Calculus: Early Transcendental Functions
Lecture Notes for Calculus 101

Feras Awad Mahmoud

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Feras Awad Mahmoud
Department of Basic Sciences
Philadelphia University
JORDAN 19392

TEXTBOOK:
This book is strongly recommended for Calculus 101 as well as a reference text for Calculus 102. The pdf soft−copy of the five chapters remain available for free download.

TYPESETTING:
The entire document was written in LaTeX, implemented for Windows using the MiKTeX 2.9 distribution. As for the text editor of my choice, I fancy WinEdt 6.0.

MORAL SUPPORT:
My wife has had a very instrumental role in providing moral support. Thank God for her patience, understanding, encouragement, and prayers throughout the long process of writing and editing.

Contents

Chapter 1

Functions

1.1 Introduction

Functions arise whenever one quantity depends on another.

Definition 1.1.1. A **function** f is a rule that assigns to each element x in a set D exactly one element called $f(x)$ in a set E.

- We usually consider functions for which the sets D and E are sets of real numbers.
- The set D is called the **domain** of the function.
- The number $f(x)$ is the value of f at x and is read f of x.
- The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain
- A symbol that represents an arbitrary number in the domain of a function f is called an **independent variable**.
- A symbol that represents a number in the range of f is called a **dependent variable**.

Since the $y-$coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 1.1).

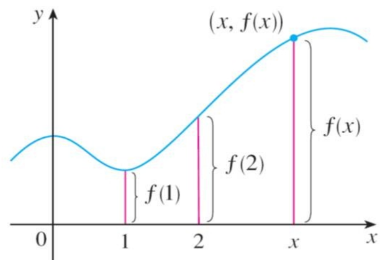


Figure 1.1:

The graph of f also allows us to picture the domain of f on the $x-$axis and its range on the $y-$axis as in Figure 1.2.

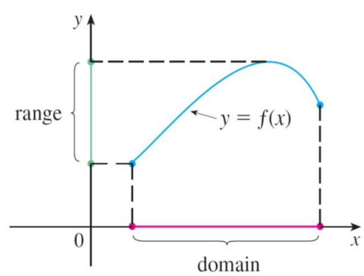


Figure 1.2:

Example 1.1. The graph of a function f is shown in Figure 1.3.

(a) Find the values of $f(1)$ and $f(7)$.

(b) What are the domain and range of f ?

Solution 1.1. a) We see from Figure 1.3 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the $x-$axis.) When $x = 7$, the graph lies on the $x-$axis, so we say that $f(7) = 0$. (In other words, $x = 7$ is a real **root** of $f(x)$.)

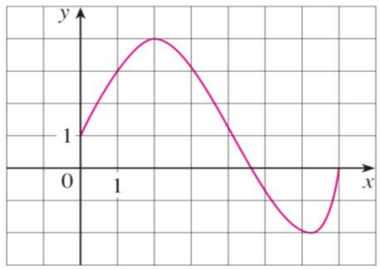


Figure 1.3:

b) We see that $f(x)$ is defined when $0 \leq x \leq 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is the closed interval $[-2, 4]$.

□

Representations of Functions There are four possible ways to represent a function:

1. verbally (by a description in words)
2. numerically (by a table of values)
3. visually (by a graph)
4. algebraically (by an explicit formula)

The Vertical Line Test The graph of a function is a curve in the $xy-$plane. But the question arises: Which curves in the $xy-$plane are graphs of functions? This is answered by the Vertical Line Test: **A curve in the $xy-$plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.** The reason for the truth of the Vertical Line Test can be seen in Figure 1.4. If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve cant represent a function because a function cant assign two different values to a.

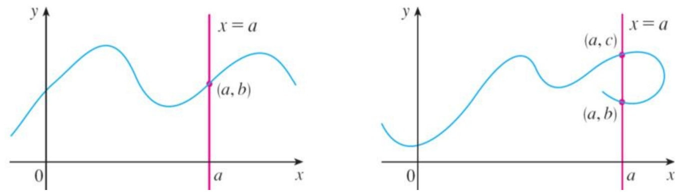


Figure 1.4:

Symmetry If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because $f(-x) = (-x)^2 = x^2 = f(x)$. If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because $f(-x) = (-x)^3 = -x^3 = f(x)$.

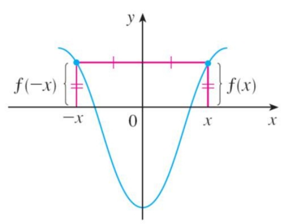


Figure 1.5: Even Function

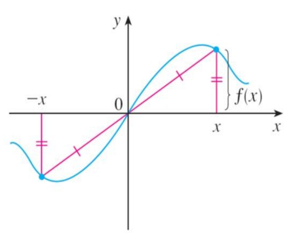


Figure 1.6: Odd Function

The geometric significance of an even function is that its graph is symmetric with respect to the $y-$axis as in Figure 1.5, while the graph of an odd function is symmetric about the origin, see Figure 1.6.

Example 1.2. Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$.

(b) $g(x) = 1 - x^4$.

(c) $h(x) = 2x - x^2$.

Solution 1.2. (a) $f(-x) = (-x)^5 + (-x) = (-1)^5 x^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$. Therefore f is an odd function.

(b) $g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$. So g is even.

(c) $h(-x) = 2(-x) - (-x)^2 = -2x - x^2$. Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd.

□

1.2 Essential Functions

There are many different types of functions that can be used to model relationships observed in the real world. In what follows, we discuss the behavior and graphs of these functions and give examples of situations appropriately modeled by such functions.

Linear Function

When we say that y is a linear function of x, we mean that the graph of the function is a line, so we can use the slope$-$intercept form of the equation of a line (see Appendix C) to write a formula for the function as $y = f(x) = mx + b$ where m is the slope of the line and b is the $y-$intercept.

Example 1.3. As dry air moves upward, it expands and cools. If the ground temperature is 20 °C and the temperature at a height of 1 km is 10 °C.

(a) Express the temperature T (in °C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(b) Draw the graph of the function in part (a). What does the slope represent?

(c) What is the temperature at a height of 2.5 km?

Solution 1.3. (a) Because we are assuming that T is a linear function of h, we can write $T = mh + b$. We are given that $T = 20$ when $h = 0$, so $20 = m \times 0 + b = b$. In other words, the $y-$intercept is $b = 20$. We are also given that $T = 10$ when $h = 1$, so $10 = m \times 1 + 20$. The slope of the line is therefore $m = 10 - 20 = -10$ and the required linear function is $T = -10h + 20$.

(b) The graph is sketched in Figure 1.7. The slope is $m = -10\,°\mathrm{C/km}$, and this represents the rate of change of temperature with respect to height.

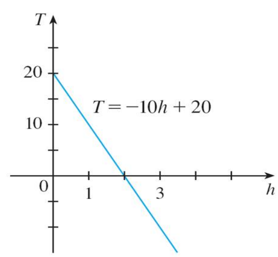


Figure 1.7:

(c) At a height of $h = 2.5$ km, the temperature is $T = -10(2.5) + 20 = -5\,°\mathrm{C}$.

□

Polynomials

A function P is called a polynomial if $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$ where n is a nonnegative integer and the numbers $a_0, a_1, a_2, ..., a_n$

are constants called the coefficients of the polynomial.

The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n. For example,

- A polynomial of degree 1 is of the form $P(x) = mx + b$ and so it is a linear function.

- A polynomial of degree 2 is of the form $P(x) = ax^2 + bx + c$ and is called a quadratic function.

- A polynomial of degree 3 is of the form $P(x) = ax^3 + bx^2 + cx + d$ and is called a cubic function.

Remark 1.2.1. A polynomial of degree n has at most n zeros (roots).

Piecewise Defined Functions

Example 1.4. A function f is defined by

$$f(x) = \begin{cases} 1 - x & : \quad x \leq -1 \\ x^2 & : \quad x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

Solution 1.4. Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leq -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2. Since $-2 \leq -1$, we have $f(-2) = 1 - (-2) = 3$. Since $-1 \leq -1$, we have $f(-1) = 1 - (-1) = 2$. Since $0 > -1$, we have $f(0) = 0^2 = 0$. The graph of this functions appears in Figure 1.8.

□

The absolute value (see Appendix B) is an example of a piecewise defined function.

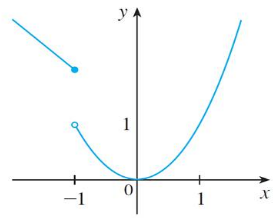


Figure 1.8:

Rational Functions

Definition 1.2.1. A function in the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials, is called a rational function. The domain of the rational function $f(x)$ is the set

$$D = \mathbb{R} - \{x \in \mathbb{R} : Q(x) = 0\}$$

Example 1.5. Find the domain of $f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$.

Solution 1.5.

$$\begin{aligned} D &= \mathbb{R} - \left\{x \in \mathbb{R} : x^2 - 4 = 0\right\} \\ &= \mathbb{R} - \{-2, 2\} \end{aligned}$$

The graph of the function is shown in Figure 1.9.

□

Example 1.6. Find the domain of $f(x) = \frac{x^2 - 9}{1 - |x|}$.

Solution 1.6.

$$\begin{aligned} D &= \mathbb{R} - \{x \in \mathbb{R} : 1 - |x| = 0\} \\ &= \mathbb{R} - \{-1, 1\} \end{aligned}$$

□

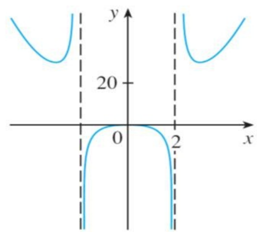


Figure 1.9:

Exercise 1.1. What is the domain of each of the following functions.

(a) $f(x) = \frac{1}{|1-x|}$

(b) $g(x) = \frac{-4x}{10+x^2}$

Root Function

Definition 1.2.2. For any integer $n \geq 2$,

$$f(x) = \sqrt[n]{g(x)}$$

is the n^{th} root function of $g(x)$. The domain of the root function depends on the value of n if it is even or odd.

n **is odd:** The domain of $f(x)$ in this case is the same as the domain of $g(x)$. The range of $f(x)$ will be $\mathbb{R}$.

n **is even:** In this case, the domain of $f(x)$ is the set

$$D = \{x \in \mathbb{R} : g(x) \geq 0\} \cap \{g(x) \text{ domain}\}$$

The range of f is $[0, \infty)$.

Example 1.7. Find the domain of $f(x) = \sqrt[3]{x^2 - 4}$.

Solution 1.7. Since f is odd root function, then

$$D = \text{ domain } (x^2 - 4) = \mathbb{R}$$

Example 1.8. Find the domain of $f(x) = \sqrt{x^2 - 4}$.

Solution 1.8. Since f is even root function, then

$$\begin{aligned} D &= \left\{x \in \mathbb{R} : x^2 - 4 \geq 0\right\} \cap \{ \text{ domain of } x^2 - 4\} \\ &= (-\infty, -2] \cup [2, \infty) \cap \mathbb{R} \\ &= (-\infty, -2] \cup [2, \infty) \end{aligned}$$

□

Example 1.9. Find the domain of $f(x) = \sqrt[6]{|x|}$.

Solution 1.9. Since f is even root function, then

$$\begin{aligned} D &= \{x \in \mathbb{R} : |x| \geq 0\} \cap \{ \text{ domain of } |x|\} \\ &= \mathbb{R} \cap \mathbb{R} \\ &= \mathbb{R} \end{aligned}$$

□

Example 1.10. Find the domain of $f(x) = \frac{1}{\sqrt{9-x^2}}$.

Solution 1.10. This function is rational and its **denominator is even root**. The root's domain is the dominant here. The domain of f is

$$\begin{aligned} D &= \left\{ \text{ domain of } \sqrt{9 - x^2}\right\} - \left\{x \in \mathbb{R} : \sqrt{9 - x^2} = 0\right\} \\ &= \left\{x \in \mathbb{R} : 9 - x^2 > 0\right\} \cap \{ \text{ domain of } 9 - x^2\} \\ &= \left\{x \in \mathbb{R} : x^2 < 9\right\} \cap \mathbb{R} \\ &= \{x \in \mathbb{R} : |x| < 3\} \cap \mathbb{R} \\ &= (-3, 3) \cap \mathbb{R} \\ &= (-3, 3) \end{aligned}$$

□

Example 1.11. Find the domain of $f(x) = \frac{1}{1-\sqrt{x}}$.

Solution 1.11. This example is quite different from the previous example. The function f here is rational but its denominator **contains** an even root. The root's domain is also the dominant here. So, the domain of f is

$$\begin{aligned} D &= \left\{ \text{ domain of } \sqrt{x} \right\} - \left\{ x \in \mathbb{R} : 1 - \sqrt{x} = 0 \right\} \\ &= \{x \in \mathbb{R} : x \geq 0\} \cap \{ \text{ domain of } x\} - \{1\} \\ &= [0, \infty) \cap \mathbb{R} - \{1\} \\ &= [0, \infty) - \{1\} \\ &= [0, 1) \cup (1, \infty) \end{aligned}$$

□

Example 1.12. Find the domain of $f(x) = \sqrt{2 - \sqrt{x}}$.

Solution 1.12. Since f is even root function that contains an even root function inside it, then both roots are dominant here. Hence, the domain of f is

$$\begin{aligned} D &= \left\{ x \in \mathbb{R} : 2 - \sqrt{x} \geq 0 \right\} \cap \{ \text{ domain of } 2 - \sqrt{x}\} \\ &= \left\{ x \in \mathbb{R} : \sqrt{x} \leq 2 \right\} \cap \{x \in \mathbb{R} : x \geq 0\} \cap \{ \text{ domain of } x\} \\ &= \{x \in \mathbb{R} : 0 \leq x \leq 4\} \cap [0, \infty) \cap \mathbb{R} \\ &= [0, 4] \cap [0, \infty) \cap \mathbb{R} \\ &= [0, 4] \end{aligned}$$

□

Exercise 1.2. Find the domain of the following.

(a) $f(x) = \frac{1}{1+\sqrt{x}}$

(b) $g(x) = \sqrt{-x}$

(c) $h(x) = \sqrt{\frac{1}{x} - 1}$

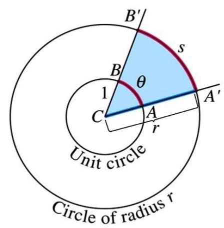


Figure 1.10:

Trigonometric Functions

In calculus the convention is that radian measure is always used (except when otherwise indicated). Figure 1.10 shows a sector of a circle with central angle θ and radius r subtending an arc with length s. Then the radian measure of the central angle $A'CB'$ is the number $\theta = \frac{s}{r}$.

Example 1.13. Find the radian measure of $60°$.

Solution 1.13. To convert degrees to radians, multiply degrees by $(\pi \text{ rad })/180°$.

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3}.$$

□

Example 1.14. Express $5\pi/4$ in degrees.

Solution 1.14. To convert radians to degrees, multiply radians by $180°/(\pi \text{ rad })$.

$$\frac{5\pi}{4} \text{ rad } = \left(\frac{5\pi}{4}\right)\left(\frac{180°}{\pi}\right) = 225°.$$

□

Nonzero radians measures can be positive or negative and can go beyond $2\pi = 360°$. The standard position of an angle occurs when we place its vertex at the origin of a coordinate system and its initial side on the positive $x-$axis as in Figure 1.11. A positive angle is obtained by rotating the initial side counterclockwise until it coincides with the terminal side. Likewise, negative angles are obtained by clockwise rotation.

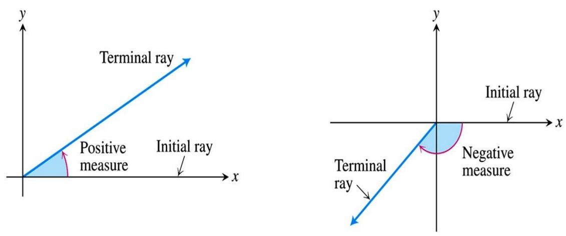


Figure 1.11:

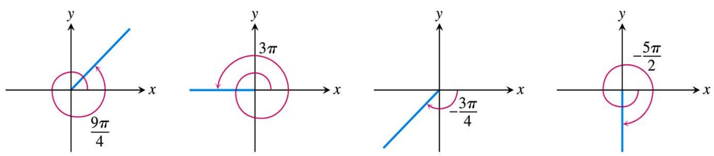


Figure 1.12:

Figure 1.12 shows several examples of angles in standard position.

For an acute angle θ the six trigonometric functions are defined as ratios of lengths of sides of a right triangle as in Figure 1.13.

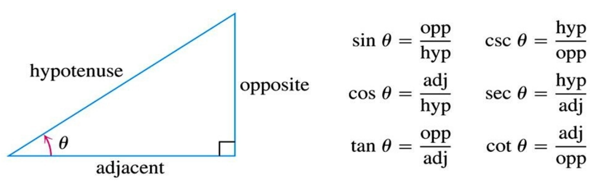


Figure 1.13:

This definition does not apply to obtuse or negative angles, so for a general angle θ in standard position we let $P(x, y)$ be any point on the terminal side of θ and we let r be the distance $|OP|$ as in Figure 1.14.

The signs of the trigonometric functions for angles in each of the four quadrants can be remembered by means of the rule **ALL STUDENTS TAKE CALCULUS** shown in Figure 1.15.

Example 1.15. Find the exact trigonometric ratios for $\theta = 2\pi/3$.

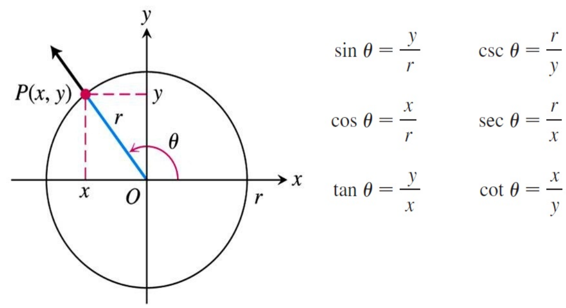


Figure 1.14:

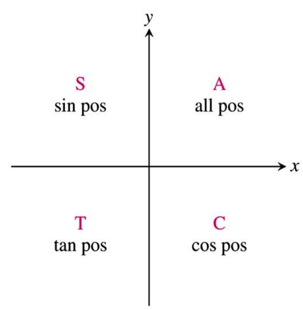


Figure 1.15:

Solution 1.15. From Figure 1.16 we see that a point on the terminal line for $\theta = 2\pi/3$ is $P\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$. Therefore, taking $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ in the definitions of the trigonometric ratios, we have

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \tan\frac{2\pi}{3} = -\sqrt{3}$$
$$\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}} \qquad \sec\frac{2\pi}{3} = -2 \qquad \cot\frac{2\pi}{3} = \frac{-1}{\sqrt{3}}$$

□

The following table gives some values of $\sin\theta$, $\cos\theta$ and $\tan\theta$.

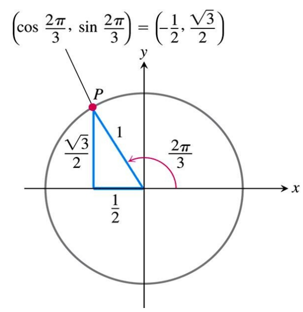
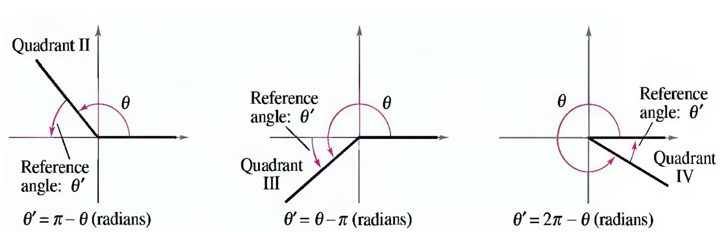


Figure 1.16:

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Reference Angles The values of trigonometric functions of angles greater than $90° = \pi/2$ or less than $0°$ can be determined from their values at corresponding acute angles called **reference angles**.

Definition 1.2.3. Let θ be an angle in standard position. Its reference angle is the acute angle θ' formed by the terminal side of θ and the $x-$axis.



Example 1.16. Find the reference angle of $\theta = 5\pi/3$.

Solution 1.16. Because $5\pi/3 = 300°$ lies in quadrant 4, the angle it makes with the $x-$axis is

$$\theta' = 2\pi - 5\pi/3 = \pi/3 = 60°.$$

□

Example 1.17. Find the reference angle of $\theta = -3\pi/4$.

Solution 1.17. First, note that $-3\pi/4 = 135°$ is coterminal with $5\pi/4 = 225°$ which lies in quadrant 3. So, the reference angle is

$$\theta' = 5\pi/4 - \pi = \pi/4 = 45°.$$

□

Example 1.18. Evaluate each of the following.

a) $\cos(4\pi/3)$
b) $\tan(-210°)$
c) $\csc(11\pi/4)$

Solution 1.18. a) Because $\theta = 4\pi/3 = 240°$ lies in quadrant 3, the reference angle is $\theta' = 4\pi/3 - \pi = \pi/3$. Moreover, the cosine is negative in quadrant 3, so

$$\cos(4\pi/3) = (-)\cos(\pi/3) = -1/2$$

b) Because $-210° + 360° = 150°$, it follows that $-210°$ is coterminal with the second-quadrant angle $150°$. Therefore, the reference angle is $\theta' = 180° - 150° = 30°$. Finally, because the tangent is negative in quadrant 2, you have

$$\tan(-210°) = (-)\tan(30°) = -\sqrt{3}/3 = -1/\sqrt{3}$$

c) Because $11\pi/4 - 2\pi = 3\pi/4$, it follows that $11\pi/4$ is coterminal with the second-quadrant angle $3\pi/4$. Therefore, the reference angle is $\theta' = \pi - 3\pi/4 = \pi/4$. Because the cosecant is positive in quadrant 2, you have

$$\csc(11\pi/4) = (+)\csc(\pi/4) = 1/\sin(\pi/4) = \sqrt{2}$$

Trigonometric Identities A trigonometric identity is a relationship among the trigonometric functions. The most identities are the following.

Part 1 The following identities are immediate consequences of the definitions of the trigonometric functions.

$$\csc\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Part 2 The following are the most useful of all trigonometric identities:

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \\ \tan^2\theta + 1 &= \sec^2\theta \\ 1 + \cot^2\theta &= \csc^2\theta \end{aligned}$$

Part 3 The following identities show that $\sin\theta$ is an **odd** function and $\cos\theta$ is an **even** function.

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta$$

Part 4 The next identities show that the sine and cosine functions are **periodic** with period 2π Since the angles θ and $\theta + 2\pi$ have the same terminal side.

$$\sin(\theta + 2\pi) = \sin\theta, \quad \cos(\theta + 2\pi) = \cos\theta$$

Part 5 The **addition** and **subtracting** formulas are the following identities.

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \cos x\sin y \\ \sin(x-y) &= \sin x\cos y - \cos x\sin y \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y \\ \cos(x-y) &= \cos x\cos y - \sin x\sin y \\ \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x\tan y} \\ \tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x\tan y} \end{aligned}$$

Part 6 The **double-angle** formulas are:

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x\\ \cos(2x) &= \cos^2 x - \sin^2 x\\ \cos(2x) &= 2\cos^2 x - 1\\ \cos(2x) &= 1 - 2\sin^2 x\end{aligned}$$

Part 7 The following are the **half-angle** formulas, which are useful in integral calculus:

$$\cos^2 x = \frac{1+\cos(2x)}{2}, \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$

Part 8 Finally, we state the **product** formulas, which are:

$$\begin{aligned}\sin x\cos y &= \frac{1}{2}[\sin(x+y)+\sin(x-y)]\\ \cos x\cos y &= \frac{1}{2}[\cos(x+y)+\cos(x-y)]\\ \sin x\sin y &= \frac{1}{2}[\cos(x-y)-\cos(x+y)]\end{aligned}$$

There are many other trigonometric identities, but those we have stated are the ones used most often in calculus.

Graphs of Trigonometric Functions Graphs of the six basic trigonometric functions using radian measure are shown in Figure 1.17. The shading for each trigonometric function indicates its periodicity.

