow, medium, fast\}$$

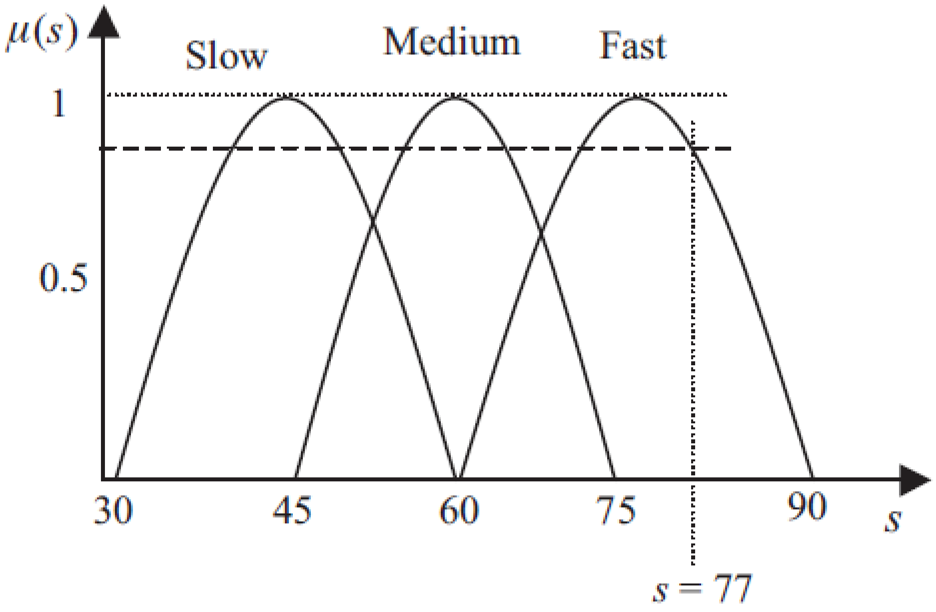

Figure 2.16 Relation between linguistic and numerical variables

Features of Linguistic Variables

Theoretically, the term set $T(X)$ is infinite but in practical applications, $T(X)$ is defined with a small number of terms so that each element of $T(X)$ defines a mapping between each element and the function $M(X)$, which associates a meaning with each term in the term set. Let the term set of the linguistic variable Speed be {slow, medium, fast} within the universe of discourse $U = [0, 120]$. The term set can be expressed as

$$T(Speed) = \{slow, medium, fast\}$$

Features of Linguistic Variables

The semantic rule of linguistic variables can be expressed using context-free grammar. For example

$$T = \{slow, very\ slow, very\ very\ slow, \ldots\}$$

Using context-free grammar the above expression can be written as

$$T \rightarrow slow$$
$$T \rightarrow very\, T$$

Here, 'very' is called a linguistic hedge, which is used to derive new linguistic variables. Linguistic hedges will be discussed further in the next section.

A linguistic variable can be a word or sentence and such natural language expressions are fuzzy, e.g., Slow OR Medium, Medium AND Fast. Figure 2.17 shows three MFs: Slow, Medium and Fast.

The linguistic variable 'Slow OR Medium' is shown graphically in Figure 2.18. It is the shaded area representing the union of the membership functions 'Slow or Medium'.