The graph of the function $f(x) = \sin x$, shown in Figure 1.17(b), is obtained by plotting points for $0 \le x \le 2\pi$ and then using the periodic nature of the function to complete the graph. Notice that the zeros of the sine function occur at the integer multiples of π, that is,

$$\sin x = 0 \text{ whenever } x = n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

Because of the identity

$$\cos x = \sin\left(x + \frac{\pi}{2}\right)$$

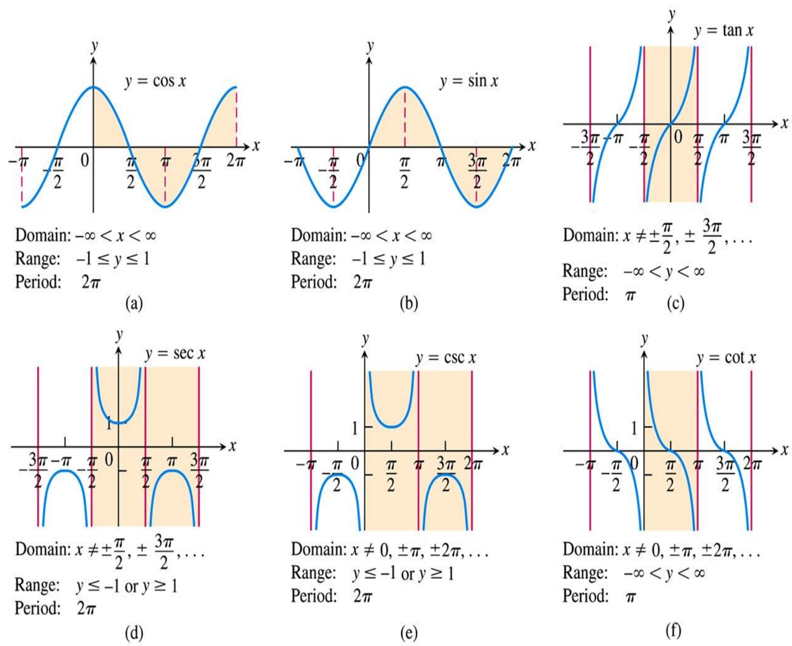


Figure 1.17:

the graph of cosine is obtained by shifting the graph of sine by an amount $\pi/2$ to the left [see Figure 1.17(a)]. Therefore, the zeros of the cosine function occur at the integer multiples of π plus $\pi/2$, that is,

$$\cos x = 0 \text{ whenever } x = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

Note that for both the sine and cosine functions the domain is $\mathbb{R} = (-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Thus, for all values of x, we have

$$\begin{aligned} -1 \le \sin x \le 1 \quad &\textbf{or we write} \quad |\sin x| \le 1 \\ -1 \le \cos x \le 1 \quad &\textbf{or we write} \quad |\cos x| \le 1 \end{aligned}$$

The tangent and cotangent have range $\mathbb{R} = (-\infty, \infty)$, whereas cosecant and secant have range $(-\infty, -1] \cup [1, \infty)$. All four functions are periodic: tangent and cotangent have period π, whereas cosecant and secant have period 2π. Since

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}$$

then the zeros of the tangent function is the same as of the sine function, and the zeros of the cotangent function is the same as of the cosine function, while the secant and cosecant functions have no zeros.

Example 1.19. Solve the following equations.

a) $\sin x = 1$

b) $\cos x = -1$

Solution 1.19. a) $\sin x$ equals 1 when $x = \pi/2$ plus multiples of 2π, see Figure 1.17(b). So,

$$\sin x = 1 \text{ whenever } x = \frac{\pi}{2} + 2n\pi \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

b) $\cos x$ equals -1 when $x = \pi$ plus multiples of 2π, see Figure 1.17(a). So,

$$\cos x = 1 \text{ whenever } x = \pi + 2n\pi \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \cdots$$

□

Exercise 1.3. Solve the following equations.

a) $\sin x = -1$

b) $\sec x = 1$

c) $\tan x = 1$

d) $\cos x = 1/2$

e) $\cos(2x) = 0$

Example 1.20. Find the domain of the following.

a) $f(x) = 1/(1 + \sin x)$

b) $f(x) = 1/(1 - 2\cos x)$

c) $f(x) = \csc x/\sqrt{2-x}$

d) $f(x) = \sin\sqrt{x}$

e) $f(x) = \sqrt{\sin x}$

Solution 1.20. a) Since f is rational function, then

$$\begin{aligned}\text{The domain of } f &= \mathbb{R} - \{x \in \mathbb{R} : 1 + \sin x = 0\}\\ &= \mathbb{R} - \{x \in \mathbb{R} : \sin x = -1\}\\ &= \mathbb{R} - \left\{x = \frac{3\pi}{2} + 2n\pi\right\}\end{aligned}$$

where $n = 0, \pm1, \pm2, \cdots$

b) Note that f is also rational function. So

$$\begin{aligned}\text{The domain of } f &= \mathbb{R} - \{x \in \mathbb{R} : 1 - 2\cos x = 0\}\\ &= \mathbb{R} - \left\{x \in \mathbb{R} : \cos x = \frac{1}{2}\right\}\\ &= \mathbb{R} - \left\{x = \frac{\pi}{3} + 2n\pi \textbf{ or } x = -\frac{\pi}{3} + 2n\pi\right\}\end{aligned}$$

where $n = 0, \pm1, \pm2, \cdots$

c) First, we write

$$f(x) = \frac{\csc x}{\sqrt{2-x}} = \frac{1}{\sin x\sqrt{2-x}}.$$

Therefore,

$$\begin{aligned}\text{The domain of } f &= \{\text{domain } \sqrt{2-x}\} - \left\{x \in \mathbb{R} : \sin x\sqrt{2-x} = 0\right\}\\ &= \{x \in \mathbb{R} : 2 - x > 0\} \cap \{\text{domain } 2 - x\}\\ &\qquad - \{x \in \mathbb{R} : \sin x = 0\}\\ &= (-\infty, 2) \cap \mathbb{R} - \{x = n\pi\}\\ &= (-\infty, 2) - \{x = n\pi\}\end{aligned}$$

where $n = 0, \pm1, \pm2, \cdots$

d) In this example, the domain of $\sin\sqrt{x}$ is the domain of $\sqrt{x}$ since the root function is the dominant here. So

$$\begin{aligned}\text{The domain of } f &= \{x \in \mathbb{R} : x \geq 0\} \cap \{\text{ domain } x\} \\ &= [0, \infty) \cap \mathbb{R} \\ &= [0, \infty)\end{aligned}$$

e) This example is quite different from the previous one in part (d).

$$\text{The domain of } f = \{x \in \mathbb{R} : \sin x \geq 0\} \cap \{\text{ domain } \sin x\}.$$

Note that, $\sin x \geq 0$ in quadrants I and II, i.e., when $x \in [0, \pi]$. Since $\sin x$ is periodic with period 2π, then it is greater than or equals to 0 when $x \in [0 + 2n\pi, \pi + 2n\pi]$ where $n = 0, \pm 1, \pm 2, \cdots$. Hence

$$\begin{aligned}\text{The domain of } f &= [0 + 2n\pi, \pi + 2n\pi] \cap \mathbb{R} \\ &= [0 + 2n\pi, \pi + 2n\pi] \\ &= [2n\pi, (1 + 2n)\,\pi]\end{aligned}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

□

Exercise 1.4. Find the domain of the following.

a) $g(x) = 1/(1 + \sin^2 x)$
b) $g(x) = \sqrt{\tan x}$
c) $g(x) = x/(1 - |\sec x|)$

1.3 Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two functions can be combined to create new functions.

Definition 1.3.1. Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

Sum: $(f+g)(x) = f(x) + g(x)$; $x \in$ domain $f \cap$ domain g

Difference: $(f-g)(x) = f(x) - g(x)$; $x \in$ domain $f \cap$ domain g

Product: $(f \times g)(x) = f(x) \times g(x)$; $x \in$ domain $f \cap$ domain g

Quotient: $(f/g)(x) = f(x)/g(x)$; $x \in$ domain $f \cap$ domain g and $g(x) \neq 0$.

Example 1.21. Find $(f/g)(x)$ and $(g/f)(x)$ for the functions given by $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4-x^2}$. Then find the domains of f/g and g/f.

Solution 1.21. The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4-x^2}}.$$

The quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4-x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains for f/g and g/f are as follows.

$$\begin{array}{lll} \text{Domain of } (f/g) & \text{is} & [0,2) \\ \text{Domain of } (g/f) & \text{is} & (0,2] \end{array}$$

□

Exercise 1.5. Given $f(x) = 2x + 1$ and $g(x) = \frac{1}{x} + 2x - 1$, find $(f-g)(x)$ and its domain. Then evaluate the difference when $x = 2$.

Definition 1.3.2. The composition of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (See Figure 1.18.)

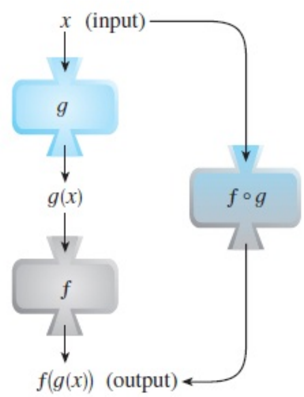


Figure 1.18:

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both g and $f(g(x))$ are defined.

Example 1.22. Find the domain of the composition $(f \circ g)(x)$ for the functions given by $f(x) = x^2 - 9$ and $g(x) = \sqrt{9 - x^2}$.

Solution 1.22. The composition of the functions is as follows.

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\sqrt{9 - x^2}\right) \\
&= \left(\sqrt{9 - x^2}\right)^2 - 9 \\
&= \left(9 - x^2\right) - 9 \\
&= -x^2
\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true.Because the domain of $-x^2$ is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $(f \circ g)$ is $[-3, 3]$.

□

Example 1.23. Let $g(x) = \sqrt{2-x}$. Find $g \circ g$ and its domain.

Solution 1.23.

$$\begin{aligned}(g \circ g)(x) &= g(g(x)) \\ &= g\left(\sqrt{2-x}\right) \\ &= \sqrt{2-\sqrt{2-x}}\end{aligned}$$

This expression is defined when both $2 - x \geq 0$ and $2 - \sqrt{2-x} \geq 0$. The first inequality means $x \leq 2$, and the second is equivalent to

$$\begin{aligned}\sqrt{2-x} &\leq 2 \\ 2-x &\leq 4 \\ x &\geq -2\end{aligned}$$

Thus $-2 \leq x \leq 2$, so the domain of $g \circ g$ is the closed interval $[-2, 2]$.

□

It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h, then g, and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Example 1.24. Find $f \circ g \circ h$ if $f(x) = x/(x+1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

Solution 1.24.

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x))) \\ &= f(g(x+3)) \\ &= f\left((x+3)^{10}\right) \\ &= \frac{(x+3)^{10}}{(x+3)^{10}+1}\end{aligned}$$

□

So far we have used composition to build complicated functions from simpler ones. But in calculus it is often useful to be able to **decompose** a complicated function into simpler ones, as in the following example.

Example 1.25. Given $F(x) = \cos^2(x+9)$, find functions f, g, and h such that $F = f \circ g \circ h$.

Solution 1.25. Since $F(x) = [\cos(x+9)]^2$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$\begin{aligned} h(x) &= x+9 \\ g(x) &= \cos x \\ f(x) &= x^2 \end{aligned}$$

Then

$$\begin{aligned} (f \circ g \circ h)(x) &= f(g(h(x))) \\ &= f(g(x+9)) \\ &= f(\cos(x+9)) \\ &= \cos^2(x+9) = F(x). \end{aligned}$$

□

1.4 Inverse Functions

Remember that, a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x+4$ from the set $A = \{1,2,3,4\}$ to the set $B = \{5,6,7,8\}$ can be written as follows.

$$f(x) = x+4 : \{(1,5),(2,6),(3,7),(4,8)\}.$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the inverse function of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x-4 : \{(5,1),(6,2),(7,3),(8,4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.19. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$\begin{aligned} f\left(f^{-1}(x)\right) &= f(x-4) = (x-4)+4 = x \\ f^{-1}\left(f(x)\right) &= f^{-1}(x+4) = (x+4)-4 = x \end{aligned}$$

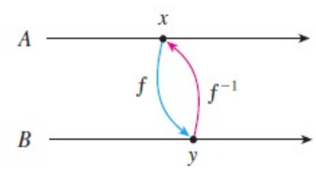


Figure 1.19:

Definition 1.4.1. A function is called a **one−to−one** function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2$$

We have the following geometric method for determining whether a function is one-to-one.

Definition 1.4.2. Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 1.26. Is the function $f(x) = x^3$ one-to-one?

Solution 1.26. If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers cant have the same cube). Therefore, by Definition 1.4.1, $f(x) = x^3$ is one-to-one. From Figure 1.20 we see that no horizontal line intersects the graph of $f(x) = x^3$ more than once. Therefore, by the Horizontal Line Test, f is one-to-one.

□

Example 1.27. Is the function $g(x) = x^2$ one-to-one?

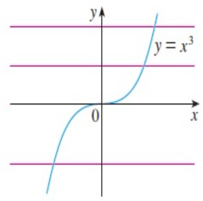


Figure 1.20:

Solution 1.27. This function is not one-to-one because, for instance, $g(1) = 1 = g(-1)$ and so 1 and -1 have the same output. From Figure 1.21 we see that there are horizontal lines that intersect the graph of g more than once. Therefore, by the Horizontal Line Test, g is not one-to-one.

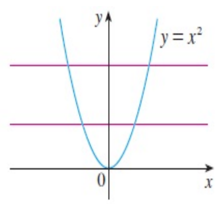


Figure 1.21:

□

Not all functions possess inverses. Only function that has the one-to-one property has inverse function according to the following definition.

Definition 1.4.3. Let f be a one-to-one function with domain **A** and range **B**. Then its **inverse function** f^{-1} has domain **B** and range **A** and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any $y \in \mathbf{B}$.

Remark 1.4.1. Do not mistake the power -1 in f^{-1} for an exponent. Thus $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$. The reciprocal $\frac{1}{f(x)}$ could, however, be written as $[f(x)]^{-1}$.

Remark 1.4.2. The letter x is traditionally used as the independent variable, so when we concentrate on f^{-1} rather than on f, we usually reverse the roles of x and y in Definition 1.4.3 and write

$$f^{-1}(x) = y \Leftrightarrow f(y) = x$$

By this formula and Definition 1.4.3 we get the following **cancelation equations:**

$$\begin{aligned} f^{-1}\left(f\left(x\right)\right) &= x \text{ for every } x \in \mathbf{A} \\ f\left(f^{-1}\left(x\right)\right) &= x \text{ for every } x \in \mathbf{B} \end{aligned}$$

Algorithm 1.1. How To Find The Inverse Function Of A One-To-One Function f:

Step 1 Write $y = f(x)$.

Step 2 Solve this equation for x in terms of y (if possible).

Step 3 To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 1.28. Find the inverse function of $f(x) = x^3 + 2$.

Solution 1.28. Note that $f(x) = x^3 + 2$ is one-to-one function (why?). According to Algorithm 1.1, first we write

$$y = x^3 + 2$$

Then we solve this equation for x:

$$\begin{aligned} x^3 &= y - 2 \\ x &= \sqrt[3]{y-2} \end{aligned}$$

Finally, we interchange x and y

$$y = \sqrt[3]{x-2} = f^{-1}(x).$$

Exercise 1.6. Show that $f(x) = x/(x+1)$; $x \neq -1$ is one-to-one function, then find its inverse.

Remark 1.4.3. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$ as illustrated by Figure 1.22.

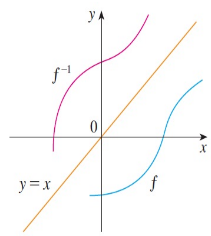


Figure 1.22:

Inverse Trigonometric Functions

When we try to find the inverse trigonometric functions, we have a slight difficulty: Because the trigonometric functions are not one-to-one, they do not have inverse functions. The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

You can see from Figure 1.23 that the sine function is not one-to-one (use the Horizontal Line Test). But the function $f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$, is one-to-one (see Figure 1.23). The inverse function of this restricted sine function exists and is denoted by $\sin^{-1}$ or arcsin. It is called the **inverse sine function** or the **arcsine function**.

The following table summarizes the definitions of the three most common inverse trigonometric functions.

Function	Domain	Range
$y = \sin^{-1} x \leftrightarrow \sin y = x$	$x \in [-1, 1]$	$y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x \leftrightarrow \cos y = x$	$x \in [-1, 1]$	$y \in [0, \pi]$
$y = \tan^{-1} x \leftrightarrow \tan y = x$	$x \in \mathbb{R}$	$y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

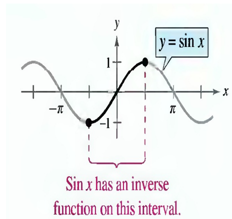


Figure 1.23:

Exercise 1.7. Find the domain of $f(x) = \sin^{-1}(x^2 - 4)$

Example 1.29. If possible, find the exact value of

1. $\sin^{-1}\left(-\frac{1}{2}\right)$
2. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
3. $\sin^{-1}(2)$
4. $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$
5. $\cos^{-1}(-1)$
6. $\tan^{-1}(0)$
7. $\tan^{-1}(1)$

Solution 1.29.

1. Because $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it follows that $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

2. Because $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it follows that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

3. It is not possible to evaluate $y = \sin^{-1} x$ at $x = 2$ because there is no angle whose sine is 2. Remember that the domain of the inverse sine function is $[-1, 1]$.

4. Because $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4} \in [0, \pi]$, it follows that $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

5. Because $\cos(\pi) = -1$ and $\pi \in [0, \pi]$, it follows that $\cos^{-1}(-1) = \pi$.
6. Because $\tan(0) = 0$ and $0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows that $\tan^{-1}(0) = 0$.
7. Because $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows that $\tan^{-1}(-1) = -\frac{\pi}{4}$.

□

The following are some important identities of inverse trigonometric functions.

1. $\sin^{-1}(-x) = -\sin^{-1}(x)$ for all $x \in [-1, 1]$
2. $\tan^{-1}(-x) = -\tan^{-1}(x)$ for all $x \in \mathbb{R}$
3. $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ for all $x \in [-1, 1]$
4. $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ for all $x \in [-1, 1]$
5. $\sin\left(\sin^{-1} x\right) = x$ for all $x \in [-1, 1]$, and $\sin^{-1}(\sin x) = x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
6. $\cos(\cos^{-1} x) = x$ for all $x \in [-1, 1]$, and $\cos^{-1}(\cos x) = x$ for all $x \in [0, \pi]$
7. $\tan(\tan^{-1} x) = x$ for all $x \in \mathbb{R}$, and $\tan^{-1}(\tan x) = x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
8. $\sin^{-1}(\sin x) = \begin{cases} \pi - x & : \frac{\pi}{2} \le x \le \frac{3\pi}{2} \\ x - 2n\pi & : x \ge \frac{3\pi}{2} \end{cases}$ where $n = 0, \pm 1, \pm 2, \cdots$.
9. $\cos^{-1}(\cos x) = 2n\pi - x$ if $x \ge \pi$ where $n = 0, \pm 1, \pm 2, \cdots$.
10. $\cos\left(\sin^{-1} x\right) = \sin(\cos^{-1} x) = \sqrt{1 - x^2}$ for all $x \in [-1, 1]$.
11. $\tan\left(\sin^{-1} x\right) = \frac{x}{\sqrt{1-x^2}}$ for all $x \in [-1, 1]$.

Example 1.30. If possible, find the exact value of

1. $\tan(\tan^{-1}(-5))$

2. $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right)$

3. $\cos\left(\cos^{-1}(\pi)\right)$

4. $\cos^{-1}\left(\cos\left(\frac{17\pi}{4}\right)\right)$

5. $\tan\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$

6. $\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

Solution 1.30. 1. Because $-5 \in \mathbb{R}$, the inverse property applies, and you have $\tan\left(\tan^{-1}(-5)\right) = -5$.

2. In this case, $\frac{5\pi}{3}$ does not lie within the range of the arcsine function $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. However, $\frac{5\pi}{3}$ is coterminal with $\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and you have

$$\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$$

3. The expression $\cos\left(\cos^{-1}(\pi)\right)$ is not defined because $\cos^{-1}(\pi)$ is not defined. Remember that the domain of the inverse cosine function is $[-1, 1]$.

4. In this case, $\frac{17\pi}{4}$ does not lie within the range of the cosine function $[0, \pi]$. However, $\frac{17\pi}{4}$ is coterminal with $2(2)\pi - \frac{17\pi}{4} = -\frac{\pi}{4}$, and you have

$$\cos^{-1}\left(\cos\left(\frac{17\pi}{4}\right)\right) = \cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$$

5. If you let $u = \cos^{-1}\left(\frac{2}{3}\right)$ then $\cos u = \frac{2}{3}$. Because $\cos u$ is positive, u is a first-quadrant angle. You can sketch and label angle u as shown in Figure 1.24. Consequently

$$\tan\left(\cos^{-1}\left(\frac{2}{3}\right)\right) = \tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{5}}{2}.$$

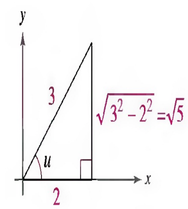


Figure 1.24:

6. If you let $u = \sin^{-1}\left(-\frac{3}{5}\right)$ then $\sin u = \frac{3}{5}$. Because $\sin u$ is negative, u is a fourth-quadrant angle. You can sketch and label angle u as shown in Figure 1.25. Consequently

$$\cos\left(\sin^{-1}\left(-\frac{3}{5}\right)\right) = \cos u = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}.$$

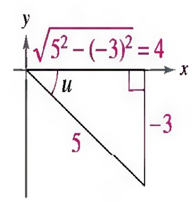


Figure 1.25:

□

Exercise 1.8. If possible, find the exact value of

1. $\sin^{-1}\left(\sin\frac{\pi}{16}\right)$
2. $\sin^{-1}\left(\sin\frac{-5\pi}{2}\right)$
3. $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$
4. $\tan\left(\sin^{-1}\frac{5}{13}\right)$

5. $\sec\left(\sin^{-1}\frac{3}{4}\right)$

6. $\cos\left(\tan^{-1} 2\right)$

7. $\sin\left(2\cos^{-1}\frac{3}{5}\right)$

Exponential and Logarithmic Functions

In this part you will study two types of non−algebraic functions: exponential functions and logarithmic functions. These functions are examples of **transcendental functions**.

Definition 1.4.4. The exponential function f with base a is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number. The domain of the exponential function $f(x) = a^{g(x)}$ is the same as the domain of $g(x)$.

Note that in the definition of an exponential function, the base $a = 1$ is excluded because it yields $f(x) = 1^x = 1$. This is a constant function, not an exponential function. You already know how to evaluate a^x for integer and rational values of x. For example, you know that $4^3 = 64$ and $4^{1/2} = \sqrt{4} = 2$.

The exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$ is different from all the functions you have studied so far because the variable x is an exponent. A distinguishing characteristic of an exponential function is its rapid increase as x increases (for $a > 1$). Many real-life phenomena with patterns of rapid growth (or decline) can be modeled by exponential functions. The basic characteristics of the exponential function are summarized below in Figure 1.26.

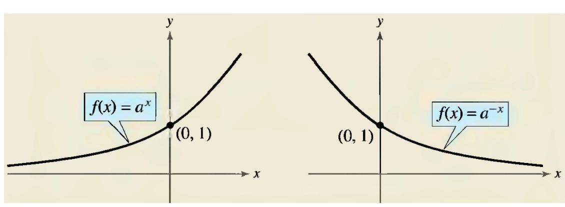


Figure 1.26:

Example 1.31. In the same coordinate plane, sketch the graph of $f(x) = 2^x$ and $g(x) = 4^x$ by hand.

Solution 1.31. The table below lists some values for each function. By plotting these points and connecting them with a smooth curve, you obtain the graphs shown in Figure 1.27. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
4^x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

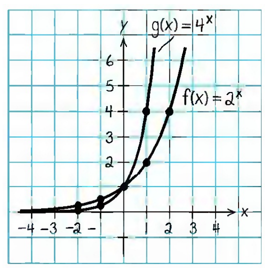


Figure 1.27:

□

Example 1.32. In the same coordinate plane, sketch the graph of $f(x) = 2^{-x}$ and $g(x) = 4^{-x}$ by hand.

Solution 1.32. The table below lists some values for each function. By plotting these points and connecting them with a smooth curve, you obtain the graphs shown in Figure 1.28. Note that both graphs are decreasing. Moreover, the graph of $g(x) = 4^{-x}$ is decreasing more rapidly than the graph of $f(x) = 2^{-x}$.

x	-3	-2	-1	0	1	2
2^x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
4^x	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

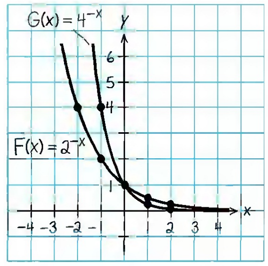


Figure 1.28:

□

The properties of exponents can also be applied to real-number exponents. For review, these properties are listed below.

- $a^x a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$
- $a^0 = 1$
- $\sqrt[n]{a^m} = a^{m/n}$
- $(ab)^x = a^x b^x$
- $(a^x)^y = a^{xy}$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $|a^2| = |a|^2 = a^2$

Example 1.33. Find the exact value of the following.

1. $(-8)^{2/3}$
2. $9^{-1/2}$

Solution 1.33. 1. $(-8)^{2/3} = \sqrt[3]{(-8)^2} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$

2. $9^{-1/2} = \frac{1}{9^{1/2}} = \left(\frac{1}{9}\right)^{1/2} = \sqrt{\frac{1}{9}} = \frac{1}{3}$

□

The Natural Base e**:** For many applications, the convenient choice for a base is the irrational number $e \approx 2.7\overline{182}$. This number is called the natural base. The function $f(x) = e^x$ is called the **natural exponential** function and its graph is shown in Figure 1.29. The graph of the exponential function has the same basic characteristics as the graph of the function $f(x) = a^x$. Be sure you see that for the exponential function $f(x) = e^x$, e is the constant $2.7\overline{182}$, whereas x is the variable.

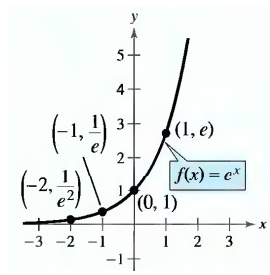


Figure 1.29:

The number e can be approximated by the expression

$$\left(1 + \frac{1}{x}\right)^x \text{ for large values of x.}$$

We learned that if a function is one-to-one-that is, if the function has the property such that no horizontal line intersects its graph more than once-the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Figure 1.26, you will see that every function of the form

$$f(x) = a^x, \qquad a > 0, a \neq 1$$

passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base** a.

Definition 1.4.5. For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function given by $f(x) = \log_a x$ (read as log base a of x) is called the logarithmic function with base a. The domain of the logarithmic function $f(x) = \log_a g(x)$ is

$$\{x \in \mathbb{R} : g(x) > 0\} \cap \text{ the domain of } g(x)$$

The equations $y = \log_a x$ and $x = a^y$ are equivalent. The first equation is in logarithmic form and the second is in exponential form. When evaluating logarithms, remember that a logarithm is an exponent. This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_z 8 = 3$ because 2 must be raised to the third power to get 8.

Example 1.34. Use the definition of logarithmic function to evaluate each logarithm at the indicated value of x.

1. $f(x) = \log_2 x$, at $x = 32$
2. $f(x) = \log_3 x$, at $x = 1$
3. $f(x) = \log_4 x$, at $x = 2$
4. $f(x) = \log_{10} x$, at $x = \frac{1}{100}$

Solution 1.34.

1. $f(32) = \log_2 32 = 5$ because $2^5 = 32$
2. $f(1) = \log_3 1 = 0$ because $3^0 = 1$
3. $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{\frac{1}{2}} = \sqrt{4} = 2$
4. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

□

The following properties follow directly from the definition of the logarithmic function with base a.

1. $\log_a 1 = 0$ because $a^0 = 1$
2. $\log_a a = 1$ because $a^1 = a$
3. $\log_a b^x = x \log_a b$
4. $\log_a a^x = x$ and $a^{\log_a x} = x$
5. $\log_a(x \times y) = \log_a x + \log_a y$
6. $\log_a(x \div y) = \log_a x - \log_a y$
7. If $\log_a x = \log_a y$, then $x = y$

Example 1.35. Find the exact value of the following.

1. $\log_2 16$
2. $\log_9 3$
3. $5^{-2\log_5 2}$
4. $\log_{10} 0.001$
5. $\log_6 9 - \log_6 5 + \log_6 20$

Solution 1.35.

1. $\log_2 16 = \log_2 2^4 = 4\log_2 2 = 4 \times 1 = 4$
2. $\log_9 3 = \log_9 9^{\frac{1}{2}} = \frac{1}{2}\log_9 9 = \frac{1}{2} \times 1 = \frac{1}{2}$
3. $5^{-2\log_5 2} = 5^{\log_5 2^{-2}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
4. $\log_{10} 0.001 = \log_{10} \frac{1}{1000} = \log_{10} \frac{1}{10^3} = \log_{10} 10^{-3} = -3\log_{10} 10 = -3 \times 1 = -3$
5. $\log_6 9 - \log_6 5 + \log_6 20 = \log_6 \left(\frac{9}{5}\right) + \log_6 20 = \log_6 \left(\frac{9}{5} \times 20\right) = \log_6 36 = \log_6 6^2 = 2\log_6 6 = 2 \times 1 = 2$

□

The Natural logarithmic Function By looking back at the graph of the natural exponential function in Figure 1.29, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as: the natural log of x.

Definition 1.4.6. For $x > 0$,

$$y = \ln x \text{ if and only if } x = e^y.$$

The function given by $f(x) = \log_e x = \ln x$ is called the **natural logarithmic function**. The domain of the natural logarithmic function $f(x) = \ln g(x)$ is

$$\{x \in \mathbb{R} : g(x) > 0\} \cap \text{ the domain of } g(x)$$

From the above definition, you can see that every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form. Note that the natural logarithm $\ln x$ is written without a base. The base is understood to be e.

Because the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 1.30.

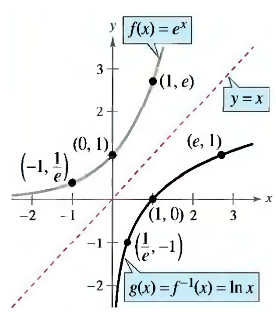


Figure 1.30:

The properties of logarithms previously listed are also valid for natural logarithms.

1. $\ln 1 = 0$ because $e^0 = 1$
2. $\ln e = 1$ because $e^1 = e$
3. $\ln a^x = x \ln a$
4. $\ln e^x = x$ and $e^{\ln x} = x$
5. $\ln(x \times y) = \ln x + \ln y$
6. $\ln(x \div y) = \ln x - \ln y$
7. If $\ln x = \ln y$, then $x = y$
8. $\log_a b = \ln b / \ln a$

Exercise 1.9. Find the exact value of $(\log_2 3)(\log_3 4)(\log_4 5)\cdots(\log_{31} 32)$.

So far in this part, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. Now, you will study procedures for solving equations involving exponential and logarithmic functions. There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties.

For $a > 0$ and $a \neq l$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties $a^x = a^y$ if and only if $x = y$, and $\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties $a^{\log_a x} = x$ and $\log_a a^x = x$.

Example 1.36. Solve the following equations.

1. $2^x = 32$
2. $\ln x - \ln 3 = 0$
3. $\left(\frac{1}{3}\right)^x = 9$
4. $e^x = 7$

5. $\ln x = -3$

6. $\log_{10} x = -1$

Solution 1.36.

Original Equation	Rewritten Equation	Solution	Property
$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
$\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
$\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse

□

Example 1.37. Solve the equation $\log_{10} x^{\frac{3}{2}} - \log_{10} \sqrt{x} = 1$.

Solution 1.37.

$$\begin{aligned}
\log_{10} x^{\frac{3}{2}} - \log_{10} \sqrt{x} &= 1 \\
\log_{10} x^{\frac{3}{2}} - \log_{10} x^{\frac{1}{2}} &= 1 \\
\frac{3}{2} \log_{10} x - \frac{1}{2} \log_{10} x &= 1 \\
\log_{10} x &= 1 \\
10^{\log_{10} x} &= 10^1 \\
x &= 10
\end{aligned}$$

□

Example 1.38. Solve the equation $e^x - 2xe^x = 0$.

Solution 1.38.

$$\begin{aligned}
e^x - 2xe^x &= 0 \\
e^x (1 - 2x) &= 0 \\
1 - 2x &= 0 \Rightarrow x = \frac{1}{2} \\
e^x &\neq 0
\end{aligned}$$

Example 1.39. Solve the equation $e^{2x} - 3e^x + 2 = 0$.

Solution 1.39.

$$\begin{aligned}
e^{2x} - 3e^x + 2 &= 0 \\
\left(e^x\right)^2 - 3e^x + 2 &= 0 \\
\left(e^x - 2\right)\left(e^x - 1\right) &= 0 \\
e^x - 2 &= 0 \Rightarrow e^x = 2 \Rightarrow x = \ln 2 \\
e^x - 1 &= 0 \Rightarrow e^x = 1 \Rightarrow x = \ln 1 = 0
\end{aligned}$$

□

Example 1.40. Solve the equation $\ln(x-2) + \ln(2x-3) = 2\ln x$.

Solution 1.40.

$$\begin{aligned}
\ln(x-2) + \ln(2x-3) &= 2\ln x \\
\ln[(x-2)(2x-3)] &= \ln x^2 \\
(x-2)(2x-3) &= x^2 \\
2x^2 - 7x + 6 &= x^2 \\
x^2 - 7x + 6 &= 0 \\
(x-1)(x-6) &= 0 \\
x - 1 &= 0 \Rightarrow x = 1 \\
x - 6 &= 0 \Rightarrow x = 6
\end{aligned}$$

You can conclude that $x = 1$ is not valid. This is because when $x = 1$, $\ln(x-2) + \ln(2x-3) = \ln(-1) + \ln(-1)$, which is invalid because -1 is not in the domain of the natural logarithmic function. So, the only solution is $x = 6$.

□

Example 1.41. Solve the inequality $e^{2-3x} \geq 4$.

Solution 1.41.

$$\begin{aligned} e^{2-3x} &\geq 4 \\ \ln e^{2-3x} &\geq \ln 4 \\ 2-3x &\geq \ln 4 \\ -3x &\geq -2+\ln 4 \\ x &\leq \frac{2}{3}-\frac{1}{3}\ln 4 \\ x &\in \left(-\infty, \frac{2}{3}-\frac{1}{3}\ln 4\right] \end{aligned}$$

□

Example 1.42. Solve the inequality $2 \leq \log_2 x < 3$.

Solution 1.42.

$$\begin{aligned} 2 \leq \log_2 x < 3 &\Rightarrow 2^2 \leq 2^{\log_2 x} < 2^3 \\ &\Rightarrow 4 \leq x < 8 \\ &\Rightarrow x \in [4, 8) \end{aligned}$$

□

Exercise 1.10. Solve the equation.

1. $\ln x^2 = 4$
2. $\log_3(5x-1) = \log_3(x+7)$
3. $\log_5 25^x = 8$
4. $\ln\left(\frac{1}{x}\right) + \ln\left(2x^3\right) = \ln 3$
5. $e^{2x} - e^x = 6$
6. $\ln(\ln x) = 1$

Example 1.43. Find the domain of $f(x) = 5^{-\sqrt{x}}$.

Solution 1.43.

$$\begin{aligned}
\text{The domain of } 5^{-\sqrt{x}} &= \text{The domain of } \sqrt{x} \\
&= \{x \in \mathbb{R} : x \geq 0\} \cap \text{The domain of } x \\
&= [0, \infty) \cap \mathbb{R} \\
&= [0, \infty)
\end{aligned}$$

□

Example 1.44. Find the domain of $f(x) = 1 - e^{\frac{1}{x}}$.

Solution 1.44.

$$\begin{aligned}
\text{The domain of } 1 - e^{\frac{1}{x}} &= \text{The domain of } \frac{1}{x} \\
&= \mathbb{R} - \{x \in \mathbb{R} : x = 0\} \\
&= \mathbb{R} - \{0\}
\end{aligned}$$

□

Example 1.45. Find the domain of $f(x) = \sqrt{e^x}$.

Solution 1.45.

$$\begin{aligned}
\text{The domain of } \sqrt{e^x} &= \{x \in \mathbb{R} : e^x \geq 0\} \cap \text{The domain of } e^x \\
&= \mathbb{R} \cap \mathbb{R} \\
&= \mathbb{R}
\end{aligned}$$

□

Example 1.46. Find the domain of $f(x) = \ln x^2$.

Solution 1.46.

$$\begin{aligned}
\text{The domain of } \ln x^2 &= \{x \in \mathbb{R} : x^2 > 0\} \cap \text{The domain of } x^2 \\
&= (\mathbb{R} - \{0\}) \cap \mathbb{R} \\
&= \mathbb{R} - \{0\}
\end{aligned}$$

□

Example 1.47. Find the domain of $f(x) = 2\ln x$.

Solution 1.47.

$$\begin{aligned}
\text{The domain of } 2\ln x &= \{x \in \mathbb{R} : x > 0\} \cap \text{The domain of } x \\
&= (0, \infty) \cap \mathbb{R} \\
&= (0, \infty)
\end{aligned}$$

□

Example 1.48. Find the domain of $f(x) = \log_4 (9 - 16x^2)$.

Solution 1.48.

$$\begin{aligned}
\text{The domain of } \log_4\left(9 - 16x^2\right) &= \left\{x \in \mathbb{R} : 9 - 16x^2 > 0\right\} \cap \text{Domain of } 9 - 16x^2 \\
&= \left\{x \in \mathbb{R} : x^2 < \frac{9}{16}\right\} \cap \mathbb{R} \\
&= \left\{x \in \mathbb{R} : \sqrt{x^2} < \sqrt{\frac{9}{16}}\right\} \cap \mathbb{R} \\
&= \left\{x \in \mathbb{R} : |x| < \frac{3}{4}\right\} \cap \mathbb{R} \\
&= \left\{x \in \mathbb{R} : -\frac{3}{4} < x < \frac{3}{4}\right\} \cap \mathbb{R} \\
&= \left(-\frac{3}{4}, \frac{3}{4}\right) \cap \mathbb{R} \\
&= \left(-\frac{3}{4}, \frac{3}{4}\right)
\end{aligned}$$

□

Example 1.49. Find the domain of $f(x) = \log_3(1 - \sqrt{x})$.

Solution 1.49.

$$\begin{aligned}
\text{The domain of } \log_3(1 - \sqrt{x}) &= \left\{x \in \mathbb{R} : 1 - \sqrt{x} > 0\right\} \cap \text{The domain of } \sqrt{x} \\
&= \left\{x \in \mathbb{R} : \sqrt{x} < 1\right\} \cap \{x \in \mathbb{R} : x \geq 0\} \cap \mathbb{R} \\
&= \{x \in \mathbb{R} : 0 \leq x < 1\} \cap [0, \infty) \cap \mathbb{R} \\
&= [0, 1) \cap [0, \infty) \cap \mathbb{R} \\
&= [0, 1)
\end{aligned}$$

Exercise 1.11. Find the domain of the following.

1. $f(x) = e^{4+x^2}$
2. $f(x) = \ln \cos x$
3. $f(x) = \log_{10}(1 - e^x)$
4. $f(x) = \frac{1}{1-e^x}$
5. $f(x) = \ln(1 + \ln x)$
6. $f(x) = \sqrt{2 - 2^x}$
7. $f(x) = \ln(4 - x)$

Example 1.50. Find a formula for the inverse of the function.

1. $f(x) = e^{2x-1}$
2. $f(x) = \ln(x + 3)$

Solution 1.50.
1. First, let $y = e^{2x-1}$. By inverse property we have $\ln y = \ln e^{2x-1}$ and then $\ln y = 2x - 1$. So, $x = (1 + \ln y)/2$. Hence, $f^{-1}(x) = (1 + \ln x)/2$.
2. Let $y = \ln(x+3)$. By inverse property we have $e^y = e^{\ln(x+3)}$ and then $e^y = x + 3$. So, $x = e^y - 3$. Hence, $f^{-1}(x) = e^x - 3$.

□

Exercise 1.12. Find a formula for the inverse of the function.

1. $g(t) = 3^{2t-1}$
2. $g(t) = \log_{10}(t + 3)$

1.5 Hyperbolic Functions

Certain even and odd combinations of the exponential functions e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve to be given special names. In many ways they are analogous to the trigonometric functions, and they have the same relationship to the hyperbola that the trigonometric functions have to the circle. For this reason they are collectively called **hyperbolic functions** and individually called **hyperbolic sine**, **hyperbolic cosine**, and so on.

Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The graphs of hyperbolic sine, cosine, and tangent can be sketched using graphical addition as in Figures 1.31. Note that sinh has do-

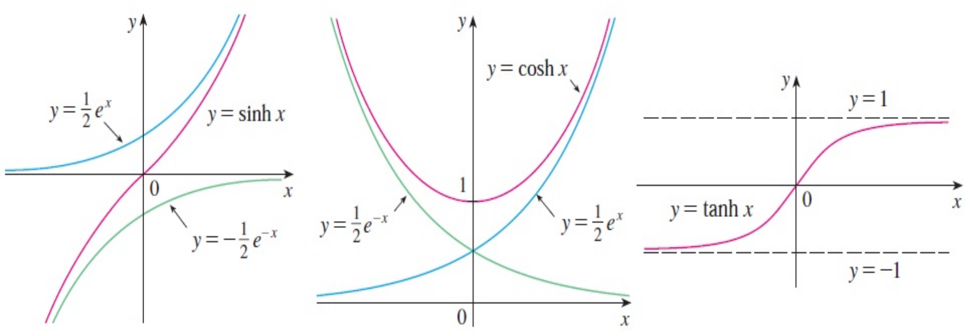


Figure 1.31:

main $\mathbb{R}$ and range $\mathbb{R}$, while cosh has domain $\mathbb{R}$ and range $[1, \infty)$. The graph of tanh has the horizontal asymptotes $y = \pm 1$.

Example 1.51. Find the numerical value of each expression.

1. $\sinh 0$
2. $\cosh(\ln 3)$
3. $\tanh 1$

Solution 1.51.
1. $\sinh 0 = (e^0 - e^0)/2 = 0$
2. $\cosh(\ln 3) = \left(e^{\ln 3} + e^{-\ln 3}\right)/2 = (3 + 1/3)/2 = 5/3$
3. $\tanh 1 = (e^1 - e^{-1}) / (e^1 + e^{-1}) = (e^2 - 1) / (e^2 + 1)$

□

Exercise 1.13. Find the numerical value of each expression.

1. $\cosh 0$
2. $\sinh(\ln 2)$
3. $\operatorname{sech} 0$

Exercise 1.14. If $\tanh x = \frac{12}{13}$, find the values of the other hyperbolic functions at x.

Hyperbolic Identities The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities. We list some of them here.

1. $\sinh(-x) = -\sinh(x)$
2. $\cosh(-x) = \cosh(x)$
3. $\cosh^2 x - \sinh^2 x = 1$
4. $\cosh x + \sinh x = e^x$
5. $\cosh x - \sinh x = e^{-x}$
6. $\sinh(2x) = 2 \sinh x \cosh x$
7. $\cosh(2x) = \sinh^2 x + \cosh^2 x$
8. $(\sinh x + \cosh x)^n = \sinh(nx) + \cosh(nx)$ where $n \in \mathbb{R}$
9. $1 - \tanh^2 x = \operatorname{sech}^2 x$
10. $\coth^2 x - 1 = \operatorname{csch}^2 x$
11. $\sinh(x + y) = \sinh x \cosh y + \sinh y \cosh x$
12. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

13. $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

14. $\tanh(\ln x) = \dfrac{x^2 - 1}{x^2 + 1}$

15. $\dfrac{1 + \tanh x}{1 - \tanh x} = e^{2x}$

Example 1.52. Prove the identity $\cosh^2 x - \sinh^2 x = 1$.

Solution 1.52.

$$\begin{aligned}\cosh^2 x - \sinh^2 x = 1 &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} = 1\end{aligned}$$

□

Exercise 1.15. If $\cosh x = \frac{5}{3}$ and $x > 0$, find the values of the other hyperbolic functions at x.

Inverse Hyperbolic Functions You can see from Figure 1.31 that sinh and tanh are one-to-one functions and so they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$. Also, Figure 1.31 shows that cosh is not one-to-one, but when restricted to the domain $[0, \infty)$ it becomes one-to-one. The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

Since the hyperbolic functions are defined in terms of exponential functions, its not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms. In particular, we have:

$$\begin{aligned}\sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right) \quad & x \in \mathbb{R} \\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right) \quad & x \geq 1 \\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad & -1 < x < 1\end{aligned}$$

Example 1.53. Show that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

Solution 1.53. Let $y = \sinh^{-1} x$. Then $x = \sinh y = \left(e^y - e^{-y}\right)/2$, so $e^y - 2x - e^{-y} = 0$, or, multiplying by e^y, we get $e^{2y} - 2xe^y - 1 = 0$. This is really a quadratic equation in e^y:

$$\left(e^y\right)^2 - 2x\left(e^y\right) - 1 = 0$$

Solving by the quadratic formula, we get

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2+1} < 0$ (because $x < \sqrt{x^2+1}$). Thus the minus sign is inadmissible and we have $e^y = x + \sqrt{x^2+1}$. Therefore,

$$y = \ln\left(e^y\right) = \ln\left(x + \sqrt{x^2+1}\right)$$

□

Example 1.54. Find the numerical value of each expression.

1. $\cosh^{-1} 1$
2. $\sinh^{-1} 1$

Solution 1.54.

1. $\cosh^{-1} 1 = \ln\left(1 + \sqrt{1^2 - 1}\right) = \ln 1 = 0$
2. $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right)$

□

Chapter 2

Limits and Continuity

2.1 An Introduction to Limits

We could begin by saying that limits are important in calculus, but that would be a major understatement. Without limits, calculus would not exist. Every single notion of calculus is a limit in one sense or another. For example: What is the slope of a curve? It is the limit of slopes of secant lines. What is the length of a curve? It is the limit of the lengths of polygonal paths inscribed in the curve. What is the area of a region bounded by a curve? It is the limit of the sum of areas of approximating rectangles.

The informal description of a limit is as follows: If $f(x)$ becomes arbitrarily close to a single number L as x approaches a from either side, the **limit** of $f(x)$, as x approaches a, is L. The existence or nonexistence of $f(x)$ at $x = a$ has no bearing on the existence of the limit of $f(x)$ as x approaches a, see Figure 2.1.

The notation for a limit is

$$\lim_{x \to a} f(x) = L$$

which is read as **the limit of** $f(x)$ **as** x **approaches** a **is** L.

Example 2.1. Guess the value of

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$$

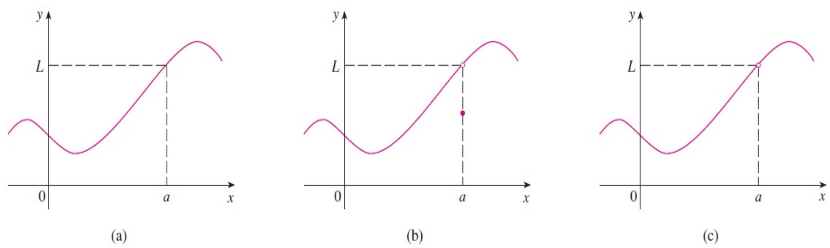


Figure 2.1:

Solution 2.1. Notice that the function $f(x) = \frac{x-1}{x^2-1}$ is not defined when $x = 1$, but that does not matter because the definition of $\lim_{x\to a} f(x)$ says that we consider values of x that are close to a but not equal to a.

The table below gives values of $f(x)$ (correct to six decimal places) for values of x that approach 1 (but are not equal to 1). On the basis of the values in the table, we make the guess that

$$\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$$

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

□

Definition 2.1.1. We write

$$\lim_{x\to a^-} f(x) = L$$

and say the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a from the left] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a. Similarly, if we require that x be greater than a, we get the **right-hand limit** of $f(x)$ as x approaches a is equal to L, and we write

$$\lim_{x\to a^+} f(x) = L$$

Thus the symbol $x \to a^-$ means that we consider only $x < a$, and the symbol $x \to a^+$ means that we consider only $x > a$. These definitions are illustrated in Figure 2.2.

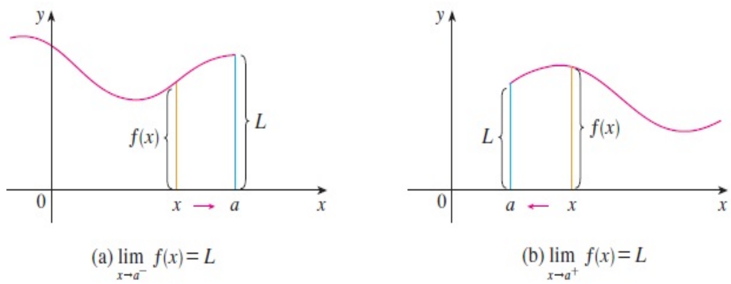


Figure 2.2:

The relationship between ordinary (two-sided) limits and one-sided limits can be stated as follows:

Theorem 2.1.1. *Let $L \in \mathbb{R}$. We say*

$$\lim_{x \to a} f(x) \ \textbf{\textit{exists and equals to}} \ L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Limits That Fail to Exist 1. If a function $f(x)$ approaches a different number from the right side of $x = c$ than it approaches from the left side, then the limit of $f(x)$ as x approaches a does not exist.