Features of Linguistic Variables

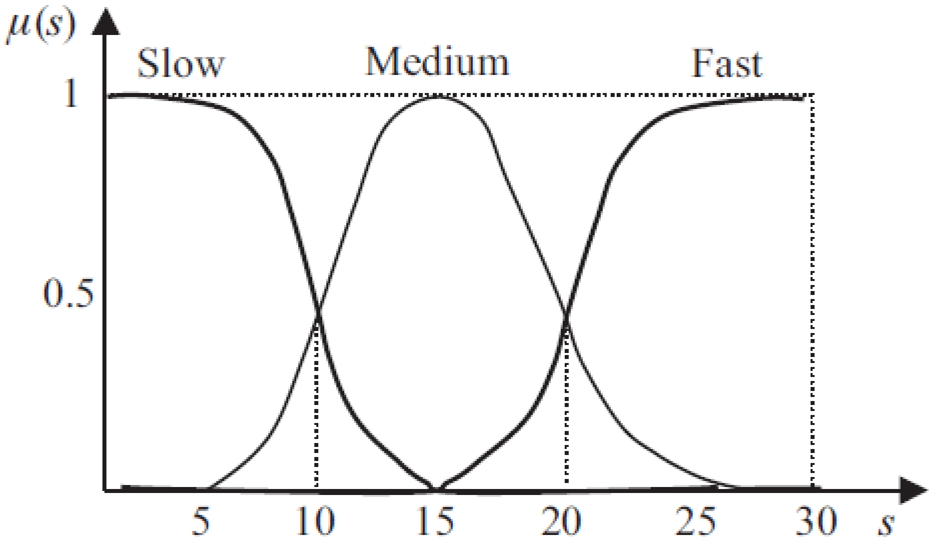

Figure 2.17 Three fuzzy sets for speed – Slow, Medium and Fast

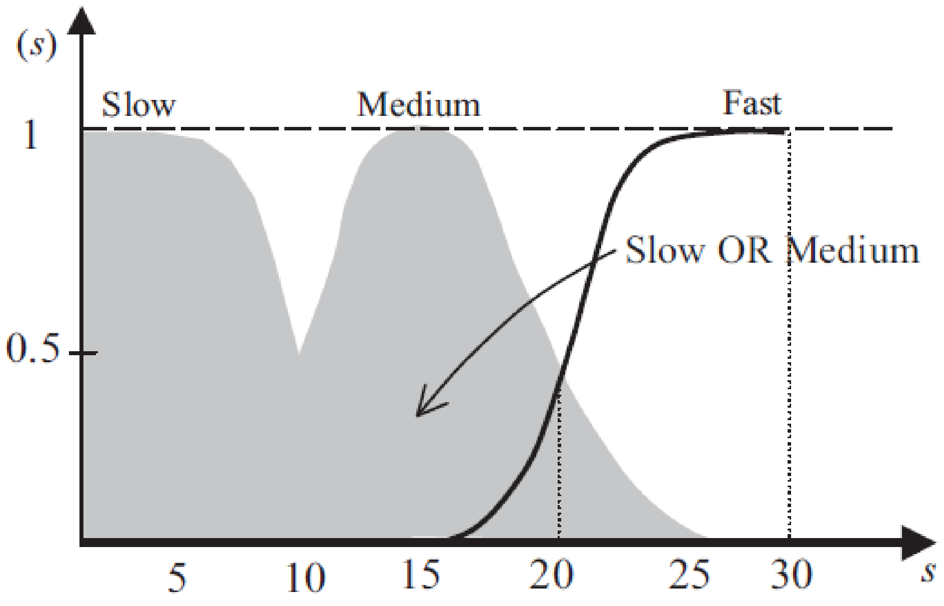

Figure 2.18 Expression for 'Slow OR Medium'

Features of Linguistic Variables

The linguistic variable 'Medium AND Fast' is shown graphically in Figure 2.19. It is the shaded area representing the intersection of the membership functions Medium AND Fast.

In the above examples, OR and AND are connectives, which play an important role in the description of linguistic variables. It can be seen from Figures 2.18 and 2.19 that they are used to derive new linguistic variables from the term sets.

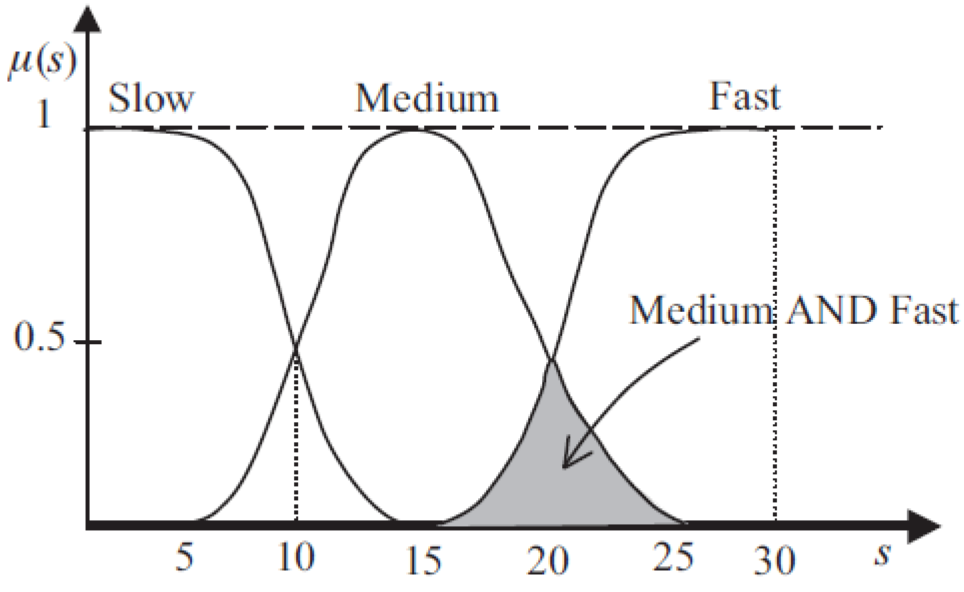

Figure 2.19 Expression for 'Medium AND Fast'

Linguistic Hedges

The purpose of the hedges is to generate a larger set of values for a linguistic variable from a small collection of primary terms. Hedges are realized on primary terms through the processes

- Intensification or concentration,
- Dilation and
- Fuzzification.

This can be represented as a quadruple $\{H, M, T, C\}$, where H is the set of hedges, M is the marker, T is the set of primary terms (e.g., slow, medium, fast, etc.) and C is the set of connectives. Parentheses are used as markers in the definition of linguistic variables to separate the term set from the hedge, e.g., Very (Small). Figure 2.20 depicts the format of the use of the different term sets, hedges and connectives for defining linguistic variables.

For example, Big but Not Very (Big). Here 'Big' is a primary term set, 'but' is a connective (which means AND in this case) and 'Very' is a hedge. 'Not' is a complement operation on the term set. Parentheses '( )' are used as a marker.

Linguistic Hedges

Example 2.8 The hedge ‘Very’ is a concentration (or intensification) operation and performed by squaring the membership values of the primary fuzzy set. The operation is shown for two primary fuzzy sets Small and U in the example below.

$$\text{Very (Small)} = \text{Small}^2 = [\mu_{\text{Small}}]^2$$

$$\text{Very (Very } (U)) = (\text{Very } ([\mu_U]))^2 = ([\mu_U]^2)^2 = [\mu_U]^4$$

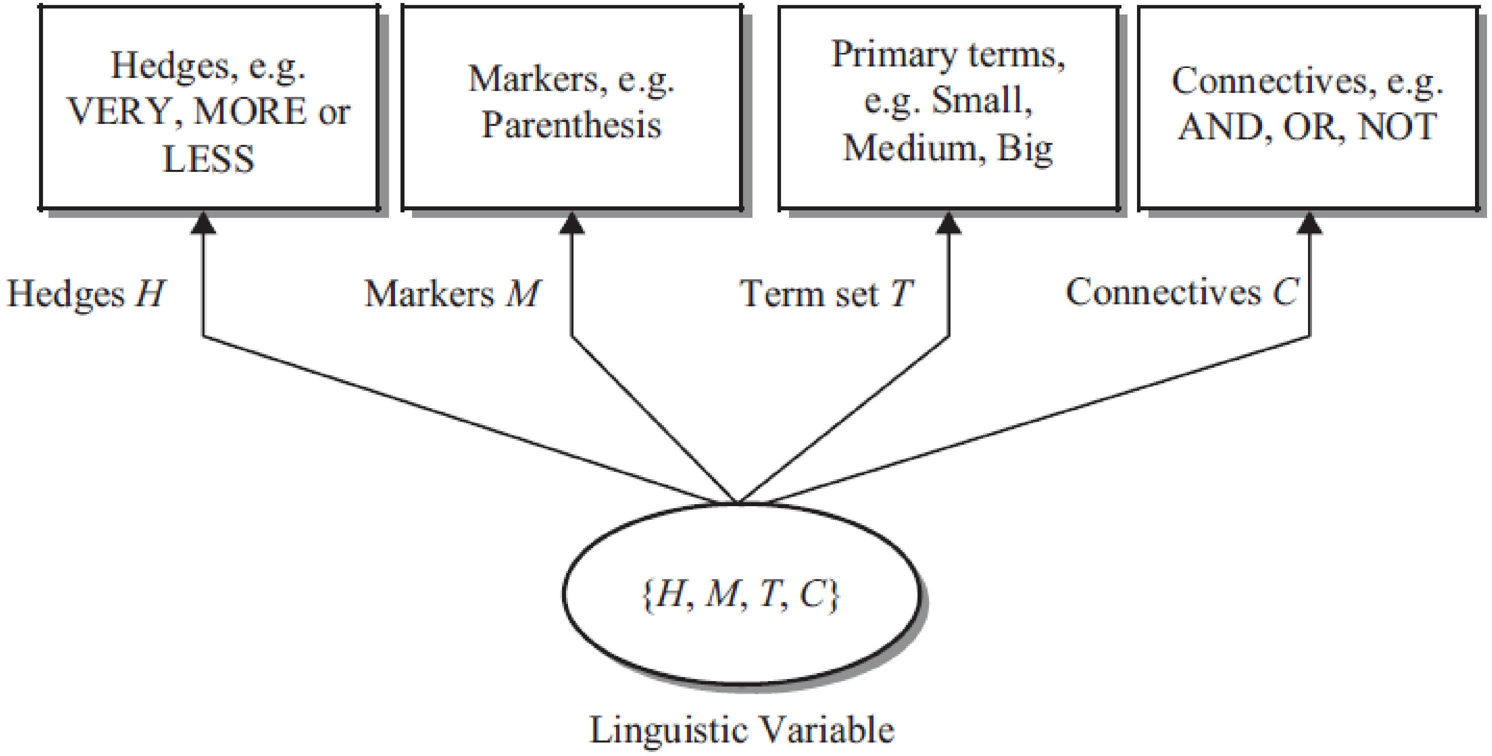

Figure 2.20 Linguistic variables and hedges

Linguistic Hedges

Example 2.9 Consider the fuzzy set A of short pencils defined by

$$A = \left\{ \frac{0.20}{p_1} + \frac{0.5}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{0.9}{p_5} \right\}$$

Then the fuzzy set for very short pencils can be expressed by the use of a hedge on the fuzzy set A:

$$Very\,(A) = [\mu_A]^2 = \left\{ \frac{0.04}{p_1} + \frac{0.25}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{0.81}{p_5} \right\}$$

The linguistic hedge 'More or less' is a dilation operation defined as More or less $(A) = A^{1/2}$. The fuzzy set for more or less short pencils can be expressed by the following:

$$More\ or\ less\,(A) = [\mu_A]^{1/2} = \left\{ \frac{0.45}{p_1} + \frac{0.71}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} + \frac{0.95}{p_5} \right\}$$

Linguistic Hedges

The application of the linguistic hedges 'Very' and 'More or less' is demonstrated through the concentration (or intensification) and dilation process as shown in Figure 2.21.

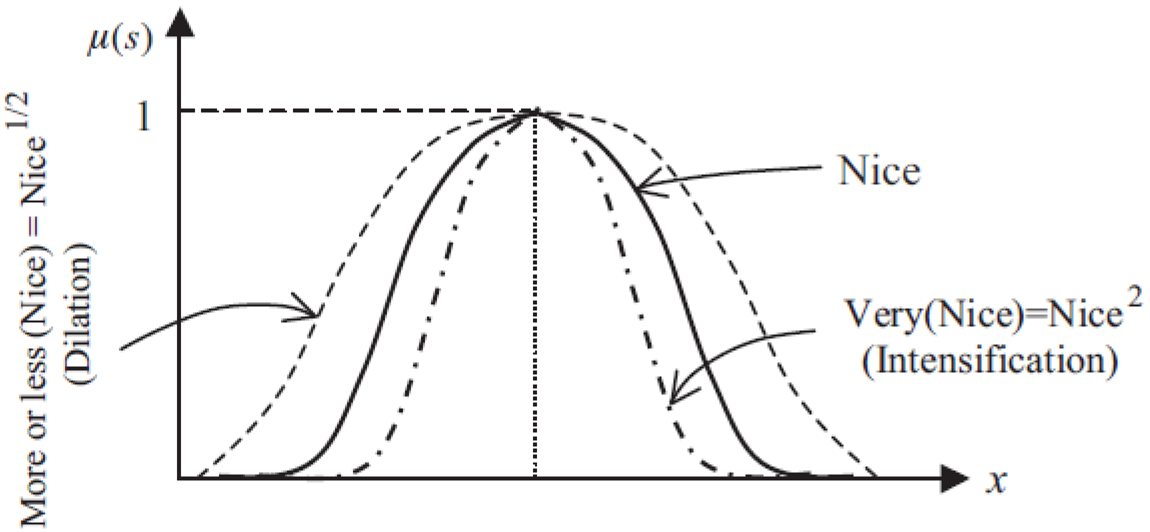

Figure 2.21 Dilation and concentration (or intensification)