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

Illustration Example 2.1. The Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

Its graph is shown in Figure 2.3. As t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$

approaches 1. There is no single number that $H(t)$ approaches as t approaches 0. Therefore,

$$\lim_{t \to 0} H(t) \text{ does not exist.}$$

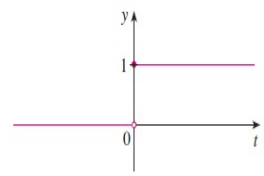


Figure 2.3:

□

2. If $f(x)$ is not approaching a real number L—that is, if $f(x)$ increases or decreases without bound—as x approaches a, you can conclude that the limit does not exist.

$$\lim_{x \to a} f(x) = \infty \text{ or } \lim_{x \to a} f(x) = -\infty$$

Illustration Example 2.2. As x becomes close to 0, x^2 also becomes close to 0, and $\frac{1}{x^2}$ becomes very large. (See the table below.) In fact, it appears from the graph of the function $f(x) = \frac{1}{x^2}$ shown in Figure 2.4 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so

$$\lim_{x \to 0} \frac{1}{x^2} \text{ does not exist.}$$

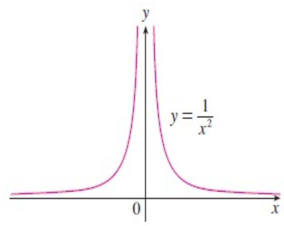


Figure 2.4:

x	$1/x^2$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10000
± 0.001	1000000

3. The limit of $f(x)$ as x approaches a also does not exist if $f(x)$ oscillates between two fixed values as x approaches a.

   Illustration Example 2.3. The values of $\sin\left(\frac{\pi}{x}\right)$ oscillate between 1 and -1 infinitely often as approaches 0, see Figure 2.5. Since the values of $f(x)$ do not approach a fixed number as x approaches 0, then

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

   does not exist.

□

Example 2.2. The graph of a function g is shown in Figure 2.6. Use it to state the values (if they exist) of the following.

1. $\lim_{x\to 2^-} g(x)$
2. $\lim_{x\to 2^+} g(x)$
3. $\lim_{x\to 2} g(x)$

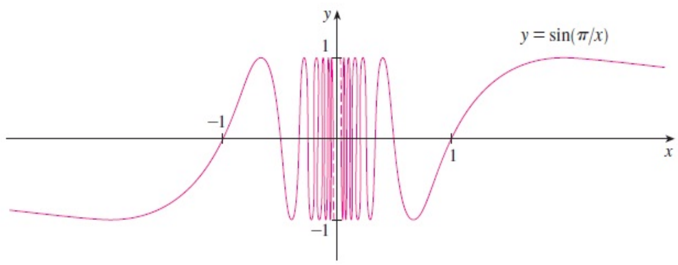


Figure 2.5:

4. $\lim_{x\to 5^-} g(x)$

5. $\lim_{x\to 5^+} g(x)$

6. $\lim_{x\to 5} g(x)$

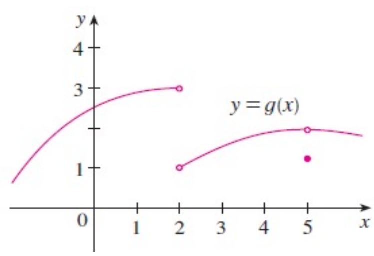


Figure 2.6:

Solution 2.2. From the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\lim_{x\to 2^-} g(x) = 3 \text{ and } \lim_{x\to 2^+} g(x) = 1$$

Since the left and right limits are different, we conclude that

$$\lim_{x\to 2} g(x)$$

does not exist. The graph also shows that

$$\lim_{x\to 5^-} g(x) = 2 \text{ and } \lim_{x\to 5^+} g(x) = 2$$

This time the left and right limits are the same and so, we have

$$\lim_{x\to 5} g(x) = 2.$$

Despite this fact, notice that $g(5) \neq 2$.

□

Exercise 2.1. For the function f whose graph is given below, state the value of each quantity, if it exists. If it does not exist, explain why.

1. $\lim_{x\to 0} f(x)$
2. $\lim_{x\to 3^-} f(x)$
3. $\lim_{x\to 3^+} f(x)$
4. $\lim_{x\to 3} f(x)$
5. $f(3)$

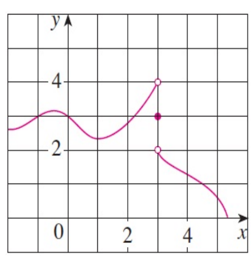


Figure 2.7:

2.2 Calculating Limits using the Limit Laws

In this section we use the following properties of limits, called the Limit Laws, to calculate limits.

Theorem 2.2.1. *Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then*

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x) \times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *if* $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$ *where n is a positive integer.*
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ *where n is a positive integer.*
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ *where n is a positive integer, and if n is even, we assume that $a > 0$.*
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ *where n is a positive integer, and if n is even, we assume that* $\lim_{x\to a} f(x) > 0$.

Example 2.3. Given that

$$\lim_{x\to 0.5} f(x) = 2 \quad \text{and} \quad \lim_{x\to 0.5} g(x) = -1$$

find

$$\lim_{x\to 0.5} [f(x) - 2g(x)]$$

Solution 2.3.

$$\begin{aligned}\lim_{x\to 0.5}[f(x)-2g(x)] &= \lim_{x\to 0.5} f(x) - \lim_{x\to 0.5}[2g(x)] \\ &= \lim_{x\to 0.5} f(x) - 2\lim_{x\to 0.5}[g(x)] \\ &= 2-2(-1)=4\end{aligned}$$

□

Example 2.4. Evaluate

$$\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x}$$

Solution 2.4.

$$\begin{aligned}\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} &= \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)} \\ &= \frac{\lim_{x\to -2}(x^3)+\lim_{x\to -2}(2x^2)-\lim_{x\to -2}(1)}{\lim_{x\to -2}(5)-\lim_{x\to -2}(3x)} \\ &= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} \\ &= = -\frac{1}{11}\end{aligned}$$

Note that if we let $f(x)=\frac{x^3+2x^2-1}{5-3x}$, then $f(-2)=-\frac{1}{11}$. In other words, we would have gotten the correct answer by directly substituting -2 for x.

□

Remark 2.2.1. **Direct Substitution Property:** If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x)=f(a)$$

For example,

$$\lim_{x\to 1}\left(x^7-3x^5+1\right)^{19}=\left[1^7-3\left(1^5\right)+1\right]^{19}=(-1)^{19}=-1$$

because $f(x) = (x^7 - 3x^5 + 1)^{19}$ is a polynomial whose domain is $\mathbb{R}$ and $1 \in \mathbb{R}$.

Functions with the Direct Substitution Property are called **continuous** at a and will be studied in Section 2.5. However, not all limits can be evaluated by direct substitution. The next examples show various ways algebraic manipulations can be used to evaluate $\lim_{x\to a} f(x)$ in situations where $f(a)$ is undefined. This usually happens when $f(x)$ is a fraction with denominator equal to 0 at $x = a$. Note, each of these limits involves a fraction whose numerator and denominator are both 0 at the point where the limit is taken.

Example 2.5. Evaluate

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3}$$

Solution 2.5. Let $f(x) = \frac{x^2-9}{x-3}$. We can not find the limit by substituting $x = 3$ because $f(3)$ is not defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{(x-3)(x+3)}{x-3}$$

The numerator and denominator have a common factor of $x - 3$. When we take the limit as approaches 3, we have $x \neq 3$ and so $x - 3 \neq 0$. Therefore we can cancel the common factor and compute the limit as follows:

$$\lim_{x\to 3} \frac{x^2 - 9}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x + 3) = 6$$

□

Exercise 2.2. Evaluate the limit, if it exists.

1. $\lim_{x\to -2} \frac{x^2+3x+2}{x^3+8}$

2. $\lim_{x\to -1} \frac{x^2+2x+1}{x^4-1}$

Example 2.6. Find

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$$

Solution 2.6. We can not apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} &= \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\\ &= \lim_{t\to 0}\frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)}\\ &= \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}\\ &= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}\\ &= \frac{1}{6}\end{aligned}$$

□

Exercise 2.3. Evaluate the limit, if it exists.

1. $\lim_{x\to 16}\frac{4-\sqrt{x}}{16x-x^2}$
2. $\lim_{x\to 8}\frac{x-8}{\sqrt[3]{x}-2}$

Example 2.7. Find

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$$

Solution 2.7. We can not apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing both the numerator and the denominator by multiplying

$$\frac{\sqrt{x}+1}{\sqrt{x}+1}\cdot\frac{\sqrt[3]{x^2}+\sqrt[3]{x}+1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}$$

which takes long time in calculations. The **Substitution** method is much better in this example. The idea is: write the problem using other variable y so that the problem will transform to nice form that can easily solve. So, let

$$x = y^6 \text{ where } 6 = \text{LCM}(2,3)$$

Also, as $x \to 1$ we have $y \to \sqrt[6]{1} = 1$. Hence,

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1} &= \lim_{y\to 1} \frac{\sqrt{y^6}-1}{\sqrt[3]{y^6}-1} \\
&= \lim_{y\to 1} \frac{y^3-1}{y^2-1} \\
&= \lim_{y\to 1} \frac{(y-1)(y^2+y+1)}{(y-1)(y+1)} \\
&= \lim_{y\to 1} \frac{y^2+y+1}{y+1} \\
&= \frac{3}{2}
\end{aligned}$$

□

Exercise 2.4. Evaluate

$$\lim_{t\to 0} \frac{e^{2t}-1}{e^t-1}$$

Example 2.8. Find

$$\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{x+4}$$

Solution 2.8. We can not apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of simplifying the numerator:

$$\begin{aligned}
\lim_{x\to -4} \frac{\frac{1}{4}+\frac{1}{x}}{x+4} &= \lim_{x\to -4} \left(\frac{1}{4}+\frac{1}{x}\right) \div (x+4) \\
&= \lim_{x\to -4} \frac{x+4}{4x} \cdot \frac{1}{x+4} \\
&= \lim_{x\to -4} \frac{1}{4x} \\
&= -\frac{1}{16}
\end{aligned}$$

Exercise 2.5. Evaluate

$$\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$$

A function f may be defined on both sides of $x = a$ but still not have a limit at $x = a$. The following example shows that even if $f(x)$ is defined at $x = a$, the limit of $f(x)$ as x approaches a may not be equal to $f(a)$.

Example 2.9. Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x+1 & \text{if} \quad x \neq 1 \\ \pi & \text{if} \quad x = 1 \end{cases}$$

Solution 2.9. Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as approaches 1 does not depend on the value of the function at 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} (x+1) = 2$$

□

Some limits are best calculated by first finding the left− and right−hand limits as shown in the following examples.

Example 2.10. If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if} \quad x > 4 \\ 8 - 2x & \text{if} \quad x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

Solution 2.10. Since $f(x) = \sqrt{x-4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x-4} = \sqrt{4-4} = 0$$

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \times 4 = 0$$

The right− and left−hand limits are equal. Thus the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

Example 2.11. If

$$f(x) = \frac{|x-2|}{x^2+x-6}$$

find: $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $\lim_{x\to 2} f(x)$

Solution 2.11. Observe that

$$|x-2| = \begin{cases} x-2 & \text{if} \quad x \geq 2 \\ -(x-2) & \text{if} \quad x < 2 \end{cases}$$

Therefore,

$$\begin{aligned} \lim_{x\to 2^+} \frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^+} \frac{x-2}{(x+3)(x-2)} \\ &= \lim_{x\to 2^+} \frac{1}{x+3} \\ &= \frac{1}{5} \\ \lim_{x\to 2^-} \frac{|x-2|}{x^2+x-6} &= \lim_{x\to 2^-} \frac{-(x-2)}{(x+3)(x-2)} \\ &= \lim_{x\to 2^-} \frac{-1}{x+3} \\ &= -\frac{1}{5} \end{aligned}$$

Since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, then the limit $\lim_{x\to 2} f(x)$ does not exist.

□

Exercise 2.6. Let

$$f(x) = \frac{x^2-1}{|x-1|}$$

find: $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1^-} f(x)$, and $\lim_{x\to 1} f(x)$

Example 2.12. Let $g(x) = \sqrt{1-x^2}$ find $\lim_{x\to 1} g(x)$.

Solution 2.12. The domain of g is $[-1, 1]$, so $g(x)$ is defined only to the right of $x = -1$ and to the left of $x = 1$. As can be seen in Figure 2.8,

$$\lim_{x\to 1^-} g(x) = 0 \quad \text{and} \quad \lim_{x\to 1^+} g(x) = 0 \quad \text{does not exist}$$

Therefore,

$$\lim_{x\to 1} g(x) = 0 \quad \text{does not exist}$$

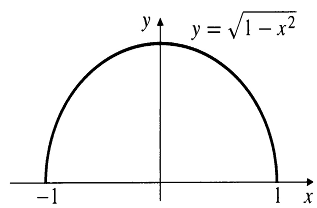


Figure 2.8:

Exercise 2.7. In the previous example, find

$$\lim_{x\to -1} g(x)$$

The following theorem will enable us to calculate some very important limits. It is called the **Squeeze Theorem** because it refers to a function g whose values are squeezed between the values of two other functions f and h that have the same limit L at a point a. Being trapped between the values of two functions that approach L, the values of f must also approach L. (See Figure 2.9.)

Theorem 2.2.2. *If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L$$

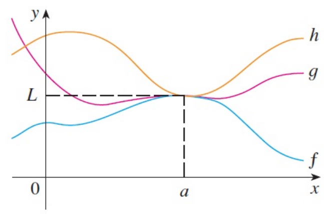


Figure 2.9:

Example 2.13. Show that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

Solution 2.13. First note that we cannot use

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

because $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. However, since

$$-1 \le \sin\frac{1}{x} \le 1$$

we have, as illustrated by Figure 2.10,

$$-x^2 \le \sin\frac{1}{x} \le x^2$$

We know that

$$\lim_{x\to 0} x^2 = 0 \quad \text{and} \quad \lim_{x\to 0} \left(-x^2\right) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin\frac{1}{x}$, and $h(x) = x^2$ in the Squeezing Theorem, we obtain

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

□

Exercise 2.8. If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find

$$\lim_{x\to 4} f(x)$$

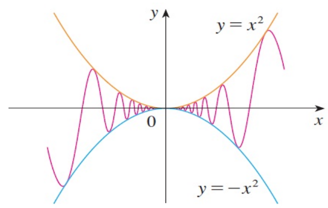


Figure 2.10:

Example 2.14. If

$$\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$$

find $\lim_{x\to 1} f(x)$.

Solution 2.14. Since $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$ then the limit exists, and since $\lim_{x\to 1}(x-1) = 0$ then $\lim_{x\to 1}[f(x)-8] = 0$. Therefore,

$$\lim_{x\to 1} f(x) = 8$$

□

Exercise 2.9. If

$$\lim_{x\to 0} \frac{f(x)}{x^2} = 5$$

find

$$\lim_{x\to 0} f(x) \quad \text{and} \quad \lim_{x\to 0} \frac{f(x)}{x}$$

2.3 Limits at Infinity and Infinite Limits

In this section we will extend the concept of limit to allow for two situations:

1. limits at infinity, where x becomes arbitrarily large, positive or negative;

2. infinite limits, which are not really limits at all but provide useful symbolism for describing the behavior of functions whose values become arbitrarily large, positive or negative.

Definition 2.3.1. If the function f is defined on an interval (a, ∞) and if we can ensure that $f(x)$ is as close as we want to the number L by taking x large enough, then we say that $f(x)$ approaches the limit L as x approaches infinity, and we write

$$\lim_{x\to\infty} f(x) = L$$

If the function f is defined on an interval $(-\infty, b)$ and if we can ensure that $f(x)$ is as close as we want to the number M by taking x negative and large enough, then we say that $f(x)$ approaches the limit M as x approaches negative infinity, and we write

$$\lim_{x\to-\infty} f(x) = M$$

Illustration Example 2.4. Consider the function

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$

whose graph is shown in Figure 2.11 and for which some values (rounded to 7 decimal places) are given in the table below. The values of $f(x)$ seem to approach 1 as x takes on larger and larger positive values, and -1 as x takes on negative values that get larger and larger in absolute value. We express this behavior by writing

$$\lim_{x\to\infty} f(x) = 1 \quad \text{and} \quad \lim_{x\to-\infty} f(x) = -1$$

x	$f(x) = x/\sqrt{x^2+1}$
-1000	-0.9999995
-100	-0.9999500
-10	-0.9950372
-1	-0.7071068
0	0
1	0.7071068
10	0.9950372
100	0.9999500
1000	0.9999995

The graph of f conveys this limiting behavior by approaching the horizontal lines $y = 1$ as x moves far to the right and $y = -1$ as x moves far to the left.

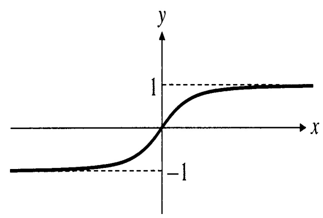


Figure 2.11:

These lines are called **horizontal asymptotes** of the graph. In general, if a curve approaches a straight line as it recedes very far away from the origin, that line is called an **asymptote** of the curve.

□

Definition 2.3.2. A line $y = b$ is a **horizontal asymptote** of the graph of a function $f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Example 2.15 (Polynomial behavior at infinity)**.** Find:

1. $\lim_{x\to\infty}(3x^3 - x^2 + 2)$
2. $\lim_{x\to-\infty}(3x^3 - x^2 + 2)$
3. $\lim_{x\to\infty}(x^4 - 5x^3 - x)$
4. $\lim_{x\to-\infty}(x^4 - 5x^3 - x)$

Solution 2.15. The highest−degree term of a polynomial dominates the other terms as $|x|$ grows large, so the limits of this term at ∞ and $-\infty$ determine the limits of the whole polynomial. So,

1. $\lim_{x\to\infty}(3x^3 - x^2 + 2) = \lim_{x\to\infty}(3x^3) = 3 \times \infty^3 = 3 \times \infty = \infty$
2. $\lim_{x\to-\infty}(3x^3 - x^2 + 2) = \lim_{x\to-\infty}(3x^3) = 3\times(-\infty)^3 = 3\times(-\infty) = -\infty$
3. $\lim_{x\to\infty}(x^4 - 5x^3 - x) = \lim_{x\to\infty}(x^4) = \infty^4 = \infty$

4. $\lim_{x\to-\infty}(x^4-5x^3-x)=\lim_{x\to-\infty}(x^4)=(-\infty)^4=\infty$

□

The only polynomials that have limits at $\pm\infty$ are constant ones, $P(x)=c$. The situation is more interesting for rational functions. Recall that a rational function is a quotient of two polynomials. The following examples show how to render such a function in a form where its limits at infinity and negative infinity (if they exist) are apparent. The way to do this is: **to divide the numerator and denominator by the highest power of x appearing in the denominator, then use the following theorem**.

Theorem 2.3.1. *If $r>0$ is a rational number such that x^r is defined for all x, then*

$$\lim_{x\to\pm\infty}\frac{1}{x^r}=0$$

Remark 2.3.1. The limits of a rational function at infinity and negative infinity either both fail to exist or both exist and are equal.

Example 2.16. Evaluate

$$\lim_{x\to\pm\infty}\frac{2x^2-x+3}{3x^2+5}$$

Solution 2.16. Divide the numerator and the denominator by x^2 , the highest power of x appearing in the denominator:

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{2x^2-x+3}{3x^2+5} &= \lim_{x\to\pm\infty}\frac{2-(1/x)+(3/x^2)}{3+(5/x^2)}\\ &= \lim_{x\to\pm\infty}\frac{2-0+0}{3+0}\\ &= \frac{2}{3}\end{aligned}$$

Therefore, $y=\frac{2}{3}$ is horizontal asymptote of $\frac{2x^2-x+3}{3x^2+5}$.

□

Example 2.17. Evaluate

$$\lim_{x\to\pm\infty} \frac{5x+2}{2x^3-1}$$

Solution 2.17. Divide the numerator and the denominator by the largest power of x in the denominator, namely, x^3.

$$\begin{aligned} \lim_{x\to\pm\infty} \frac{5x+2}{2x^3-1} &= \lim_{x\to\pm\infty} \frac{(5/x^2)+(2/x^3)}{2-(1/x^3)} \\ &= \lim_{x\to\pm\infty} \frac{0+0}{2-0} \\ &= 0 \end{aligned}$$

Therefore, $y = 0$ (the $x-$axis) is horizontal asymptote of $\frac{5x+2}{2x^3-1}$.

□

Example 2.18. Evaluate

$$\lim_{x\to\infty} \frac{x^3+1}{x^2+1}$$

Solution 2.18. Divide the numerator and the denominator by x^2, the largest power of x in the denominator:

$$\begin{aligned} \lim_{x\to\infty} \frac{x^3+1}{x^2+1} &= \lim_{x\to\infty} \frac{x+(1/x^2)}{1+(1/x^2)} \\ &= \lim_{x\to\infty} \frac{\infty+0}{1+0} \\ &= \infty \end{aligned}$$

Also,

$$\lim_{x\to-\infty} \frac{x^3+1}{x^2+1} = -\infty$$

Therefore the limits are not exist, and the function $\frac{x^3+1}{x^2+1}$ has no horizontal asymptotes.

□

Summary of limits at $\pm\infty$ for rational functions Let

$$\begin{aligned} P_m(x) &= a_m x^m + \cdots + a_0 \\ \text{and } Q_n(x) &= b_n x^n + \cdots + b_0 \end{aligned}$$

be polynomials of degree m and n, respectively, so that $a_m \neq 0$ and $b_n \neq 0$. Then

$$\lim_{x\to\pm\infty} \frac{P_m(x)}{Q_n(x)}$$

1. equals 0 if $m < n$,
2. equals $\frac{a_m}{b_n}$ if $m = n$,
3. does not exist if $m > n$.

The technique used in the previous examples can also be applied to more general kinds of functions. The function in the following example is not rational, and the limit seems to produce a meaningless $\infty - \infty$ until we resolve matters by rationalizing the numerator.

Example 2.19. Evaluate

$$\lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right)$$

Solution 2.19. We are trying to find the limit of the difference of two functions, each of which becomes arbitrarily large as x increases to infinity. We rationalize the expression by multiply ing the numerator and the denominator (which is 1) by the conjugate expression, $\sqrt{x^2+x}+x$:

$$\begin{aligned} \lim_{x\to\infty} \left(\sqrt{x^2+x} - x\right) &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x} - x\right)\left(\sqrt{x^2+x} + x\right)}{\sqrt{x^2+x}+x} \\ &= \lim_{x\to\infty} \frac{x^2+x-x^2}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x} \\ &= \lim_{x\to\infty} \frac{x}{x\sqrt{1+\frac{1}{x}}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}+1} = \frac{1}{2} \end{aligned}$$

Here $\sqrt{x^2} = |x| = x$ because $x > 0$ as $x \to \infty$.

Exercise 2.10. Evaluate

$$\lim_{x\to-\infty}\left(\sqrt{x^2+x}-x\right)$$

Example 2.20. Using the Squeezing Theorem, find the horizontal asymptote of the curve

$$f(x) = 2 + \frac{\sin x}{x}$$

Solution 2.20. We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and

$$\lim_{x\to\pm\infty}\left|\frac{1}{x}\right| = 0,$$

we have

$$\lim_{x\to\pm\infty}\left|\frac{\sin x}{x}\right| = 0$$

by the Squeezing Theorem. Hence,

$$\lim_{x\to\pm\infty}\left(2+\frac{\sin x}{x}\right) = 2+0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right.

□

Example 2.21. Find the horizontal asymptotes (if any) of the following functions.

1. $f(x) = \tan^{-1} x$
2. $g(x) = e^x$

Solution 2.21. 1. In fact,

$$\lim_{x\to\infty}\tan^{-1}x = \frac{\pi}{2} \text{ and } \lim_{x\to-\infty}\tan^{-1}x = -\frac{\pi}{2}$$

so both of the lines $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$ are horizontal asymptotes.

2. The graph of the natural exponential function e^x has the line $y = 0$ (the $x-$axis) as a horizontal asymptote. (The same is true of any exponential function with base $a > 1$.) In fact

$$\lim_{x\to-\infty} e^x = 0 \text{ while } \lim_{x\to\infty} e^x = \infty$$

so the line $y = 0$ is horizontal asymptote of $g(x) = e^x$.

□

Example 2.22. Use the definitions of the hyperbolic functions to find each of the following limits.

1. $\lim_{x\to\infty} \tanh x$
2. $\lim_{x\to-\infty} \sinh x$

Solution 2.22.

1. $\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x - e^{-x}} = \lim_{x\to\infty} \frac{e^x}{e^x} = 1$
2. $\lim_{x\to-\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \lim_{x\to\infty} \frac{-e^{-x}}{2} = -\infty$

□

Exercise 2.11. Use the definitions of the hyperbolic functions to find each of the following limits.