Linguistic Hedges

Table 2.1 Hedges and their meaning

Hedge	Meaning
About, around, near, roughly	Approximates a scalar
Above, more than	Restricts a fuzzy region
Almost, definitely, precisely	Contrasts intensification
Below, less than	Restricts a fuzzy region
Generally, usually	Contrasts diffusion
Neighbouring, close to	Approximates narrowly
Not	Negation or complement
Quite, rather, somewhat	Dilutes a fuzzy region
Very, extremely	Intensifies a fuzzy region

Linguistic Hedges

A linguistic variable can be used with more than one hedge, for example

Almost very fast but generally below 100 km/hr.

Close to 100 m but not very high.

Not more than about zero.

'But' is a connective here which is equivalent to AND. The equivalent versions of the above linguistic variables with markers can be expressed as

Almost (Very (Fast)) AND Generally (below 100 km/hr).

Close (100 m) AND Not (Very (High)).

Not (More than (About zero)).

Linguistic Hedges

The operation of multiple hedges can result in the same primary fuzzy set. For example, the operation of the hedges ‘More or less very nice’ is represented by the following expression. It can be seen that the operation of the hedges on the primary term (fuzzy) set ‘Nice’ resulted in the same primary fuzzy set:

$$\text{More or less (Very (Nice))} = \text{More or less } (\text{Nice}^2) = (\text{Nice}^2)^{1/2} = \text{Nice}$$

Some widely used hedges and their meanings are given in Table 2.1.

Linguistic variables and hedges allow us to construct mathematical models for expressions of natural language. These models can then be used to write process rules and computer programs and simulate real-world processes and behaviour.

Fuzzy Relations

The concept of a relation has a natural extension to fuzzy sets and plays an important role in the theory of such sets and their applications. A fuzzy relation R from the fuzzy set A in X to the fuzzy set B in Y is a fuzzy set defined by the Cartesian product AxB in the Cartesian product space XxY. R is characterized by the membership function expressing various degrees of strength of relations:

$$R = A \times B = \sum \mu_R(x, y)/(x, y) = \sum \min\left(\mu_A(x), \mu_B(y)\right) \tag{2.20}$$

$$R = A \times B = \sum \mu_R(x, y)/(x, y) = \sum \mu_A(x)^* \mu_B(y) \tag{2.21}$$

In Equations (2.20) and (2.21) the sum does not mean a mathematical summation operation, it means all possible combinations of all elements.

R is also called the relational matrix. The Cartesian product is implemented in the same fashion, as is the cross product of two vectors. For example, fuzzy set A with 4 elements (a column vector of dimension 4×1) and fuzzy set B with 5 elements (a row vector of dimension 1×5) will provide the resulting fuzzy relation R which is represented by a matrix of dimension 4×5.

Fuzzy Relations

Example 2.10 Let A and B be two fuzzy sets defined by

$$A = \{1/1 + 0.8/2 + 0.6/3 + 0.5/4\}$$
$$B = \{0.5/1 + 1/2 + 0.3/3 + 0/4\}$$

The fuzzy relation (i.e., the Cartesian product of A and B using the min operation) will be

$$\mathbf{R} = \mathbf{A} \times \mathbf{B} = \begin{bmatrix} \{1, .5\} & \{1, 1\} & \{1, .3\} & \{1, 0\} \\ \{.8, .5\} & \{.8, 1\} & \{.8, .3\} & \{.8, 0\} \\ \{.6, .5\} & \{.6, 1\} & \{.6, .3\} & \{.6, 0\} \\ \{.5, .5\} & \{.5, 1\} & \{.5, .3\} & \{.5, 0\} \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0.3 & 0 \\ 0.5 & 0.8 & 0.3 & 0 \\ 0.5 & 0.6 & 0.3 & 0 \\ 0.5 & 0.5 & 0.3 & 0 \end{bmatrix}$$

The fuzzy relation using the product operation will be

$$\mathbf{R} = \mathbf{A} \times \mathbf{B} = \begin{bmatrix} \{1, .5\} & \{1, 1\} & \{1, .3\} & \{1, 0\} \\ \{.8, .5\} & \{.8, 1\} & \{.8, .3\} & \{.8, 0\} \\ \{.6, .5\} & \{.6, 1\} & \{.6, .3\} & \{.6, 0\} \\ \{.5, .5\} & \{.5, 1\} & \{.5, .3\} & \{.5, 0\} \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0.3 & 0 \\ 0.4 & 0.8 & 0.24 & 0 \\ 0.3 & 0.6 & 0.18 & 0 \\ 0.25 & 0.5 & 0.15 & 0 \end{bmatrix}$$

Compositional Rule of Inference

If R is a fuzzy relation in $X \times Y$ and A is a fuzzy set in X then the fuzzy set B in Y is given by

$$B = A \circ R \tag{2.22}$$

B is inferred from A using the relation matrix R which defines the mapping between X and Y and the operation '$\circ$' is defined as the max/min operation.