1. $\lim_{x\to-\infty} \tanh x$
2. $\lim_{x\to\infty} \sinh x$
3. $\lim_{x\to\infty} \text{sech } x$

Oblique Asymptotes If the degree of the numerator of a rational function is one greater than the degree of the denominator, the graph has an **oblique (slanted) asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$ Heres an example.

Example 2.23. Find the oblique asymptote for the graph of

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

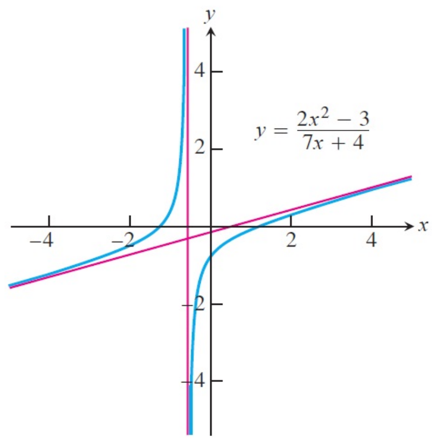


Figure 2.12:

Solution 2.23. By long division, we find

$$\begin{aligned} f(x) &= \frac{2x^2-3}{7x+4} \\ &= \underbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}_{\text{linear function } g(x)} + \underbrace{\frac{-115}{49(7x+4)}}_{\text{remainder}} \end{aligned}$$

As $x \to \pm\infty$ the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the (slanted) line

$$g(x) = \frac{2}{7}x - \frac{8}{49}$$

an asymptote of the graph of f (Figure 2.12). The line is an asymptote both to the right and to the left.

□

Infinite Limits and Vertical Asymptotes Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large, positive or negative. We continue our analysis of graphs of rational functions using vertical asymptotes and dominant terms for numerically large values of x.

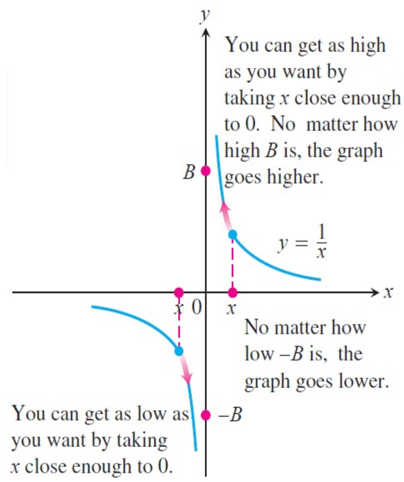


Figure 2.13:

Illustration Example 2.5. Let us look again at the function $f(x) = \frac{1}{x}$. As $x \to 0^+$ the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.13). Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1}{x} = \infty$$

In writing this, we are not saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, we are saying that $\lim_{x\to 0^+} \frac{1}{x}$ does not exist because $\frac{1}{x}$ becomes arbitrarily large and positive as $x \to 0^+$. As $x \to 0^-$, the values of $f(x) = \frac{1}{x}$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.13.) We write

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There is no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ does not exist because its values become arbitrarily large and negative.

□

Example 2.24. Find

$$\lim_{x\to 1^+} \frac{1}{x-1} \quad \text{and} \quad \lim_{x\to 1^-} \frac{1}{x-1}$$

Solution 2.24. Think about the number $x-1$ and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $\frac{1}{x-1} \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $\frac{1}{x-1} \to -\infty$.

□

Rational functions can behave in various ways near zeros of their denominators. See the following illustration example.

Illustration Example 2.6.

(a) $$\lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$$

(b) $$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$$

(c) $$\lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$$

(d) $$\lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$$

(e) $$\lim_{x\to 2} \frac{x-3}{x^2-4} = \lim_{x\to 2} \frac{x-3}{(x-2)(x+2)} \text{ does not exist}$$

(f) $$\lim_{x\to 2} \frac{2-x}{(x-2)^3} = \lim_{x\to 2} \frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2} \frac{-1}{(x-2)^2} = -\infty$$

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancelation still leaves a zero in the denominator.

□

Definition 2.3.3. A line $x = a$ is a **vertical asymptote** of the graph of a function $f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty$$

Example 2.25. Find the horizontal and vertical asymptotes of the curve

$$f(x) = \frac{x+3}{x+2}$$

Solution 2.25. We are interested in the behavior as $x \to \pm\infty$ and as $x \to -2$ where the denominator is zero. Since

$$\lim_{x\to\pm\infty} \frac{x+3}{x+2} = \lim_{x\to\pm\infty} \frac{1+\frac{3}{x}}{1+\frac{2}{x}} = 1$$

and

$$\lim_{x\to-2^+} \frac{x+3}{x+2} = \infty \quad \text{and} \quad \lim_{x\to-2^+} \frac{x+3}{x+2} = -\infty$$

then: the horizontal asymptote is $y = 1$ and the vertical asymptote is $x = -2$.

□

Example 2.26. Find the horizontal and vertical asymptotes of the curve

$$g(x) = \frac{-8}{x^2-4}$$

Solution 2.26. We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$ where the denominator is zero. Notice that g is an even function of x, so its graph is symmetric with respect to the $y-$axis. Since

$$\lim_{x\to\pm\infty} \frac{-8}{x^2-4} = 0$$

the line $y = 0$ is a horizontal asymptote of the graph of g. Since

$$\lim_{x\to 2^+} \frac{-8}{x^2-4} = -\infty \quad \text{and} \quad \lim_{x\to 2^-} \frac{-8}{x^2-4} = \infty$$

and

$$\lim_{x\to -2^+} \frac{-8}{x^2-4} = \infty \quad \text{and} \quad \lim_{x\to -2^-} \frac{-8}{x^2-4} = -\infty$$

then the lines $x = -2$ and $x = 2$ are vertical asymptotes of the graph of g.

□

Example 2.27. Find the asymptotes of the graph of

$$f(x) = \frac{x^2-3}{2x-4}$$

Solution 2.27. We are interested in the behavior as $x \to \pm\infty$ and as $x \to 2$ where the denominator is zero. First, we divide $(2x-4)$ into (x^2-3) to obtain

$$f(x) = \underbrace{\frac{x}{2}+1}_{\text{linear}} + \underbrace{\frac{1}{2x-4}}_{\text{remainder}}$$

Since

$$\lim_{x\to 2^+} f(x) = \infty \quad \text{and} \quad \lim_{x\to 2^+} f(x) = -\infty$$

the line $x = 2$ is a vertical asymptote. As $x \to \pm\infty$, the remainder approaches 0 and

$$f(x) \to \frac{x}{2} + 1$$

The line $y = \frac{x}{2} + 1$ is an oblique asymptote, see Figure 2.14.

□

Exercise 2.12. Find the asymptotes of the function.

1. $f(x) = \frac{x^2+x-6}{x^2-9}$
2. $g(x) = \frac{x^3+1}{x^2}$

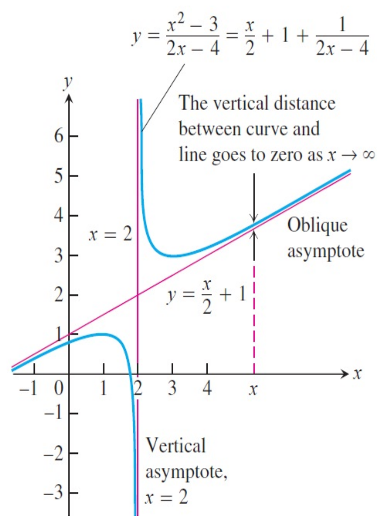


Figure 2.14:

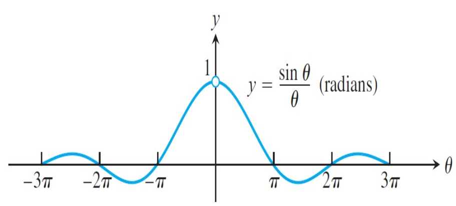


Figure 2.15:

2.4 Limits Involving $(\sin\theta)/\theta$

A central fact about $(\sin\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.15 and confirm it algebraically using the Squeezing Theorem.

Theorem 2.4.1.

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \textit{ in radians.})$$

Corollary 2.4.2.

$$\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1 \qquad (\theta \textit{ in radians.})$$

Corollary 2.4.3.

$$(1)\qquad \lim_{\theta\to 0}\frac{\sin a\theta}{b\theta}=\lim_{\theta\to 0}\frac{a\theta}{\sin b\theta}=\lim_{\theta\to 0}\frac{\sin a\theta}{\sin b\theta}=\frac{a}{b}$$

$$(2)\qquad \lim_{\theta\to 0}\frac{\tan a\theta}{b\theta}=\lim_{\theta\to 0}\frac{a\theta}{\tan b\theta}=\lim_{\theta\to 0}\frac{\tan a\theta}{\tan b\theta}=\frac{a}{b}$$

$$(3)\qquad \lim_{\theta\to 0}\frac{\tan a\theta}{\sin b\theta}=\lim_{\theta\to 0}\frac{\sin a\theta}{\tan b\theta}=\frac{a}{b}$$

Example 2.28. Show that

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0$$

Solution 2.28. Using the half−angle formula $\cos x = 1-2\sin^2(x/2)$, we calculate

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\left[1-2\sin^2(x/2)\right]}{x}\\
&= \lim_{x\to 0}\left[\frac{2\sin\left(\frac{1}{2}x\right)}{x}\sin\left(\frac{1}{2}x\right)\right]\\
&= 2\times\lim_{x\to 0}\frac{\sin\left(\frac{1}{2}x\right)}{x}\times\lim_{x\to 0}\sin\left(\frac{1}{2}x\right)\\
&= 2\times\frac{1/2}{1}\times 0 = 0
\end{aligned}$$

Exercise 2.13. Show that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta\sin\theta}=\frac{1}{2}$$

by:

a) using the half−angle formula as in the previous example,

b) multiplying both the nominator and denominator by $1+\cos\theta$.

Example 2.29. Evaluate

$$\lim_{x\to 0}\frac{5\sin 3x+\tan 7x}{3x+x^2}$$

Solution 2.29.

$$\begin{aligned}\lim_{x\to 0}\frac{5\sin 3x+\tan 7x}{3x+x^2} &= \lim_{x\to 0}\frac{5\frac{\sin 3x}{x}+\frac{\tan 7x}{x}}{\frac{3x}{x}+\frac{x^2}{x}}\\ &= \lim_{x\to 0}\frac{5\frac{\sin 3x}{x}+\frac{\tan 7x}{x}}{3+x}\\ &= \frac{5\times\frac{3}{1}+\frac{7}{1}}{3+0}=\frac{22}{3}\end{aligned}$$

□

Exercise 2.14. Evaluate

$$\lim_{x\to 0}\frac{4x}{\tan 3x+\sin 2x}$$

Example 2.30. Evaluate

$$\lim_{t\to 0}\frac{\sin(t^2)}{t}$$

Solution 2.30.

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(t^2)}{t} &= \lim_{t\to 0}\left[\frac{\sin(t^2)}{t^2}\times t\right]\\ &= \lim_{t\to 0}\frac{\sin(t^2)}{t^2}\times\lim_{t\to 0}t\\ &= 1\times 0=0\end{aligned}$$

Example 2.31. Evaluate

$$\lim_{t\to 0}\frac{\sin^2 t}{3t^2}$$

Solution 2.31.

$$\begin{aligned}\lim_{t\to 0}\frac{\sin^2 t}{3t^2} &= \frac{1}{3}\lim_{t\to 0}\left[\frac{\sin t}{t}\right]^2\\ &= \frac{1}{3}\left[\lim_{t\to 0}\frac{\sin t}{t}\right]^2=\frac{1}{3}\times(1)^2=\frac{1}{3}\end{aligned}$$

□

Example 2.32. Evaluate

$$\lim_{t \to 5} \frac{\tan(t-5)}{t^2-25}$$

Solution 2.32.

$$\begin{aligned}
\lim_{t \to 5} \frac{\tan(t-5)}{t^2-25} &= \lim_{t \to 5} \frac{\tan(t-5)}{(t-5)(t+5)} \\
&= \lim_{t \to 5} \frac{\tan(t-5)}{t-5} \times \lim_{t \to 5} \frac{1}{t+5} = 1 \times \frac{1}{5+5} = \frac{1}{10}
\end{aligned}$$

□

Exercise 2.15. Evaluate

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x-1} \quad \text{and} \quad \lim_{x \to 0} \frac{\tan x}{|x|}$$

2.5 Continuous Functions

We noticed in Section 2.2 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called **continuous at** a. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word continuity in everyday language. (A continuous process is one that takes place gradually, without interruption or abrupt change.)

Definition 2.5.1. A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Notice that Definition 2.5.1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$.

We say that f is **discontinuous** at a (or has a **discontinuity** at a) if f is not continuous at a.

Example 2.33. Figure 2.16 shows the graph of a function f. At which numbers is f **discontinuous**? Why?

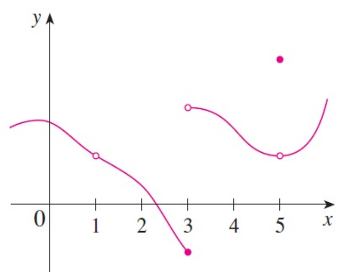


Figure 2.16:

Solution 2.33. It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined. The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x\to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3. What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x\to 5} f(x)$ exists (because the left and right limits are the same). But $\lim_{x\to 5} f(x) \neq f(5)$. So f is discontinuous at 5.

Exercise 2.16. From the graph of g in Figure 2.17 below, state the intervals on which g is continuous.

Example 2.34. Where are each of the following functions discontinuous?

$$\text{(a)} \quad f(x) = \frac{x^2 - x - 2}{x - 2} \qquad\qquad \text{(b)} \quad g(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases}$$

Solution 2.34. (a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we will see why f is continuous at all other numbers.

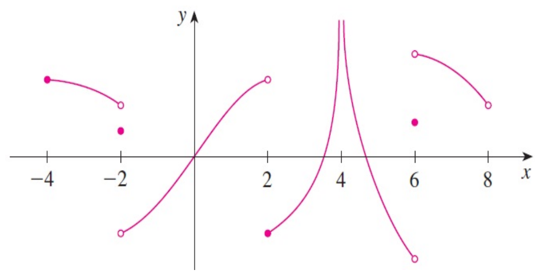


Figure 2.17:

(b) Here $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x \to 2}(x+1) = 3$$

exists. But

$$\lim_{x \to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

$\square$

Exercise 2.17. Explain why the function

$$f(x) = \begin{cases} \cos x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 - x^2 & \text{if} \quad x > 0 \end{cases}$$

is discontinuous at $a = 0$?

Definition 2.5.2. A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition 2.5.3. A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Example 2.35. Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

Solution 2.35. If $-1 < a < 1$, then using the Limit Laws, we have

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x \to a} \sqrt{1 - x^2} \\
&= 1 - \sqrt{1 - a^2} \\
&= f(a)
\end{aligned}$$

Thus, by Definition 2.5.1, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x \to -1^+} f(x) = 1 = f(-1) \quad \text{and} \quad \lim_{x \to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 2.5.3, f is continuous on $[-1, 1]$.

□

Instead of always using Definitions 2.5.1, 2.5.2, and 2.5.3 to verify the continuity of a function as we did in the previous example, it is often convenient to use the next two theorems, which shows how to build up complicated continuous functions from simple ones.

Theorem 2.5.1. *If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:*

1. $f + g$ ***2.*** $f - g$ ***3.*** cf
4. $f \times g$ ***5.*** $\frac{f}{g}$ *if* $g(a) \neq 0$

Note that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, fg, and f/g (if g is never 0).

Theorem 2.5.2. *All functions, except possibly piecewise functions, are continuous on its domain.*

Example 2.36. Find the points of discontinuity of the following.

1. $f(x) = \frac{1}{3-|x|}$
2. $f(x) = 2\ln\sqrt{1-x}$
3. $f(x) = 5 + 2^{1/x}$
4. $f(x) = \begin{cases} 2x-3 & \text{if} \quad x \leq 4 \\ 1+\frac{16}{x} & \text{if} \quad x > 4 \end{cases}$

Solution 2.36.

1. Since the domain of $f(x) = \frac{1}{3-|x|}$ is $\mathbb{R} - \{-3, 3\}$ then f is continuous on $\mathbb{R} - \{-3, 3\}$ and discontinuous on $\{-3, 3\}$.
2. Since the domain of $f(x) = 2\ln\sqrt{1-x}$ is $(-\infty, 1)$ then f is continuous on $(-\infty, 1)$ and discontinuous on $\mathbb{R} - (-\infty, 1) = [1, \infty)$.
3. Since the domain of $f(x) = 5 + 2^{1/x}$ is $\mathbb{R} - \{0\}$ then f is continuous on $\mathbb{R} - \{0\}$ and discontinuous only when $x = 0$.
4. Note that f is piecewise function. Since $f(x) = 2x-3$ when $x < 4$ then it is continuous because $2x-3$ is a polynomial continuous everywhere and has no discontinuity point. When $x > 4$, $f(x) = 1 + \frac{16}{x}$ also continuous, although it is discontinuous at $x = 0$ but $0 \not> 4$. So, the only possible discontinuity points of $f(x)$ is $x = 4$. However, $f(x)$ is continuous at $x = 4$ because:

   a) $f(4) = 2 \times 4 - 3 = 5$ is defined,

   b) Since

   $$\begin{aligned} \lim_{x\to 4^+} f(x) &= \lim_{x\to 4^+} \left(1 + \frac{16}{x}\right) = 5 \\ \lim_{x\to 4^-} f(x) &= \lim_{x\to 4^-} (2x-3) = 5 \end{aligned}$$

   then $\lim_{x\to 4} f(x)$ exists and equals 5.

   c) $f(4) = \lim_{x\to 4} f(x) = 5$.

Hence, $f(x)$ is continuous on $\mathbb{R}$ and has no discontinuity point.

□

Exercise 2.18. Where is the function

$$g(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$$

continuous?

Example 2.37. For what value of the constant c is the function continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if} \quad x < 2 \\ x^3 - cx & \text{if} \quad x \geq 2 \end{cases}$$

Solution 2.37. Because f is continuous on $\mathbb{R} = (-\infty, \infty)$ then it is continuous at $x = 2$ the piecewise point of f. Therefore, $\lim_{x \to 2} f(x)$ exists, and

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^-} f(x) \\ \lim_{x \to 2^+} \left(x^3 - cx\right) &= \lim_{x \to 2^-} \left(cx^2 + 2x\right) \\ 8 - 2c &= 4c + 4 \\ 6c &= 4 \\ c &= \frac{4}{6} = \frac{2}{3} \end{aligned}$$

□

Exercise 2.19. Find the values of a and b that make the function continuous everywhere.

(a) $$f(x) = \begin{cases} (\sin ax)/5x & \text{if} \quad x \neq 0 \\ 2 & \text{if} \quad x = 0 \end{cases}$$

(b) $$g(x) = \begin{cases} (x^2 - 4)/(x - 2) & \text{if} \quad x < 2 \\ ax^2 - bx + 3 & \text{if} \quad 2 \leq x < 3 \\ 2x - a + b & \text{if} \quad x \geq 3 \end{cases}$$

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

Theorem 2.5.3. *If g is continuous at a and f is continuous at $g(a)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.*

Theorem 2.5.4. *If f is continuous at b and $\lim_{x\to a} g(x) = b$ then*

$$\lim_{x\to a} f(g(x)) = f(b)$$

In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Example 2.38. Evaluate

$$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right)$$

Solution 2.38. Because $\sin^{-1}$ is a continuous function, we can apply Theorem 2.5.4:

$$\begin{aligned}
\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) &= \sin^{-1}\left(\lim_{x\to 1}\frac{1-\sqrt{x}}{1-x}\right)\\
&= \sin^{-1}\left(\lim_{x\to 1}\frac{1-x}{(1-\sqrt{x})(1+\sqrt{x})}\right)\\
&= \sin^{-1}\left(\lim_{x\to 1}\frac{1}{1+\sqrt{x}}\right)\\
&= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}
\end{aligned}$$

Exercise 2.20. Use continuity to evaluate $\lim_{x\to\pi} \sin(x + \sin x)$.

CHAPTER 3

The Derivative

The **derivative is a limit** measures the rate at which a function changes and is one of the most important ideas in calculus. Derivatives are used widely in science, economics, medicine, and computer science to calculate velocity and acceleration, to explain the behavior of machinery, to estimate the drop in water levels as water is pumped out of a tank, and to predict the consequences of making errors in measurements. Finding derivatives by evaluating limits can be lengthy and difficult. We develop techniques to make calculating derivatives easier.

3.1 The Derivative as a Function

The problem of finding the tangent line to a curve involves finding a type of limit. This special type of limit is called a **derivative**.

Definition 3.1.1. The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{w\to x}\frac{f(w)-f(x)}{w-x} \end{aligned}$$

provided the limit exists.

The domain of f' is the set of points in the domain of f for which the limit exists, and the domain may be the same or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative) at** x. If f' exists at every point in the domain of f, we call f **differentiable**.

Notations There are many ways to denote the derivative of a function $y = f(x)$ where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

The symbols d/dx and D indicate the operation of differentiation and are called **differentiation operators**. We read dy/dx as the derivative of y with respect to x, and df/dx and $(d/dx)f(x)$ as the derivative of f with respect to x. The **prime** notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The dy/dx symbol should not be regarded as a ratio.

Be careful not to confuse the notation $D(f)$ as meaning the domain of the function f instead of the derivative function f'. The distinction should be clear from the context.

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}$$

Calculating Derivatives from the Definition The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$ we use the notation

$$\frac{d}{dx}f(x)$$

as another way to denote the derivative $f'(x)$.

Example 3.1. Differentiate $f(x) = x^2$.

Solution 3.1.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{x^2+2xh-h^2-x^2}{h} \\ &= \lim_{h\to 0}\frac{2xh-h^2}{h} \\ &= \lim_{h\to 0}\frac{h(2x-h)}{h} = \lim_{h\to 0}(2x-h) = 2x \end{aligned}$$

□

Example 3.2. Find the derivative of $g(x) = \sqrt{x}$ at $x = 4$.

Solution 3.2.

$$\begin{aligned} g'(x) &= \lim_{w\to x}\frac{f(w)-f(x)}{w-x} \\ &= \lim_{w\to x}\frac{\sqrt{w}-\sqrt{x}}{w-x} \\ &= \lim_{w\to x}\frac{\sqrt{w}-\sqrt{x}}{\left(\sqrt{w}-\sqrt{x}\right)\left(\sqrt{w}+\sqrt{x}\right)} \\ &= \lim_{w\to x}\frac{1}{\sqrt{w}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \\ g'(4) &= \frac{1}{2\times\sqrt{4}} = \frac{1}{4} \end{aligned}$$

□

Exercise 3.1. Find the derivative of $f(t) = 1/t$ at $t = 5$.

Differentiable on an Interval; One-Sided Derivatives A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right}-\textbf{hand derivative at } a$$

$$f'_-(b) = \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left}-\textbf{hand derivative at } b$$

exist at the endpoints.

Right−hand and left−hand derivatives may be defined at any point of a function's domain. The usual relation between one-sided and two-sided limits holds for these derivatives. **A function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal**.

$$f'(x_0) \text{ exists } \Leftrightarrow f'_+(x_0) = f'_-(x_0) \text{ and both are exist.}$$

Example 3.3. Show that the function $g(x) = |x|$ is not differentiable at $x = 0$.

Solution 3.3. There can be no derivative at the origin because the one−sided derivatives differ there:

$$\begin{aligned}
g'_+(0) &= \lim_{h\to 0^+} \frac{g(0+h) - g(0)}{h} \\
&= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} \\
&= \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1 \\
g'_-(0) &= \lim_{h\to 0^-} \frac{g(0+h) - g(0)}{h} \\
&= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} \\
&= \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1
\end{aligned}$$

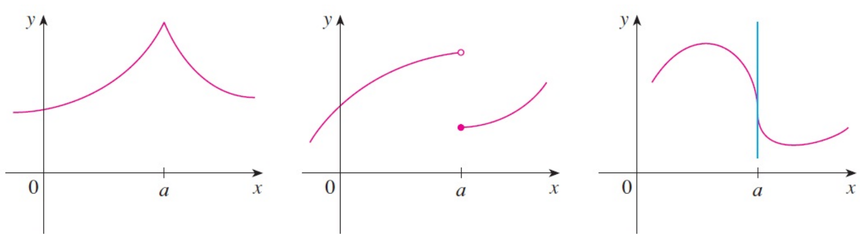
□

Exercise 3.2. Show that the function $f(x) = \sqrt{x}$ is not differentiable at $x = 0$.