Compositional Rule of Inference

Example 2.11 Let A be a fuzzy set defined by

$$A = \{0.9/1 + 0.4/2 + 0/3\}$$

with the fuzzy relation R given by the following relational matrix:

$$\mathbf{R} = \mathbf{A} \times \mathbf{B} = \begin{bmatrix} 1 & 0.8 & 0.1 \\ 0.8 & 0.6 & 0.3 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

Then the fuzzy output B in Y using the max/min operation will be

$$\mathbf{B} = \mathbf{A} \circ \mathbf{R} = \begin{bmatrix} \frac{0.9}{1} & \frac{0.4}{2} & \frac{0}{3} \end{bmatrix} \circ \begin{bmatrix} 1 & 0.8 & 0.1 \\ 0.8 & 0.6 & 0.3 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

Compositional Rule of Inference

$$\mathbf{B} = \begin{bmatrix} \{0.9, 1\} & \{0.9, 0.8\} & \{0.9, 0.1\} \\ \{0.4, 0.8\} & \{0.4, 0.6\} & \{0.4, 0.3\} \\ \{0, 0.6\} & \{0, 0.3\} & \{0, 0.1\} \end{bmatrix}$$

Taking the minimum values row-wise, we obtain

$$\mathbf{B} = \begin{bmatrix} 0.9 & 0.8 & 0.1 \\ 0.4 & 0.4 & 0.3 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking the maximum values column-wise, we obtain the fuzzy set B from the compositional relation:

$$\mathbf{B} = \begin{bmatrix} 0.9 & 0.8 & 0.3 \end{bmatrix}$$

Fuzzy If–Then Rules

Fuzzy sets and their operations are the subjects and verbs of fuzzy logic. If–Then rule statements are used to formulate the conditional statements that comprise fuzzy logic. A single fuzzy If–Then rule assumes the form

$$\text{If } <\text{fuzzy proposition}> \text{ Then } <\text{fuzzy proposition}> \tag{2.23}$$

For example,

$$\text{If } <x \text{ is } A_1> \text{ Then } <y \text{ is } B_2>$$

where A_1 and B_2 are linguistic variables defined by fuzzy sets on the ranges (i.e., the universe of discourse) X and Y, respectively. The If part of the rule 'x is A_1' is called the antecedent or premise and the Then part of the rule 'y is B_2' is called the consequent. In other words, the conditional statement can be expressed in mathematical form:

$$\text{If } A_1 \text{ Then } B_2 \text{ or } A_1 \rightarrow B_2 \tag{2.24}$$

Fuzzy If–Then Rules

Example 2.12 The speed and pressure of a steam engine can be expressed with the following linguistic conditional statement:

If *Speed* is *Slow* Then *Pressure* should be *High*

Graphically, this statement is represented in Figure 2.22.

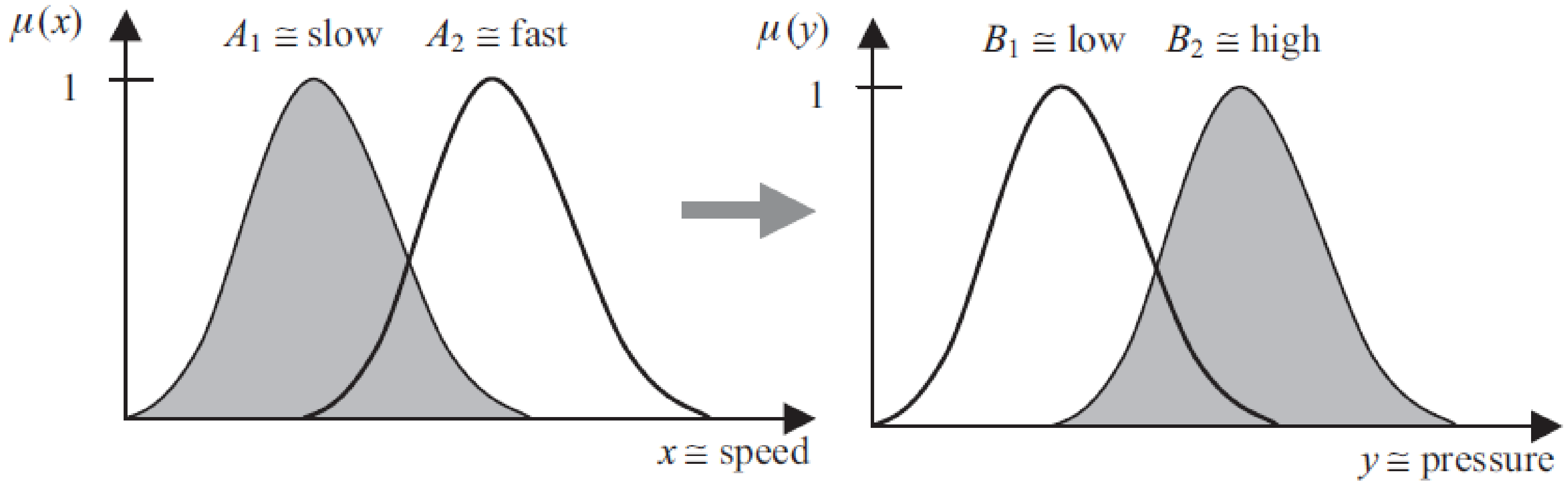

Figure 2.22 If–Then rule

Rule Forms

In general, three forms exist for any linguistic variables:

(i) Assignment statement
e.g., x is *not large* AND *not very small*.

(ii) Conditional statement
e.g., IF x is *very big* THEN y is *medium*.

(iii) Unconditional statement
e.g., set pressure *high*.

Compound Rules

A linguistic statement expressed by a human might involve compound rule structures. By using basic properties and operations defined for fuzzy sets, any compound rule structure may be decomposed and reduced to a number of simple canonical rules.