When Does a Function Not Have a Derivative at a Point?

Differentiability is a **smoothness** condition on the graph of f. A function whose graph is otherwise smooth will fail to have a derivative at a point for several reasons, such as at points where the graph has (see Figure 3.1)

1. a *corner*, where the one-sided derivatives differ.
2. a *cusp*, where the derivative approaches ∞ from one side and $-\infty$ from the other.
3. a *vertical tangent*, where the derivative approaches ∞ from both sides or $-\infty$ from both sides.
4. a discontinuity.



(a) A corner (b) A discontinuity (c) A vertical tangent

Figure 3.1:

Differentiable Functions Are Continuous A function is continuous at every point where it has a derivative.

Theorem 3.1.1. *If f has a derivative at $x = c$, then f is continuous at $x = c$.*

Corollary 3.1.2. *If f is discontinuous at $x = c$, then f is not differentiable at $x = c$.*

Note that a function need not have a derivative at a point where it is continuous. Also, a function that is not differentiable at a point need not be discontinuous at that point. For example, $|x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$.

Example 3.4. The figure below shows the graph of a function over a closed interval D. At what domain points does the function appear to be

1. differentiable?
2. continuous but not differentiable?
3. neither continuous nor differentiable?

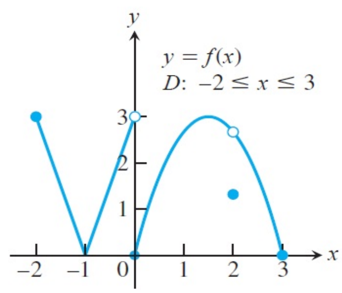


Figure 3.2:

Solution 3.4. 1. f is differentiable on $[-2,3]-\{-1,0,2\}$

2. f is continuous but not differentiable at $x=1$ because

$$f(-1)=\lim_{x\to -1} f(x)=0$$

but there is a corner at $x=1$.

3. (c) f is neither continuous nor differentiable at $x=0$ because

$$\lim_{x\to 0} f(x)$$

does not exist, and $x=2$ because

$$f(2)\neq \lim_{x\to 2} f(x)$$

□

3.2 Differentiation Rules and Higher Derivatives

This section introduces rules that allow us to differentiate a great variety of functions. By these rules, we can differentiate functions without having to apply the definition of the derivative each time.

Rule 1 If f has the constant value $f(x) = c$ then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

The first rule of differentiation is that the derivative of every constant function is zero. For example,

$$\frac{d}{dx}(8) = 0, \qquad \frac{d}{dx}(\pi) = 0, \qquad \frac{d}{dx}\left(\sqrt{e}\right) = 0.$$

Rule 2 If n is a real number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

The second rule tells how to differentiate x^n if n is a real number. For example,

(1) $\dfrac{d}{dx}(x) = 1$

(2) $\dfrac{d}{dx}\left(x^2\right) = 2x$

(3) $\dfrac{d}{dx}\left(x^3\right) = 3x^2$

(4) $\dfrac{d}{dx}\left(x^4\right) = 4x^3$

(5) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = \dfrac{d}{dx}\left(x^{-2}\right) = -2x^{-3} = \dfrac{-2}{x^3}$

(6) $\dfrac{d}{dx}\left(\sqrt[3]{x^4}\right) = \dfrac{d}{dx}\left(x^{4/3}\right) = \dfrac{4}{3}x^{1/3} = \dfrac{4}{3}\sqrt[3]{x}$

Rule 3 If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

The third rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant. For example,

$$\frac{d}{dx}\left(3x^2\right) = 3\frac{d}{dx}\left(x^2\right) = 3 \times (2x) = 6x.$$

Rule 4 If u and v are differentiable functions of x, then their sum (or difference) $u \pm v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

This rule says that the derivative of the sum (or difference) of two differentiable functions is the sum of their derivatives. For example, if $f(x) = x^3 + \frac{4}{3}x^2 - 5x + 1$ then

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(x^3 + \frac{4}{3}x^2 - 5x + 1\right) \\
&= \frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\
&= \frac{d}{dx}\left(x^3\right) + \frac{4}{3}\frac{d}{dx}\left(x^2\right) - 5\frac{d}{dx}(x) + \frac{d}{dx}(1) \\
&= 3x^2 + \frac{4}{3} \times 2x - 5 \times 1 + 0 \\
&= 3x^2 + \frac{8}{3}x - 5
\end{aligned}$$

Rule 5 While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives. The derivative of a product of two functions is the sum of two products, as we now explain. If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

For example,

$$\begin{aligned}\frac{d}{dx}\left[x\left(1-x^2\right)\right] &= x\frac{d}{dx}\left(1-x^2\right)+\left(1-x^2\right)\frac{d}{dx}(x)\\ &= x(-2x)+\left(1-x^2\right)(1)\\ &= -2x^2+1-x^2\\ &= 1-3x^2\end{aligned}$$

Rule 6 Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives. What happens instead is the Quotient Rule: If u and v are differentiable at x and if $v(x) \neq 0$ then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

For example,

$$\begin{aligned}\frac{d}{dt}\left(\frac{t^2-1}{t^2+1}\right) &= \frac{\left(t^2+1\right)\frac{d}{dt}\left(t^2-1\right)-\left(t^2-1\right)\frac{d}{dt}\left(t^2+1\right)}{\left(t^2+1\right)^2}\\ &= \frac{\left(t^2+1\right)(2t)-\left(t^2-1\right)(2t)}{\left(t^2+1\right)^2}\\ &= \frac{2t^3+2t-2t^3+2t}{\left(t^2+1\right)^2}\\ &= \frac{4t}{\left(t^2+1\right)^2}\end{aligned}$$

Rule 7 The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x$$

The following is an example of derivative of function involving the sine function.

$$\frac{d}{dx}\left(2x^3\sin x\right) = 2x^3\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(2x^3) = 2x^3\cos x + 6x^2\sin x$$

Rule 8 The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

The following is an example of derivative of function involving the cosine function.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{\cos x}{1-\sin x}\right) &= \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2} \\
&= \frac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2} \\
&= \frac{1-\sin x}{(1-\sin x)^2} \\
&= \frac{1}{1-\sin x}
\end{aligned}$$

Rule 9 Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \ \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \sec^2 x \\
\frac{d}{dx}(\cot x) &= -\csc^2 x \\
\frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\csc x) &= -\csc x \cot x
\end{aligned}$$

Rule 10 The Derivative Rule for Inverses: If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)}$$

Example 3.5. Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$

Solution 3.5.

$$\left(f^{-1}\right)'(6) = \frac{1}{f'\left(f^{-1}(6)\right)} = \frac{1}{f'(2)} = \frac{1}{3 \times 2^2} = \frac{1}{12}$$

□

Exercise 3.3. Let $f(x) = 2x + 3$. Find $f^{-1}(x)$, then evaluate df/dx at $x = -1$ and df^{-1}/dx at $x = f(-1)$ to show that

$$\left(f^{-1}\right)'(-1) = \frac{1}{f'\left(f^{-1}(-1)\right)}$$

Rule 11 For every positive value of x, we have

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \text{ and in general } \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

For example,

$$\begin{aligned}
\frac{d}{dt}(t \ln t) &= t\frac{d}{dt}(\ln t) + \ln t \frac{d}{dt}(t) = t \times \frac{1}{t} + \ln t = 1 + \ln t \\
\frac{d}{dt}(2 \log_{10} t) &= 2\frac{d}{dt}(\log_{10} t) = 2 \times \frac{1}{t \ln 10} = \frac{2}{t \ln 10}
\end{aligned}$$

Rule 12 The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. So, by Rule 10, we have

$$\frac{d}{dx}(e^x) = e^x, \text{ and in general } \frac{d}{dx}(a^x) = a^x \ln a.$$

For example,

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\tan t}{10^t}\right) &= \frac{10^t \frac{d}{dt}(\tan t) - \tan t \frac{d}{dt}(10^t)}{\left(10^t\right)^2} \\
&= \frac{10^t \sec^2 t - 10^t (\ln 10) \tan t}{10^{2t}} = \frac{\sec^2 t - (\ln 10) \tan t}{10^t}
\end{aligned}$$

Rule 13 The derivatives of the inverse trigonometric functions are summarized as follows.

$$
\begin{aligned}
&(1) \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \text{ where } |x| < 1\\
&(2) \quad \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}} \text{ where } |x| < 1\\
&(3) \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\\
&(4) \quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}\\
&(5) \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}} \text{ where } |x| > 1\\
&(6) \quad \frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}} \text{ where } |x| > 1
\end{aligned}
$$

Rule 14 Derivatives of Hyperbolic Functions.

$$
\begin{aligned}
&(1) \quad \frac{d}{dx}(\sinh x) = \cosh x\\
&(2) \quad \frac{d}{dx}(\cosh x) = \sinh x\\
&(3) \quad \frac{d}{dx}(\tanh x) = \text{sech}^2 x\\
&(4) \quad \frac{d}{dx}(\coth x) = -\text{csch}^2 x\\
&(5) \quad \frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x\\
&(6) \quad \frac{d}{dx}(\text{csch } x) = -\text{csch } x \coth x
\end{aligned}
$$

Rule 15 Derivatives of Inverse Hyperbolic Functions.

$$
\begin{aligned}
&(1) \quad \frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}\\
&(2) \quad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}
\end{aligned}
$$

$$
\begin{aligned}
&(3) \qquad \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \\
&(4) \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2} \\
&(5) \qquad \frac{d}{dx}(\text{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}} \\
&(6) \qquad \frac{d}{dx}(\text{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}
\end{aligned}
$$

Exercise 3.4. Find the derivative f' with respect to x of the following functions.

1. $f(x) = (1+x)\sqrt{x}$
2. $f(x) = 2^x - x^2$
3. $f(x) = e^x \tan^{-1} x$
4. $f(x) = 2x^2/(7x+5)$
5. $f(x) = x \sinh x - \cosh x$

Example 3.6. If $f(x) = \sqrt{x}g(x)$ where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

Solution 3.6.

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left[\sqrt{x}g(x)\right] = \sqrt{x}g'(x) + g(x) \times \frac{1}{2\sqrt{x}} \\
f'(4) &= \sqrt{4}g'(4) + g(4) \times \frac{1}{2\sqrt{4}} \\
&= 2 \times 3 + 2 \times \frac{1}{4} = \frac{13}{2}
\end{aligned}
$$

□

Example 3.7. For what values of a and b such that

$$
f(x) = \begin{cases} x^2 + a & \text{if } \ x \le 1 \\ bx & \text{if } \ x > 1 \end{cases}
$$

is differentiable at $x = 1$.

Solution 3.7. Since f is differentiable at $x = 1$, then: f is continuous at $x = 1$, and $f'(1)$ exists. Therefore,

$$\begin{aligned}
f'(1) \text{ exists} &\Rightarrow f'_+(1) = f'_-(1) \\
&\Rightarrow \frac{d}{dx}(bx)\bigg|_{x=1} = \frac{d}{dx}\left(x^2 + a\right)\bigg|_{x=1} \\
&\Rightarrow b = 2 \\
f \text{ continuous at } x = 1 &\Rightarrow \lim_{x\to 1} f(x) \text{ exists} \\
&\Rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) \\
&\Rightarrow \lim_{x\to 1^+} (bx) = \lim_{x\to 1^-} \left(x^2 + a\right) \\
&\Rightarrow b = 1 + a \Rightarrow 2 = 1 + a \Rightarrow a = 1
\end{aligned}$$

□

Exercise 3.5. For what values of m and b such that

$$g(x) = \begin{cases} x^2 & \text{if } \ x \le 2 \\ mx + b & \text{if } \ x > 2 \end{cases}$$

is differentiable at $x = 2$.

Remark 3.2.1.

$$\lim_{h\to 0} \frac{f(x + ah) - f(x)}{bh} = \frac{a}{b} f'(x)$$

Example 3.8. If $f(x) = x + \ln x$, evaluate

$$\lim_{h\to 0} \frac{f(2 - 2h) - f(2)}{3h}$$

Solution 3.8. Note that $f'(x) = 1 + 1/x$. Therefore,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(2 - 2h) - f(2)}{3h} &= \frac{-2}{3} f'(2) \\
&= \frac{-2}{3} \times \left(1 + \frac{1}{2}\right) \\
&= \frac{-2}{3} \times \frac{3}{2} = -1
\end{aligned}$$

Exercise 3.6. Each limit represents the derivative of some function f at some number c. State such an f and c in each case and then evaluate the limit.

$$(1) \qquad \lim_{x\to 1} \frac{x^9 - 1}{x - 1}$$

$$(2) \qquad \lim_{h\to 0} \frac{\sqrt[4]{16+h} - 2}{h}$$

Higher Derivatives If $f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative of** f because it is the derivative of the first derivative. Notationally,

$$f''(x) = \frac{d^2}{dx^2} f(x) = \frac{d}{dx}\left[\frac{d}{dx} f(x)\right]$$

If f'' is differentiable, its derivative

$$f'''(x) = \frac{d^3}{dx^3} f(x) = \frac{d}{dx}\left[\frac{d^2}{dx^2} f(x)\right]$$

is the **third derivative of** f **with respect to** x. The names continue as you imagine, with

$$f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = \frac{d}{dx}\left[\frac{d^{(n-1)}}{dx^{(n-1)}} f(x)\right]$$

denoting the $n^{\textbf{th}}$ **derivative of** f **with respect to** x **for any positive integer** n.

Example 3.9. If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$

Solution 3.9.

$$\begin{aligned} f(x) &= x^3 - x \\ f'(x) &= \frac{d}{dx}\left(x^3 - x\right) = 3x^2 - 1 \\ f''(x) &= \frac{d}{dx}\left(3x^2 - 1\right) = 6x \\ f'''(x) &= \frac{d}{dx}(6x) = 6 = 3! \\ f^{(4)}(x) &= \frac{d}{dx}(6) = 0 \end{aligned}$$

□

Example 3.10. Find the 27th derivative of $\cos x$.

Solution 3.10. The first few derivatives of $f(x) = \cos x$ are as follows:

$$\begin{aligned} f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \\ f^{(5)}(x) &= -\sin x \end{aligned}$$

We see that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \cos x$ whenever n is a multiple of 4. Therefore

$$f^{(24)}(x) = \cos x$$

and, differentiating three more times, we have

$$f^{(27)}(x) = \sin x$$

□

Exercise 3.7. Let $f(x) = (x+1)/x$. Evaluate: $f''(x)$ and $f'''(2)$.

Exercise 3.8. Evaluate

$$\frac{d^{87}}{dx^{87}} \sin x$$

Example 3.11. Find a formula of the general nth derivative of $g(x) = xe^x$.

Solution 3.11. To find the nth derivative of a function, good idea to start doing first few derivatives, and try to guess a formula from these derivatives. For the function $g(x) = xe^x$ we have

$$\begin{aligned}
g'(x) &= x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) = xe^x + e^x = (x+1)e^x \\
g''(x) &= (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) = (x+1)e^x + e^x = (x+2)e^x \\
g'''(x) &= (x+2)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+2) = (x+2)e^x + e^x = (x+3)e^x \\
g^{(4)}(x) &= (x+3)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+3) = (x+3)e^x + e^x = (x+4)e^x \\
&\vdots \\
g^{(n)}(x) &= (x+n)e^x
\end{aligned}$$

□

Exercise 3.9. Find a formula of the general nth derivative of:

1. $f(x) = x^n$
2. $f(x) = 1/x$

3.3 The Chain Rule

Suppose you are asked to differentiate the function $F(x) = \sqrt{1+x^2}$. The differentiation formulas you learned in the previous sections of this chapter do not enable you to calculate $F'(x)$. Observe that F is a composite function. In fact, if we let $y = f(u) = \sqrt{u}$ and let $u = g(x) = 1 + x^2$, then we can write $y = F(x) = f(g(x))$, that is, $F = f \circ g$. We know how to differentiate both f and g, so it would be useful to have a rule that tells us how to find the derivative of $F = f \circ g$ in terms of the derivatives of f and g.

It turns out that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This fact is one of the most important of the differentiation rules and is called the **Chain Rule**.

The Chain Rule If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example 3.12. Given that $g(2) = 1/2$, $g'(2) = -1/4$, and $f'(1/2) = 1$, find $(f \circ g)'(2)$.

Solution 3.12.

$$(f \circ g)'(2) = f'(g(2)) \cdot g'(2) = f'\left(\frac{1}{2}\right) \times \frac{-1}{4} = 1 \times \frac{-1}{4} = \frac{-1}{4}$$

□

Example 3.13. Let $y = 5 - e^t$ and $t = 2x^2 - 3$. Find dy/dx at $x = 1$.

Solution 3.13. Here, y is a function of t and t is a function of x, so y is a function of x. Note that when $x = 1$ we have $t = 2 \times (1)^2 - 3 = -1$. Hence,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \left(-e^t\right) \times (4x)$$

and

$$\left.\frac{dy}{dx}\right|_{\substack{x=1\\t=-1}} = \left(-e^{-1}\right) \times (4) = \frac{-4}{e}$$

□

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts, an **inner part** and an **outer part**. The Chain Rule tells you that the derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) times the derivative of the inner function. That is,

$$y' = f'(u) \cdot u'$$

Example 3.14. Differentiate $\sin(x^2+x)$ with respect to x.

Solution 3.14.

$$\frac{d}{dx}\sin\underbrace{\left(x^2+x\right)}_{\text{inside}} = \cos\underbrace{\left(x^2+x\right)}_{\substack{\text{inside}\\ \text{left alone}}} \times \underbrace{(2x+1)}_{\substack{\text{derivative of}\\ \text{the inside}}}$$

□

Example 3.15. Differentiate $\sqrt{x^2+1}$ with respect to x.

Solution 3.15. Since

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{d}{dx}\left(x^2+1\right) = 2x,$$

then

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2\sqrt{x^2+1}} \times 2x = \frac{x}{\sqrt{x^2+1}}$$

□

Example 3.16. Differentiate $(x^3-1)^{100}$ with respect to x.

Solution 3.16. Since

$$\frac{d}{dx}x^{100} = 100x^{99} \quad \text{and} \quad \frac{d}{dx}\left(x^3-1\right) = 3x^2,$$

then

$$\frac{d}{dx}\left(x^3-1\right)^{100} = 100\left(x^3-1\right)^{99} \times 3x^2 = 300x^2\left(x^3-1\right)^{99}$$

□

Example 3.17. Differentiate $\sin^3 x$ with respect to x.

Solution 3.17. Note that $\sin^3 x = (\sin x)^3$. So, since

$$\frac{d}{dx}x^3 = 3x^2 \quad \text{and} \quad \frac{d}{dx}(\sin x) = \cos x,$$

then

$$\frac{d}{dx}\sin^3 x = \frac{d}{dx}(\sin x)^3 = 3(\sin x)^2 \times \cos x = 3\sin^2 x\cos x$$

□

Example 3.18. Differentiate $2^{x\cos x}$ with respect to x.

Solution 3.18. Since

$$\frac{d}{dx}2^x = 2^x \ln 2 \quad \text{and} \quad \frac{d}{dx}(x\cos x) = \cos x - x\sin x,$$

then

$$\frac{d}{dx}2^{x\cos x} = 2^{x\cos x}\ln 2 \times (\cos x - x\sin x)$$

□

Example 3.19. Differentiate $\ln(\cos(5x^2))$ with respect to x.

Solution 3.19. Since

$$\frac{d}{dx}\ln x = \frac{1}{x}, \qquad \frac{d}{dx}\cos x = -\sin x, \quad \text{and} \quad \frac{d}{dx}5x^2 = 10x$$

then

$$\begin{aligned}\frac{d}{dx}\ln\left(\cos\left(5x^2\right)\right) &= \frac{1}{\cos(5x^2)} \times -\sin\left(5x^2\right) \times 10x \\ &= \frac{-10x\sin(5x^2)}{\cos(5x^2)} \\ &= -10x\tan\left(5x^2\right)\end{aligned}$$

□

Example 3.20. Differentiate

$$\ln\left(\frac{t^2\sin t}{\sqrt{t+1}}\right)$$

with respect to t.

Solution 3.20. First simplify the expression as follows, then differentiate with respect to t.

$$\begin{aligned}
\ln\left(\frac{t^2\sin t}{\sqrt{t+1}}\right) &= \ln\left(t^2\right)+\ln(\sin t)-\ln\left(\sqrt{t+1}\right)\\
&= 2\ln t+\ln(\sin t)-\ln(t+1)^{\frac{1}{2}}\\
&= 2\ln t+\ln(\sin t)-\frac{1}{2}\ln(t+1)\\
\frac{d}{dt}\left[\ln\left(\frac{t^2\sin t}{\sqrt{t+1}}\right)\right] &= 2\frac{d}{dt}(\ln t)+\frac{d}{dt}[\ln(\sin t)]-\frac{1}{2}\frac{d}{dt}[\ln(t+1)]\\
&= \frac{2}{t}+\frac{\cos t}{\sin t}-\frac{1}{2}\frac{1}{t+1}\\
&= \cot t+\frac{4+3t}{2t(1+t)}
\end{aligned}$$

□

Example 3.21. Find the derivative of $f(x)=\cosh(\ln x)$. Simplify where possible.

Solution 3.21.

$$\begin{aligned}
f'(x) &= \sinh(\ln x)\times\frac{d}{dx}\ln x\\
&= \frac{e^{\ln x}-e^{-\ln x}}{2}\times\frac{1}{x}\\
&= \frac{x-1/x}{2}\times\frac{1}{x}=\frac{x^2-1}{2x^2}
\end{aligned}$$

□

Example 3.22. Find the derivative of $f(x)=\sinh^{-1}(\tan x)$. Simplify where possible.

Solution 3.22.

$$f'(x)=\frac{1}{\sqrt{1+\tan^2 x}}\times\frac{d}{dx}\tan x=\frac{\sec^2 x}{\sqrt{\sec^2 x}}=\sec x$$

□

Exercise 3.10. Calculate f' of each of the following functions.

1. $f(x) = (x^4 - 3x^2 + 5)^3$
2. $f(x) = \sin^{-1}(\tan x)$
3. $f(x) = 2x\sqrt{1 + x^2}$
4. $f(x) = e^{\sin(2x)}$
5. $f(x) = 3^{x \ln x}$
6. $f(x) = \log_{10}(x^2 e^x)$
7. $f(x) = [\ln(x + 2e^x)]^2$
8. $f(x) = e^{\cosh(3x)}$
9. $f(x) = \text{sech}^{-1}(e^{-x})$
10. $f(x) = x \sinh^{-1} x + \ln\sqrt{1 - x^2}$

Example 3.23. Find $g'(9)$ if $g(3x) = 6x^2$.

Solution 3.23. Since $g(3x) = 6x^2$ then

$$\begin{aligned} \frac{d}{dx}[g(3x)] &= \frac{d}{dx}\left(6x^2\right) \\ 3 \times g'(3x) &= 12x \\ g'(3x) &= 4x \end{aligned}$$

Now, let $x = 3$ to obtain $g'(9) = 4 \times 3 = 12$.

□

Example 3.24. Let

$$\frac{d}{dx}\left[f\left(x^2\right)\right] = x^2.$$

Find $f'(x^2)$.

Solution 3.24.

$$\begin{aligned}
\frac{d}{dx}\left[f\left(x^2\right)\right] &= x^2 \\
2xf'\left(x^2\right) &= x^2 \\
f'\left(x^2\right) &= \frac{x^2}{2x} = \frac{1}{2}x \text{ where } x \neq 0
\end{aligned}$$

□

Exercise 3.11. Find

$$\frac{d^2}{dt^2}\tan^{-1}\left(3x^2\right)$$

3.4 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form that expresses $y = f(x)$ explicitly in terms of the variable x. Another situation occurs when we encounter equations like

$$x^2 + y^2 = 25 \quad \text{or} \quad y^2 - x = 0.$$

These equations define an **implicit relation** between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by **implicit differentiation**. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'.

Implicit Differentiation Technique

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Example 3.25. Find dy/dx if $y^2 = x$.

Solution 3.25. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$\begin{aligned} y^2 &= x \\ \frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}(x) \\ 2y\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{2y} \end{aligned}$$

□

Example 3.26. Find dy/dx of circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

Solution 3.26. The circle is not the graph of a single function of x. But we can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\begin{aligned} x^2 + y^2 &= 25 \\ \frac{d}{dx}\left(x^2 + y^2\right) &= \frac{d}{dx}(25) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ 2y\frac{dy}{dx} &= -2x \\ \frac{dy}{dx} &= -\frac{x}{y} \Rightarrow \left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4} \end{aligned}$$

□

Example 3.27. Find dy/dx if $y^2 = x^2 + \sin(xy)$.

Solution 3.27.

$$
\begin{aligned}
y^2 &= x^2 + \sin(xy) \\
\frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}\left(x^2 + \sin(xy)\right) \\
2y\frac{dy}{dx} &= 2x + \cos(xy)\frac{d}{dx}(xy) \\
2y\frac{dy}{dx} &= 2x + \cos(xy)\left(x\frac{dy}{dx} + y\right) \\
2y\frac{dy}{dx} &= 2x + x\cos(xy)\frac{dy}{dx} + y\cos(xy) \\
2y\frac{dy}{dx} - x\cos(xy)\frac{dy}{dx} &= 2x + y\cos(xy) \\
[2y - x\cos(xy)]\frac{dy}{dx} &= 2x + y\cos(xy) \\
\frac{dy}{dx} &= \frac{2x + y\cos(xy)}{2y - x\cos(xy)}
\end{aligned}
$$

□

Exercise 3.12. Find dy/dx of $x = \sin^{-1}(2y)$ by implicit differentiation.

Implicit differentiation can also be used to find **higher** derivatives. Here is an example.

Example 3.28. Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution 3.28. To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$
\begin{aligned}
\frac{d}{dx}\left(2x^3 - 3y^2\right) &= \frac{d}{dx}(8) \\
6x^2 - 6yy' &= 0 \\
-6yy' &= -6x^2 \\
y' &= \frac{x^2}{y}, \qquad \text{when } y \neq 0
\end{aligned}
$$

We now apply the Quotient Rule to find y''.

$$
y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'
$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\frac{x^2}{y^2} = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$

□

Exercise 3.13. Find d^2y/dx^2 of $x^3 + y^3 = 16$ at the point $(2, 2)$ by implicit differentiation.

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

Example 3.29. Differentiate

$$y = x^{\sqrt{x}}$$

Solution 3.29. Using logarithmic differentiation, we have

$$\begin{aligned}
\ln y &= \ln\left(x^{\sqrt{x}}\right) = \sqrt{x}\ln x \\
\frac{d}{dx}(\ln y) &= \frac{d}{dx}\left(\sqrt{x}\ln x\right) \\
\frac{y'}{y} &= \sqrt{x}\cdot\frac{1}{x} + \ln x \cdot \frac{1}{2\sqrt{x}} \\
y' &= y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{2+\ln x}{2\sqrt{x}}\right)
\end{aligned}$$

Exercise 3.14. Find dy/dx if $y = (\sin x)^x$.

3.5 Tangent Line

The problem of finding the tangent line to a curve involve calculating a derivative. If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the **tangent** t to be the line through P with slope m. See Figure 3.3.

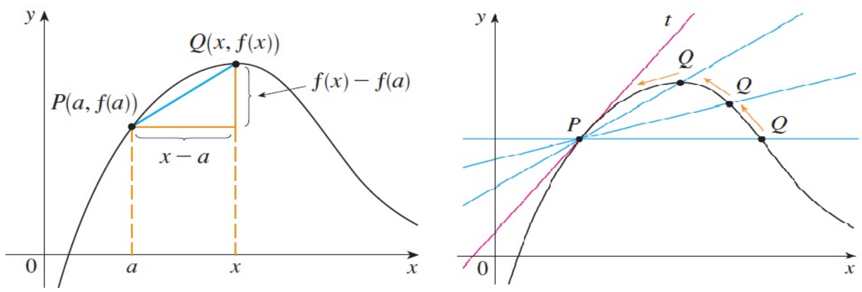


Figure 3.3:

Definition 3.5.1. The **tangent line** to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through the point with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

provided that this limit exists.

Example 3.30. Find an equation of the tangent line to the parabola $f(x) = x^2$ at the point with $x-$coordinate is $x = 1$.

Solution 3.30. Here, the tangent point is

$$(1, f(1)) = \left(1, 1^2\right) = (1, 1)$$

Since $f'(x) = 2x$ then the slope is $m = f'(1) = 2 \times 1 = 2$. Using the point$-$slope form of the equation of a line, we find the equation of the tangent line at $P(1, 1)$ as follows.

$$\begin{aligned} y - y_0 &= m\,(x - x_0) \\ y - 1 &= 2(x - 1) \\ y &= 2x - 1 \end{aligned}$$

□

Example 3.31. Find an equation of the tangent line to the hyperbola $g(x) = 3/x$ at the point $(3, 1)$.

Solution 3.31. Since $f'(x) = -3/x^2$, then the slope of the tangent at $(3, 1)$ is

$$m = f'(3) = \frac{-3}{9} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point $(3, 1)$ is

$$\begin{aligned} y - y_0 &= m\,(x - x_0) \\ y - 1 &= -\frac{1}{3}(x - 3) \\ y &= -\frac{1}{3}x + 2 \end{aligned}$$

□

Exercise 3.15. Find equations of the tangent line to the curve $f(t) = t^4 + 2e^t$ at the point $(0, 2)$.

Example 3.32. Find equations of the tangent line to the curve $f(x) = \sin(\sin x)$ at the point with $x-$coordinate is $x = \pi$.

Solution 3.32. Here, the tangent point is

$$(\pi, f(\pi)) = (\pi, \sin(\sin \pi)) = (\pi, \sin 0) = (\pi, 0)$$

Since $f'(x) = \cos(\sin x) \cos x$ (by using Chain Rule), then the slope is

$$m = f'(\pi) = \cos(\sin \pi) \cos \pi = \cos(0) \times -1 = 1 \times -1 = -1$$

Using the point$-$slope form of the equation of a line, we find the equation of the tangent line at $P(\pi, 0)$ as follows.

$$\begin{aligned} y - y_0 &= m\,(x - x_0) \\ y - 0 &= -1(x - \pi) \\ y &= -x + \pi \end{aligned}$$

□

Exercise 3.16. Find equations of the tangent line to the curve $g(x) = (1+2x)^{10}$ at the point with $x-$coordinate is $x = 0$.

Example 3.33.

1. Find y' if $x^3 + y^3 = 6xy$.
2. Find the tangent to $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
3. At what points in the first quadrant is the tangent line horizontal?

Solution 3.33.

1. Differentiating both sides of $x^3 + y^3 = 6xy$ with respect to x, regarding y as a function of x, and using the Chain Rule on the term y^3 and the Product Rule on the term $6xy$, we get

$$\begin{aligned} 3x^2 + 3y^2y' &= 6xy' + 6y \\ y^2y' - 2xy' &= 2y - x^2 \\ \left(y^2 - 2x\right) y' &= 2y - x^2 \\ y' &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

2. When $x = y = 3$,

$$y' = \frac{2 \times 3 - 3^2}{3^2 - 2 \times 3} = -1$$

this is a reasonable value for the slope at $(3, 3)$. So, an equation of the tangent to the curve at $(3, 3)$ is

$$\begin{aligned} y - y_0 &= m\,(x - x_0) \\ y - 3 &= -1(x - 3) \\ y &= -x + 6 \end{aligned}$$

3. The tangent line is horizontal if $y' = 0$. Using the expression for y' from part the first part, we see that $y' = 0$ when $2y - x^2 = 0$ (provided that $y^2 - 2x \neq 0$). Substituting $y = \frac{1}{2}x^2$ in the equation of the curve, we get

$$x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$$

which simplifies to $x^6 = 16x^3$. Now, solve the last equation:

$$\begin{aligned} x^6 &= 16x^3 \\ x^6 - 16x^3 &= 0 \\ x^3\left(x^3 - 16\right) &= 0 \\ x^3 &= 0 \Rightarrow x = 0 \\ x^3 - 16 &= 0 \Rightarrow x = 16^{1/3} \end{aligned}$$

If $x = 0$ then $y = 0$, and if $x = 16^{1/3} = 2^{4/3}$ then

$$y = \frac{1}{2}\left(2^{4/3}\right)^2 = 2^{5/3}$$

Thus the tangent is horizontal at $(0, 0)$ and at $\left(2^{4/3}, 2^{5/3}\right)$.

□

Exercise 3.17. The curve with equation $y^2 = 5x^4 - x^2$ is called a **kampyle of Eudoxus**. Find an equation of the tangent line to this curve at the point $(1, 2)$.

Chapter 4

Applications of Differentiation

This chapter studies some of the important applications of derivatives. We learn how derivatives are used to find extreme values of functions, to determine and analyze the shapes of graphs, to calculate limits of fractions whose numerators and denominators both approach zero or infinity, and to find numerically where a function equals zero.

4.1 Indeterminate Forms and L'Hôspital's Rule

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as l'Hôspital's Rule, after Guillaume de l'Hôspital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

Indeterminate Form $\mathbf{0/0}, \infty/\infty$ If the functions $f(x)$ and $g(x)$ are both zero or both $\pm\infty$ at $x = a$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$ or ∞/∞, a meaningless expressions (**indeterminate forms**), that we cannot evaluate. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancelation, rearrangement of terms, or other algebraic manipulations. L'Hôspital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

Theorem 4.1.1. *Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

(In other words, we have an indeterminate form of type $0/0$ or ∞/∞.) Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the limit on the right side exists or $\pm\infty$.

- L'Hôspital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of and before using L'Hôspital's Rule.
- L'Hôspital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity.

Example 4.1. Find

$$\lim_{x\to 1}\frac{\ln x}{x-1}$$

Solution 4.1. Since $\lim_{x\to 1}\ln x = 0$ and $\lim_{x\to 1}(x-1) = 0$, then we can apply L'Hôspital's Rule:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{1/x}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

□

Example 4.2. Calculate

$$\lim_{x\to\infty}\frac{e^x}{x^2}$$

Solution 4.2. We have $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so L'Hôspital's Rule gives:

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(e^x\right)}{\frac{d}{dx}\left(x^2\right)} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate, but a second application of L'Hôspital's Rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

□

Example 4.3. Calculate

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$$

Solution 4.3. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} \sqrt[3]{x} = \infty$ as $x \to \infty$, L'Hôspital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying L'Hôspital's Rule a second time as we did in the previous example, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$

□

Example 4.4. Find

$$\lim_{x\to 0} \frac{\tan x - x}{x^3}$$

Solution 4.4. Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use L'Hôspital's Rule:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply L'Hôspital's Rule again:

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}\\ &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}\\ &= \frac{2}{6}\times\lim_{x\to 0}\sec^2 x\times\lim_{x\to 0}\frac{\tan x}{x}\\ &= \frac{1}{3}\times 1\times 1 = \frac{1}{1}\end{aligned}$$

□

Exercise 4.1. Find the limit.

(1) $\lim_{t\to 0}\frac{e^t - 1}{t^3}$

(2) $\lim_{t\to\infty}\frac{\ln(\ln t)}{t}$

(3) $\lim_{t\to 0}\frac{\sin^{-1} t}{t}$

(4) $\lim_{t\to\infty}\frac{\sinh t}{e^t}$

Indeterminate Products $\mathbf{0}\cdot\pm\infty$ If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then it is not clear what the value of $\lim_{x\to a}[f(x)g(x)]$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be $\pm\infty$. Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type** $0\cdot\infty$. We can deal with it by writing the product as a quotient:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/f}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞ so that we can use L'Hôspital's Rule.

Example 4.5. Evaluate

$$\lim_{x\to 0^+} x\ln x$$

Solution 4.5. The given limit is indeterminate because, as $x \to 0^+$, the first factor x approaches 0 while the second factor $\ln x$ approaches $-\infty$. Writing x as $\frac{1}{1/x}$ we have $1/x \to \infty$ as $x \to 0^+$, so L'Hôspital's Rule gives:

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

□

Exercise 4.2. Find the limit.

(1) $\lim_{t\to\infty} t\sin\left(\frac{\pi}{t}\right)$

(2) $\lim_{t\to\infty} t^2 e^{-t}$

(3) $\lim_{t\to 0^+} \sin t \ln t$

Indeterminate Differences $\infty - \infty$ If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be ∞ (g wins) or will they compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator, or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.

Example 4.6. Evaluate

$$\lim_{x\to(\pi/2)^-}(\sec x - \tan x)$$

Solution 4.6. First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\begin{aligned}
\lim_{x\to(\pi/2)^-}(\sec x - \tan x) &= \lim_{x\to(\pi/2)^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) \\
&= \lim_{x\to(\pi/2)^-}\frac{1-\sin x}{\cos x} \\
&= \lim_{x\to(\pi/2)^-}\frac{-\cos x}{-\sin x} = 0
\end{aligned}$$

□

Exercise 4.3. Find the limit.

(1) $\lim_{t\to\infty}\left(\sqrt{t^2+1} - t\right)$

(2) $\lim_{t\to\infty}\left(te^{1/t} - t\right)$

(3) $\lim_{t\to\infty}(t - \ln t)$

Indeterminate Powers $\mathbf{0^0}, \infty^{\mathbf{0}}, \mathbf{1}^\infty$ These several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

Each of these three cases can be treated by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

and then

$$\lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)\ln f(x)}$$

where the indeterminate product $g(x)\ln f(x)$ is of type $0\cdot\infty$.

Example 4.7. Calculate

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$$

Solution 4.7. First notice that as $x \to 0^+$, we have $1+\sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$(1+\sin 4x)^{\cot x} = e^{\cot x \ln(1+\sin 4x)}$$

Then

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = e^{\lim_{x\to 0^+} \cot x \ln(1+\sin 4x)}$$

Since

$$\begin{aligned}\lim_{x\to 0^+} \cot x \ln(1+\sin 4x) &= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \\ &= \lim_{x\to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4\end{aligned}$$

then

$$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = e^4$$

□

Example 4.8. Find

$$\lim_{x\to 0^+} x^x$$

Solution 4.8. Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed by writing the function as an exponential

$$x^x = e^{x \ln x}$$

and then

$$\lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x \ln x} = e^0 = 1$$

□

Exercise 4.4. Find the limit.

(1) $\displaystyle \lim_{t\to 0}(1-2t)^{1/t}$

(2) $\displaystyle \lim_{t\to \infty}\left(1+\frac{a}{t}\right)^{bt}$

4.2 The Mean Value Theorem

We know that constant functions have zero derivatives, but could there be a complicated function, with many terms, the derivatives of which all cancel to give zero? What is the relationship between two functions that have identical derivatives over an interval? What we are really asking here is what functions can have a particular kind of derivative. These and many other questions we study in this chapter are answered by applying the **Mean Value Theorem**. To arrive at this theorem we first need **Rolle's Theorem**.

Rolle's Theorem Drawing the graph of a function gives strong geometric evidence that between any two points where a differentiable function crosses a horizontal line there is at least one point on the curve where the tangent is horizontal (Figure 4.1). More precisely,we have the following theorem.

Theorem 4.2.1. *Suppose that f is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If*

$$f(a) = f(b),$$

then there is at least one number $c \in (a, b)$ at which

$$f'(c) = 0.$$

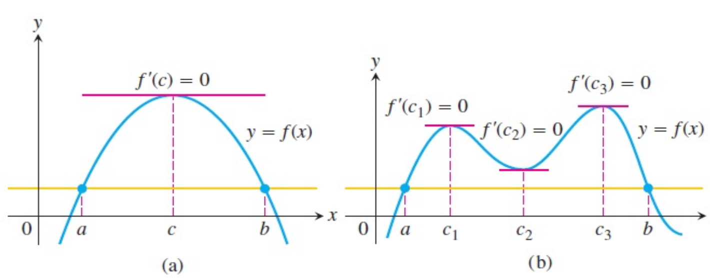


Figure 4.1:

Example 4.9. Verify that the function

$$f(x) = \frac{x^3}{3} - 3x$$

satisfies the three hypotheses of Rolle's Theorem on the interval $[-3, 3]$. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

Solution 4.9. First, you can see that $f(-3) = 0 = f(3)$. Since f is a polynomial, it is differentiable on $(-3, 3)$ and continuous on $[-3, 3]$. Thus, by Rolle's Theorem, there is a number $c \in (-3, 3)$ such that

$$\begin{aligned} f'(c) &= 0 \\ c^2 - 3 &= 0 \\ c^2 &= 3 \\ c &= \pm\sqrt{3} \in (-3, 3) \end{aligned}$$

□

Exercise 4.5. Verify that the function

$$f(x) = \cos 2t$$

satisfies the three hypotheses of Rolle's Theorem on the interval $\left[\frac{\pi}{8}, \frac{7\pi}{8}\right]$. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

Mean Value Theorem The Mean Value Theorem, which was first stated by Joseph Louis Lagrange, is a slanted version of Rolle's Theorem (Figure 4.2). There is a point where the tangent is parallel to chord AB.

Theorem 4.2.2. *Suppose that f is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one number $c \in (a, b)$ at which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

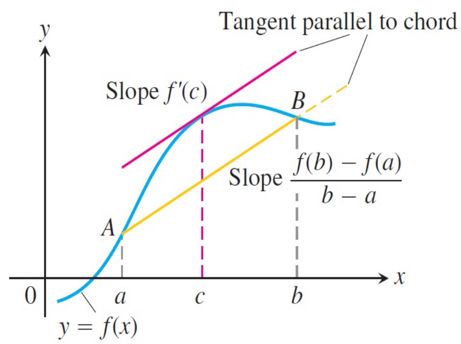


Figure 4.2:

Example 4.10. Verify that the function

$$f(x) = x^3 - x$$

satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Solution 4.10. Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Therefore, by the Mean Value Theorem, there is a number $c \in (0, 2)$ such that

$$\begin{aligned} f'(c) &= \frac{f(2) - f(0)}{2 - 0} \\ 3c^2 - 1 &= \frac{6 - 0}{2} \\ 3c^2 - 1 &= 3 \\ c^2 &= \frac{4}{3} \\ c &= \pm\frac{2}{\sqrt{3}} \end{aligned}$$

But must lie in $(0, 2)$, so $c = 2/\sqrt{3}$ only since $-2/\sqrt{3} \notin (0, 2)$.

□

Exercise 4.6. Verify that the function

$$f(x) = e^{-2x}$$

satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 3]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

Example 4.11. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Solution 4.11. We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval $[0, 2]$. There exists a number $c \in (0, 2)$ such that

$$\begin{aligned} f'(c) &= \frac{f(2) - f(0)}{2 - 0} \\ 2f'(c) &= f(2) - (-3) \\ 2f'(c) - 3 &= f(2) \end{aligned}$$

We are given that $f'(x) \leq 5$ for all x, so in particular we know that $f'(c) \leq 5$. Multiplying both sides of this inequality by 2, we have $2f'(c) \leq 10$, so

$$f(2) = 2f'(c) - 3 \leq 10 - 3 = 7.$$

Thus, the largest possible value for $f(2)$ is 7.

□

Exercise 4.7. If $g(1) = 10$ and $g'(t) \geq 2$ for $1 \leq t \leq 4$, how small can $g(4)$ possibly be?

Mathematical Consequences At the beginning of the section, we asked what kind of function has a zero derivative over an interval. The first corollary of the Mean Value Theorem provides the answer.

Corollary 4.2.3. *If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.*

Example 4.12. Prove the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$.

Solution 4.12. Although calculus is not needed to prove this identity, the proof using calculus is quite simple. If $f(x) = \tan^{-1} x + \cot^{-1} x$, then

$$f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

for all values of x. Therefore $f(x) = C$, a constant. To determine the value of C, we put $x = 1$ because we can evaluate exactly. Then

$$C = f(1) = \tan^{-1} 1 + \cot^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Thus $\tan^{-1} x + \cot^{-1} x = \pi/2$.

□

At the beginning of this section, we also asked about the relationship between two functions that have identical derivatives over an interval. The next corollary tells us that their values on the interval have a constant difference.

Corollary 4.2.4. *If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant on (a, b).*

Exercise 4.8. Prove the identity

$$2\sin^{-1} x = \cos^{-1}\left(1 - 2x^2\right) \text{ for } x \geq 0$$

4.3 Extreme Values of Functions

This section shows how to locate and identify extreme values of a continuous function from its derivative. Once we can do this, we can solve a variety of optimization problems in which we find the optimal (best) way to do something in a given situation.

Definition 4.3.1. Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if $f(x) \leq f(c)$ for all $x \in D$ and an **absolute minimum** value on D at c if $f(x) \geq f(c)$ for all $x \in D$.

Absolute maximum and minimum values are called **absolute extrema** (plural of the Latin extremum). Absolute extrema are also called **global extrema**, to distinguish them from local extrema defined later in this section.

Illustration Example 4.1. For example, on the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.3).

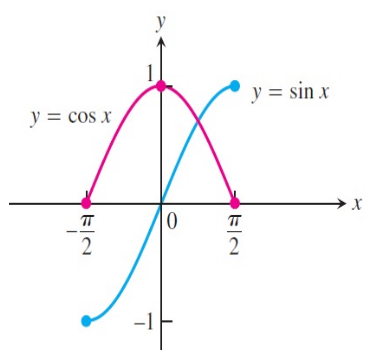


Figure 4.3:

Illustration Example 4.2. The absolute extrema of the following functions on their domains can be seen in Figure 4.4. Each function has the same defining equation $y = x^2$, but the domains vary. Notice that a function might not have a maximum or minimum if the domain is unbounded or fails to contain an endpoint.

Function rule	Domain D	Absolute extrema on D
$y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$
$y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$ Absolute minimum of 0 at $x = 0$
$y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$ No absolute minimum
$y = x^2$	$(0, 2)$	No absolute extrema

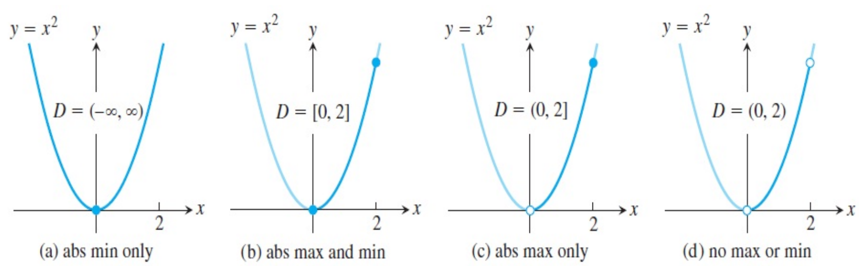


Figure 4.4:

The following theorem asserts that a function which is continuous at every point of a closed interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function.

Theorem 4.3.1. *If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$ and $m \le f(x) \le M$ for every other x in $[a, b]$.*

Local (Relative) Extreme Values Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is smaller than at any other point nearby. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

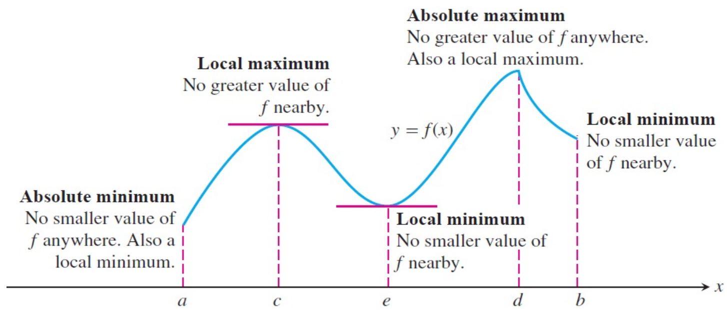


Figure 4.5:

Definition 4.3.2. A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c$$

We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a local maximum or local minimum value at an endpoint c if the appropriate inequality holds for all x in some half-open interval in its domain containing c. In Figure 4.5, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called relative extrema.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, a list of all local maxima will automatically include the absolute maximum if there is one. Similarly, a list of all local minima will include the absolute minimum if there is one.

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

Theorem 4.3.2. *If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then*

$$f'(c) = 0.$$

The above Theorem says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are:

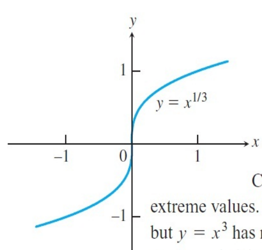
1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

Definition 4.3.3. An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Thus the only domain points where a function can assume extreme values are critical points and endpoints.

A differentiable function may have a critical point at $x = c$ without having a local extreme value there. For instance, the function $f(x) = x^3$ has a critical point at the origin and zero value there, but is positive to the right of the origin and negative to the left. So it cannot have a local extreme value at the origin (see Figure 4.6).



Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

Figure 4.6:

The question is: How to find the absolute extrema of a continuous function f on a finite closed interval?

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

Example 4.13. Find the absolute maximum and minimum values of $f(x) = 10x\,(2 - \ln x)$ on the interval $[1, e^2]$.

Solution 4.13. Figure 4.7 suggests that f has its absolute maximum value near $x = 3$ and its absolute minimum value of 0 at $x = e^2$. Let's verify this observation. We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values. The first derivative is

$$f'(x) = 10\,(2 - \ln x) - 10x\left(\frac{1}{x}\right) = 10\,(1 - \ln x)$$

The only critical point in the domain $[1, e^2]$ is the point $x = e$, where $f'(x) = 0$. The values of f at this one critical point and at the endpoints are:

$$f(e) = 10e, \quad f(1) = 20, \quad f\left(e^2\right) = 0$$

We can see from this list that the function's absolute maximum value is $10e$ it occurs at the critical interior point $x = e$. The absolute minimum value is 0 and occurs at the right endpoint $x = e^2$.