Conjunctive antecedents: A multiple conjunctive antecedent can have the following form:

$$\text{IF } x \text{ is } A_1 \text{ AND } x \text{ is } A_2 \ldots \text{ AND } x \text{ is } A_n \text{ THEN } y \text{ is } B_S \tag{2.25}$$

Equation (2.25) can be rewritten as

$$\text{IF } x \text{ is } A_S \text{ THEN } y \text{ is } B_S \tag{2.26}$$

where $A_S = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n$ and A_S is expressed by means of a membership function based on the definition of fuzzy intersection operation as

$$\mu_{A_S}(x) = \min\left[\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)\right] \tag{2.27}$$

Compound Rules

Disjunctive antecedents: Similarly, a multiple disjunctive antecedent can have the following form:

$$\text{IF } x \text{ is } A_1 \text{ OR } x \text{ is } A_2 \ \ldots \text{ OR } x \text{ is } A_n \text{ THEN } y \text{ is } B_S \tag{2.28}$$

Equation (2.28) can be rewritten as

$$\text{IF } x \text{ is } A_S \text{ THEN } y \text{ is } B_S \tag{2.29}$$

where $A_S = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n$ and A_S is expressed by means of a membership function based on the definition of fuzzy union operation as

$$\mu_{A_S}(x) = \max\left[\mu_{A_1}(x), \mu_{A_2}(x), \ldots, \mu_{A_n}(x)\right] \tag{2.30}$$

Aggregation of Rules

Most rule-based systems have more than one rule. The process of obtaining the overall consequent from the individual consequents contributed by each rule is the aggregation of rules. In the case of a system of rules that must be jointly satisfied, the rules are connected by AND connectives. The aggregated output y is found by fuzzy intersection of the entire individual rule consequent y_i, where $i = 1, 2, 3, \ldots, r$:

$$y = y_1 \; AND \; y_2 \; AND \; \cdots AND \; y_r \tag{2.31}$$

or

$$y = y_1 \cap y_2 \cap \cdots \cap y_r$$

The output is defined by means of a membership function based on the definition of fuzzy intersection operation as

$$\mu_y(y) = \min\left[\mu_{y_1}(y), \mu_{y_2}(y), \ldots, \mu_{y_r}(y)\right] \text{ for } y \in Y \tag{2.32}$$

Aggregation of Rules

For the case of a disjunctive system of rules where at least one rule must be satisfied, the rules are connected by OR connectives. The aggregated output y is found by fuzzy union of all the individual rule consequents y_i, where $i = 1, 2, 3, \ldots, r$:

$$y = y_1 \; OR \; y_2 \; OR \; \cdots OR \; y_r \tag{2.33}$$

or

$$y = y_1 \cup y_2 \cup \cdots \cup y_r$$

The output is defined by means of a membership function based on the definition of fuzzy union operation as

$$\mu_y(y) = \max\left[\mu_{y_1}(y), \mu_{y_2}(y), \ldots, \mu_{y_r}(y)\right] \text{ for } y \in Y \tag{2.34}$$

Aggregation of Rules

Example 2.13 Let us consider a fuzzy system with two inputs x_1 and x_2 (antecedents) and a single output y (consequent). Inputs x_1 and x_2 have three linguistic variables small, medium and big with a triangular membership function. Output y has two linguistic variables small and big with a triangular membership function as shown in Figure 2.23. The rule base consists of the following two rules:

Rule 1: IF x_1 is *small* AND x_2 is *medium* THEN y is *big*

Rule 2: IF x_1 is *medium* AND x_2 is *big* THEN y is *small*

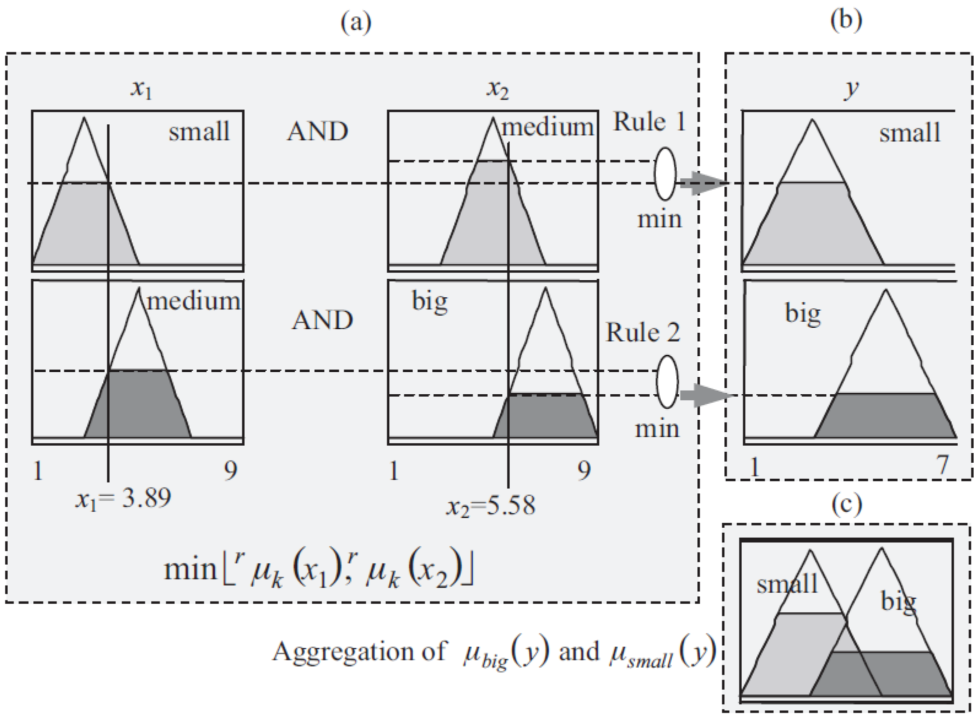

Figure 2.23 Max/min inference method

Aggregation of Rules

The inputs $x_1 = 3.89$ and $x_2 = 5.58$ are crisp values for which the membership values $\mu_k(x_1)$ and $\mu_k(x_2)$ (k denotes the MFs small, medium or big) are calculated for triangular membership functions. The aggregated outputs for r rules are given by

$$r1 : \mu_{big}(y) = \max \lfloor \min \lfloor \mu_{small}(x_1), \mu_{medium}(x_2) \rfloor \rfloor$$
$$r2 : \mu_{small}(y) = \max \left\lfloor \min \left\lfloor \mu_{medium}(x_1), \mu_{big}(x_2) \right\rfloor \right\rfloor$$

In this example, $r = 1, 2$. The minimum membership values of $\lfloor \mu_{small}(x_1), \mu_{medium}(x_2) \rfloor$ and $\left\lfloor \mu_{medium}(x_1), \mu_{big}(x_2) \right\rfloor$ for the antecedents are calculated and propagate through to the consequent part. This operation is shown in Figure 2.23(a). The membership function for the consequent of each rule is then truncated by taking the maximum values, i.e. $\max \lfloor \min \lfloor \mu_{small}(x_1), \mu_{medium}(x_2) \rfloor \rfloor$ and $\max \left\lfloor \min \left\lfloor \mu_{medium}(x_1), \mu_{big}(x_2) \right\rfloor \right\rfloor$ are computed, which is shown in Figure 2.23(b). The truncated membership functions for each rule, i.e. $\mu_{big}(y)$ and $\mu_{small}(y)$ are aggregated using the graphical equivalent of either conjunctive or disjunctive rules. The aggregation operation max results in an aggregated membership function comprising the outer envelope of the individual truncated membership forms from each rule. This operation is shown in Figure 2.23(c).

It has been demonstrated in Figure 2.23 that any numeric value (or crisp value) has to be converted into a fuzzy input and then a conclusion can be drawn using the rule of inference on consequent fuzzy sets. There are three distinct steps in the process. They are described in the following sections.

Aggregation of Rules

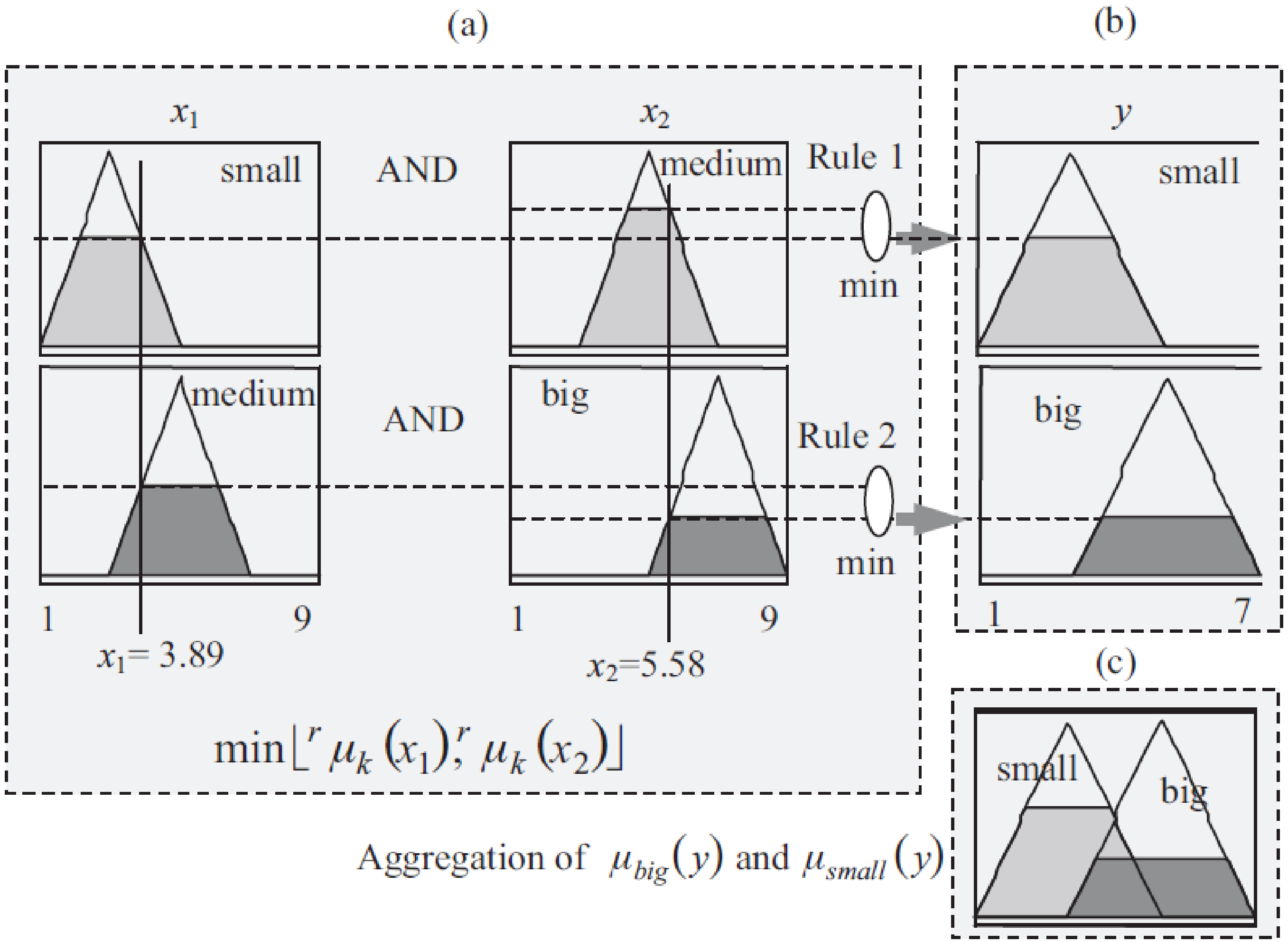

Figure 2.23 Max/min inference method

Fuzzification

The process that allows converting a numeric value (or crisp value) into a fuzzy input is called fuzzification. There are two methods of fuzzification.

- Singleton fuzzification: This maps a real value $x_i \in X$ into a fuzzy singleton A_{x_i} which has membership value 1 at $x = x_i$ and 0 at all other points in X. This is expressed as

$$\mu_{A_{x_i}}(x) = \begin{cases} 1 & \text{if} \quad x = x_i \\ 0 & \text{otherwise} \end{cases} \tag{2.35}$$

  Singleton fuzzification greatly simplifies computation but is generally used in implementations where there is no noise. There is no widespread use of singleton fuzzification in fuzzy systems and applications.

Fuzzification

- A_{x_i} is fuzzy: This maps a real $x_i \in X$ into a fuzzy set A_{x_i} in X described by a membership function:

$$\mu_{A_{x_i}}(x) = \begin{cases} 1 & \text{if} \quad x = x_i \\ [0, 1] & \text{decreases from 1 as } x \text{ moves from } x_i \end{cases} \tag{2.36}$$

In other words, fuzzification actually provides a membership grade of a real (or crisp) value $x_i \in X$ as its belongingness to a fuzzy set A_{x_i}. The fuzzy set can be described by various

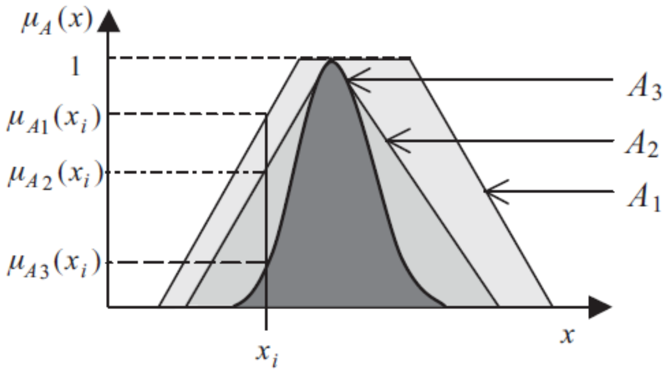

Figure 2.24 Fuzzification in different types of MFs

membership functions discussed in Section 2.4. Figure 2.24 shows the fuzzification of $x_i \in X$ using three different types of membership functions: trapezoidal (A_1), triangular (A_2) and Gaussian (A_3). It demonstrates that $x = x_i \in X$ has a different fuzzified value, i.e. membership grade, depending on the type (i.e., shape) of the membership function. The membership grades are $\mu_{A_1}(x_i)$ for trapezoidal MF, $\mu_{A_2}(x_i)$ for triangular MF and $\mu_{A_3}(x_i)$ for Gaussian MF.

Fuzzification

Example 2.14 Let A be a fuzzy set defined by the bell-shaped MF (with centre $m = 5$, width $\sigma = 1$ and shape parameter $a = 1$ in Equation (2.6)) as follows:

$$\mu_A(x) = \frac{1}{1 + \left|\frac{x-5}{1}\right|^2}$$

The fuzzification of the value $x_i = 6$ will yield the grade of membership as

$$\mu_A(x_i) = \frac{1}{1 + \left|\frac{6-5}{1}\right|^2} = 0.5$$

It is obvious that the shape of the MF plays an important role in fuzzification and in any subsequent process. Fuzzification using well-defined MFs can suppress noise in the inputs of a fuzzy system (Wang, 1997).

Defuzzification

Defuzzification is the reverse process of fuzzification. Mathematically, the defuzzification of a fuzzy set is the process of conversion of a fuzzy quantity into a crisp value. This is necessary when a crisp value is to be provided from a fuzzy system to the user. For example, if we develop a fuzzy system for blood pressure control, we will probably want to tell the user what blood pressure is expected to be in the next time instant.

Fuzzy control engineers have many different ways of defuzzifying. However, there are quite simple methods in use. It is intuitive that fuzzification and defuzzification should be reversible.

Defuzzification

That is, if we fuzzify a number into a fuzzy set and immediately defuzzify it, we should be able to get the same number back again.

There are many defuzzification methods available in the literature. Very often standard defuzzification methods fail in some applications. It is, therefore, important to select the appropriate defuzzification method for a particular application. Unfortunately, there is no standard rule for selecting a particular defuzzification method for an application. The choice of the most appropriate method depends on the application. A good study on the selection of appropriate defuzzification methods has been reported by Runker (1997). In the next few sections, some widely used methods of defuzzification are presented.

Max-membership method: Also known as the height method, the max-membership method is both simple and quick. This method takes the peak value of each fuzzy set and builds the weighted sum of these peak values. This method is given by the algebraic expression in Equation (2.38).

$$x^* = \frac{\sum_{k=1}^{m} c_k . h_k}{\sum_{k=1}^{n} h_k} \tag{2.37}$$

Defuzzification using the max-membership method is shown in Figure 2.25(a). c_k is the peak value of the fuzzy sets and h_k is the height of the clipped fuzzy sets, as shown in the figure.