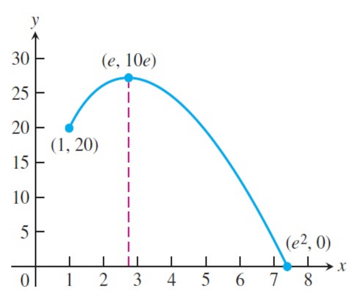


Figure 4.7:

Example 4.14. Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution 4.14. We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values. The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

$$f(0) = 0, \quad f(-2) = \sqrt[3]{4}, \quad f(3) = \sqrt[3]{9}$$

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9}$ and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$ where the graph has a cusp (Figure 4.8).

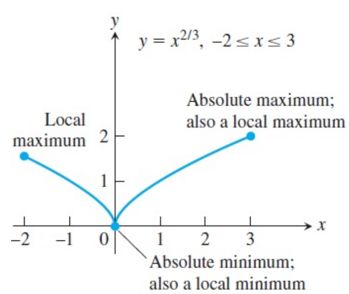


Figure 4.8:

Exercise 4.9. Find the absolute maximum and minimum values of $g(t) = t^2$ on $[-2, 1]$.

Example 4.15. If the function $f(x) = x^3 + ax^2 + bx$ has the local minimum value $-\frac{2}{9}\sqrt{3}$ at $x = 1/\sqrt{3}$, what are the values of a and b.

Solution 4.15. Since f has the local minimum value $-\frac{2}{9}\sqrt{3}$ at $x = 1/\sqrt{3}$ then we know two things:

$$\begin{aligned}
f\left(\frac{1}{\sqrt{3}}\right) &= -\frac{2}{9}\sqrt{3} \\
\left(\frac{1}{\sqrt{3}}\right)^3 + a\left(\frac{1}{\sqrt{3}}\right)^2 + b\left(\frac{1}{\sqrt{3}}\right) &= -\frac{2}{9}\sqrt{3} \\
\frac{1}{3\sqrt{3}} + \frac{a}{3} + \frac{b}{\sqrt{3}} &= -\frac{2}{9}\sqrt{3} \\
1 + a\sqrt{3} + 3b &= -2 \\
\sqrt{3}a + 3b &= -3 \quad \textbf{this is equation 1}
\end{aligned}$$

and

$$\begin{aligned}
f'\left(\frac{1}{\sqrt{3}}\right) &= 0 \\
3\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2a}{\sqrt{3}}\right) + b &= 0 \\
2a + \sqrt{3}b &= -\sqrt{3} \quad \textbf{this is equation 2}
\end{aligned}$$

By solving equations 1 and 2 we obtain: $a = 0$ and $b = -1$.

□

Exercise 4.10. For what values of the numbers a and b does the function

$$g(t) = ate^{bt^2}$$

have the maximum value $g(2) = 1$?

Exercise 4.11. Find the absolute maximum and minimum values of each function on the given interval.

1. $g(t) = -1/t$; $-2 \leq t \leq -1$
2. $g(t) = \sqrt[3]{t}$; $-1 \leq t \leq 8$
3. $g(t) = \sin t$; $-\frac{\pi}{2} \leq t \leq \frac{5\pi}{6}$

4.4 Monotonic Functions

In sketching the graph of a differentiable function, it is useful to know where it **increases** (rises from left to right) and where it **decreases** (falls from left to right) over an interval. This section gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function to identify whether local extreme values are present.

Increasing Functions and Decreasing Functions If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

Definition 4.4.1. Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$ then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$ then f is said to be **decreasing** on I.

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for every pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of it is sometimes said that f is strictly increasing or decreasing on I. The interval I may be finite (also called bounded) or infinite (unbounded) and never consists of a single point.

As another corollary to the Mean Value Theorem, functions with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions. A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

Corollary 4.4.1. *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*

* *If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.*
* *If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Example 4.16. Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and on which f is decreasing.

Solution 4.16. The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3\left(x^2 - 4\right) \\ &= 3(x-2)(x+2) \end{aligned}$$

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f to create non-overlapping open intervals $(\infty, -2)$, $(-2, 2)$ and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval. The behavior of f is determined then by applying Corollary 4.4.1 to each subinterval. The results are summarized in the following Figure.

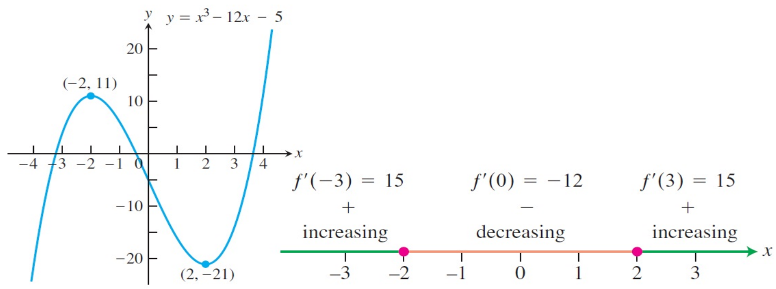


Figure 4.9:

First Derivative Test for Local Extrema In Figure 4.10, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its

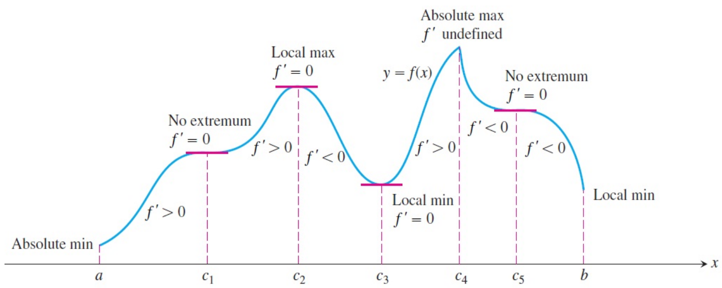


Figure 4.10:

right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

First Derivative Test for Local Extrema: Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Example 4.17. Find the critical points of

$$f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution 4.17. The function f is continuous at all x since it is the product of two continuous functions. The first derivative

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) \\ &= \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} \\ &= \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}} \end{aligned}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value. The critical points partition the $x-$axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points, as summarized in the following Figure. Thus, f decreases on

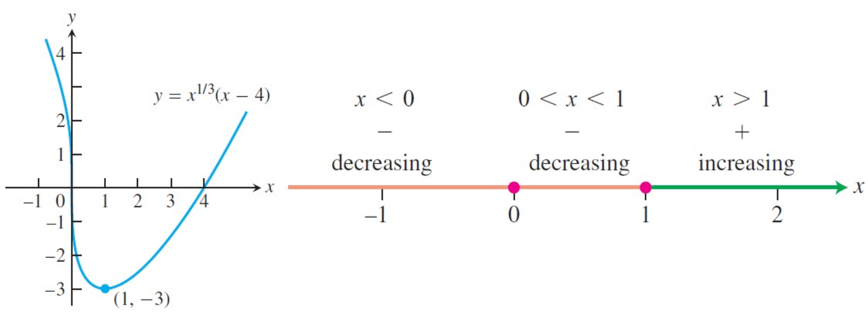


Figure 4.11:

$(-\infty, 1]$ and increases on $[1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive). The value of the local minimum is $f(1) = (1)^{1/3}(1-4) = -3$. This is also an absolute minimum since f is decreasing on $(-\infty, 1]$ and increases on $[1, \infty)$. Figure 4.11 shows this value in relation to the function's graph. Note that

$$\lim_{x \to 0} f'(x) = -\infty$$

so the graph of f has a vertical tangent at the origin.

Example 4.18. Find the critical points of

$$f(x) = \left(x^2 - 3\right) e^x.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution 4.18. The function f is continuous and differentiable for all real numbers, so the critical points occur only at the zeros of f'. Using the Derivative Product Rule, we find the derivative

$$\begin{aligned} f'(x) &= \left(x^2 - 3\right) \frac{d}{dx} e^x + e^x \frac{d}{dx} \left(x^2 - 3\right) \\ &= \left(x^2 - 3\right) e^x + e^x (2x) \\ &= \left(x^2 + 2x - 3\right) e^x \end{aligned}$$

Since e^x is never zero, the first derivative is zero if and only if

$$\begin{aligned} x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \end{aligned}$$

The zeros $x = -3$ and $x = 1$ partition the $x-$axis into intervals as follows. We can see from the Figure 4.12 that there is a local

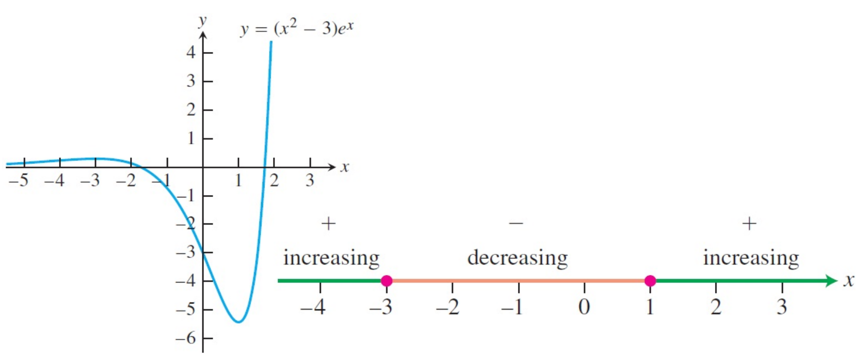


Figure 4.12:

maximum at $x = -3$ and a local minimum at $x = 1$. The local minimum value is also an absolute minimum. There is no absolute maximum. The function increases on $(-\infty, -3] \bigcup [1, \infty)$ and decreases on $[-3, 1]$. Figure 4.12 also shows the graph.

Exercise 4.12. For each of the following functions: Find the open intervals on which the function is increasing and decreasing, and identify the function's local and absolute extreme values, if any, saying where they occur.

$$(1) \qquad g(t) = t - 6\sqrt{t-1}$$
$$(2) \qquad g(t) = t \ln t$$

Exercise 4.13. Show that $g(t) = t - \ln t$ is increasing for $t > 1$.

4.5 Concavity and Curve Sketching

We have seen how the first derivative tells us where a function is increasing, where it is decreasing, and whether a local maximum or local minimum occurs at a critical point. In this section we see that the second derivative gives us information about how the graph of a differentiable function bends or turns. With this knowledge about the first and second derivatives, coupled with our previous understanding of symmetry and asymptotic behavior, we can now draw an accurate graph of a function. By organizing all of these ideas into a coherent procedure, we give a method for sketching graphs and revealing visually the key features of functions. Identifying and knowing the locations of these features is of major importance in mathematics and its applications to science and engineering, especially in the graphical analysis and interpretation of data.

Concavity As you can see in Figure 4.13, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the **concavity** of the curve.

Definition 4.5.1. The graph of a differentiable function $y = f(x)$ is

1. **concave up** on an open interval I if f' is increasing on I;

2. **concave down** on an open interval I if f' is decreasing on I.

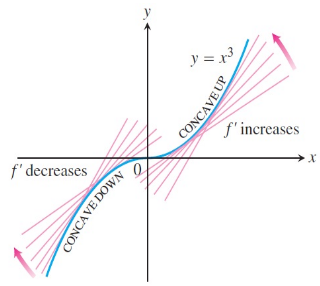


Figure 4.13:

The Second Derivative Test for Concavity: Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Illustration Example 4.3. The curve $y = x^3$ (Figure 4.13) is concave down on $(\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$. While, The curve $y = x^2$ (Figure 4.14) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive.

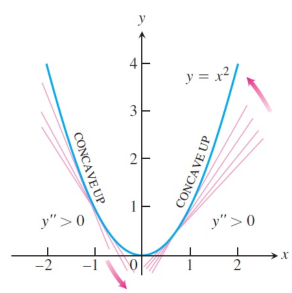


Figure 4.14:

Example 4.19. Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution 4.19. The first derivative of $y = 3 + \sin x$ is $y' = \cos x$ and the second derivative is $y'' = -\sin x$. The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$ where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$ where $y'' = -\sin x$ is positive (Figure 4.15).

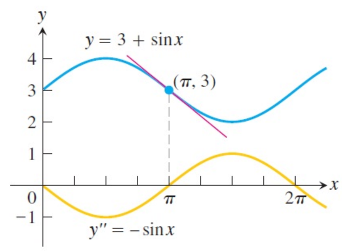


Figure 4.15:

Points of Inflection The curve $y = 3 + \sin x$ in the previous example changes concavity at the point $(\pi, 3)$. Since the first derivative $y = \cos$ exists for all x, we see that the curve has a tangent line of slope -1 at the point $(\pi, 3)$. This point is called a **point of inflection** of the curve. Notice from Figure 4.15 that the graph crosses its tangent line at this point and that the second derivative $y = -\sin x$ has value 0 when $x = \pi$. In general, we have the following definition.

Definition 4.5.2. A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection.**

Remark 4.5.1. At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

The next example illustrates a function having a point of inflection where the first derivative exists, but the second derivative fails to exist.

Example 4.20. Find the inflection points, if any, of the function $f(x) = x^{5/3}$.

Solution 4.20. The graph of $f(x) = x^{5/3}$ has a horizontal tangent at the origin because $f'(x) = \frac{5}{3}x^{2/3} = 0$ when $x = 0$. However, the second derivative

$$f''(x) = \frac{d}{dx}\left(\frac{5}{3}x^{2/3}\right) = \frac{10}{9}x^{-1/3} = \frac{10}{9\sqrt[3]{x}}$$

fails to exist at $x = 0$. Nevertheless, $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so the second derivative changes sign at $x = 0$ and there is a point of inflection at the origin. The graph is shown in Figure 4.16.

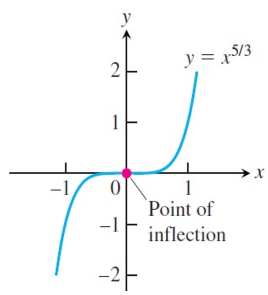


Figure 4.16:

Here is an example showing that an inflection point need not occur even though both derivatives exist and $f'' = 0$.

Example 4.21. Find the inflection points, if any, of the function $f(x) = x^4$.

Solution 4.21. The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.17). Even though the second derivative $y'' = 12x^2$ is zero there, it does not change sign.

□

As our final illustration, we show a situation in which a point of inflection occurs at a vertical tangent to the curve where neither the first nor the second derivative exists.

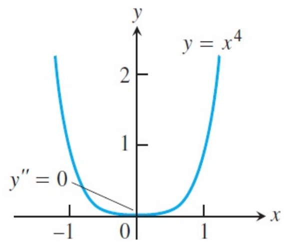


Figure 4.17:

Example 4.22. Find the inflection points, if any, of the function $f(x) = x^{1/3}$.

Solution 4.22. The graph of $f(x) = x^{1/3}$ has a point of inflection at the origin because the second derivative is positive for $x < 0$ and negative for $x > 0$:

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

However, both $y' = \frac{1}{3}x^{-2/3}$ and y'' fail to exist at $x = 0$, and there is a vertical tangent there. See Figure 4.18.

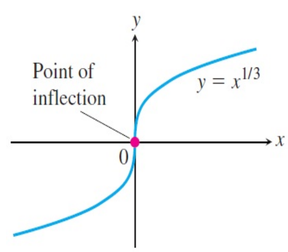


Figure 4.18:

Second Derivative Test for Local Extrema Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and nature of local extrema.

Theorem 4.5.1. *Suppose f'' is continuous on an open interval that contains $x = c$.*

1. *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.*
2. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.*
3. *If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.*

This test requires us to know f'' only at c itself and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if f'' does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Example 4.23. Discuss the curve $y = x^4 - 4x^3$ with respect to increasing, decreasing, concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution 4.23. The function f is continuous since it is a polynomial. The domain of f is $(-\infty, \infty)$ and the domain of $f'(x) = 4x^3 - 12x^2$ is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3),$$

the first derivative is zero at $x = 0$ and $x = 3$. We use these critical points to define intervals where f is increasing or decreasing.

Interval	$x < 0$	$0 < x < 3$	$x > 3$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

- Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.
- Using the table above, we see that f is decreasing on $(-\infty, 3]$, and increasing on $[3, \infty)$.
- $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$. We use these points to define intervals where f is concave up or concave down.

Interval	$x < 0$	$0 < x < 2$	$x > 2$
Sign of f''	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(\infty, 0) \bigcup (2, \infty)$ and concave down on $(0, 2)$. The general shape of the curve is shown Figure 4.19.

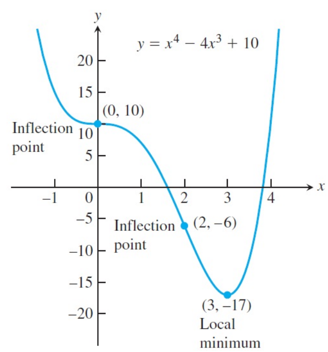


Figure 4.19:

Exercise 4.14. Find the intervals of increasing, decreasing and concavity, the extremum points and the points of inflections, of each of the following functions. Use this information to sketch the curve.

$$(1) \quad g(t) = \tan^{-1} t \qquad\qquad (2) \quad g(t) = \frac{4 + t^2}{2t}$$

Exercise 4.15. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.

Example 4.24. The graph of the derivative f' of a function f is shown in Figure 4.21. On what intervals is f increasing or decreasing, concave up or down? At what values of x does f have a extreme and inflection points?

Solution 4.24. Since $f'(x) > 0$ on $(1, 5)$, f is increasing on this interval. Since $f'(x) < 0$ on $(0, 1)$ and $(5, 6)$, f is decreasing on these

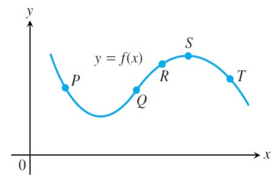


Figure 4.20:

intervals. Since $f'(x) = 0$ at $x = 1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 1$. Since $f'(x) = 0$ at $x = 5$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 5$.

Since f' increases on $(0, 3)$ then $f''(x) > 0$ on this interval, and since f' decreases on $(3, 6)$ then $f''(x) < 0$ on this interval. Hence, f is concave up on $(0, 3)$ and concave down on $(3, 6)$, where the point $(3, f(3))$ is the inflection point of the graph of f.

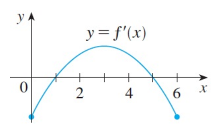


Figure 4.21:

Exercise 4.16. The graph of the derivative f' of a function f is shown in Figure 4.22. On what intervals is f increasing or decreasing, concave up or down? At what values of x does f have a extreme and inflection points?

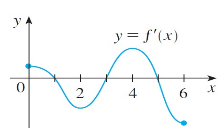


Figure 4.22:

Chapter **5**

Integration

The *integral* is of fundamental importance in statistics, the sciences, and engineering. As with the derivative, the integral also arises as a limit, this time of increasingly fine approximations. We use it to calculate quantities ranging from probabilities and averages to energy consumption and the forces against a dams floodgates. In this chapter we focus on the integral concept and its use in computing areas of various regions with curved boundaries.

5.1 Antiderivatives

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative. More generally, starting with a function f, we want to find a function F whose derivative is f. If such a function F exists, it is called an **antiderivative of** f. We will see next that antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

Definition 5.1.1. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *anti-differentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

Example 5.1. Find an antiderivative for $f(x) = 2x$.

Solution 5.1. We need to think backward here: What function do we know has a derivative equal to $2x$? It is not difficult to this question if we keep the Power Rule in mind. In fact, if $F(x) = x^2$, then $F'(x) = 2x = f(x)$. But the function $G(x) = x^2 - 5$ also satisfies $G'(x) = 2x = f(x)$. Therefore both F and G are antiderivatives of f. Indeed, any function of the form $H(x) = x^2 + C$, where C is a constant, is an antiderivative of $f(x) = x^2$.

□

Theorem 5.1.1. *If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is*

$$F(x) + C$$

where C is an arbitrary constant.

Example 5.2. Find an antiderivative F of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution 5.2. Since the derivative of x^3 is $3x^2$, the general antiderivative $F(x) = x^3 + C$ gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C. Substituting $x = 1$ into $F(x)$ gives

$$\begin{aligned} F(1) &= 1^3 + C \\ -1 &= 1 + C \\ C &= -2 \end{aligned}$$

So, $F(x) = x^3 - 2$ is the antiderivative satisfying $F(1) = -1$.

5.2 Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

Definition 5.2.1. The collection of all antiderivatives of is called the indefinite integral of f with respect to x, and is denoted by

$$\int f(x)dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

After the integral sign in the notation we just defined, the integrand function is always followed by a differential to indicate the variable of integration. Using this notation, we have, for example:

$$\begin{aligned} \int 2x dx &= x^2 + C \\ \int \cos x &= \sin x + C \end{aligned}$$

Exercise 5.1. Verify the formula

$$\int \frac{\tan^{-1} x}{x^2} dx = \ln x - \frac{1}{2} \ln\left(1 + x^2\right) - \frac{\tan^{-1} x}{x} + C.$$

In the following we list antiderivatives in the notation of indefinite integrals. Each formula is true because the derivative of the function in the right appears in the left. In particular, these formulas are the **rules of integration**.

Rule 1 The first formula says that the integration of a constant times a function is the constant times the integration of the function.

$$\int c f(x) dx = c \int f(x) dx$$

Rule 2 The second formula says that the integration of a sum (difference) is the sum (difference) of the integrations.

$$\int \left[f(x) \pm g(x)\right] dx = \int f(x) dx \pm \int g(x) dx$$

Rule 3 The integration of constant functions is the constant itself times x.

$$\int k dx = kx + C \text{ where } k \text{ is a constant}$$

For example,

$$\begin{aligned} \int \frac{1}{2} dx &= \frac{1}{2} x + C \\ \int \pi dx &= \pi x + C \end{aligned}$$

Rule 4 The integration of power functions is given by

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ where } n \neq -1$$

In general,

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \text{ where } n \neq -1$$

Example 5.3. Find the general indefinite integral

$$\int \left(x^2 + \frac{1}{x^2}\right) dx$$

Solution 5.3.

$$\begin{aligned}
\int \left(x^2 + \frac{1}{x^2}\right) dx &= \int x^2 dx + \int x^{-2} dx \\
&= \frac{1}{3}x^3 + \frac{1}{-1}x^{-1} + C \\
&= \frac{1}{3}x^3 - \frac{1}{x} + C
\end{aligned}$$

□

Example 5.4. Find the general indefinite integral

$$\int (x-1)(x+3)dx$$

Solution 5.4.

$$\begin{aligned}
\int (x-1)(x+3)dx &= \int \left(x^2 + 2x - 3\right) dx \\
&= \int x^2 dx + 2\int x dx - \int 3dx \\
&= \frac{1}{3}x^3 + 2 \times \frac{1}{2}x^2 - 3x + C \\
&= \frac{1}{3}x^3 + x^2 - 3x + C
\end{aligned}$$

Example 5.5. Find the general indefinite integral

$$\int \sqrt[3]{x^2}dx$$

Solution 5.5.

$$\begin{aligned}\int \sqrt[3]{x^2}dx &= \int x^{2/3}dx = \frac{1}{5/3}x^{5/3} + C \\ &= \frac{3}{5}\sqrt[3]{x^5} + C\end{aligned}$$

□

Example 5.6. Find the general indefinite integral

$$\int \frac{x^2 - 3\sqrt{x}}{x}dx$$

Solution 5.6.

$$\begin{aligned}\int \frac{x^2 - 3\sqrt{x}}{x}dx &= \int \left(\frac{x^2}{x} - \frac{3\sqrt{x}}{x}\right) dx \\ &= \int \left(x - 3x^{-1/2}\right) dx \\ &= \int xdx - 3\int x^{-1/2}dx \\ &= \frac{1}{2}x^2 - 3 \times \frac{1}{1/2}x^{1/2} + C \\ &= \frac{1}{2}x^2 - 6\sqrt{x} + C\end{aligned}$$

□

Example 5.7. Find the general indefinite integral

$$\int (1-2x)^{99}dx$$

Solution 5.7.

$$\int (1-2x)^{99}dx = \frac{(1-2x)^{100}}{-2 \times 100} + C = -\frac{(1-2x)^{100}}{200} + C$$

Rule 5 The fifth rule is the first rule we learn about the integration of the division of two functions.

$$\int \frac{1}{x} dx = \ln|x| + C$$

In general,

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

Example 5.8. Find the general indefinite integral

$$\int \frac{e^x}{9+e^x} dx$$

Solution 5.8. Since $\frac{d}{dx}(9+e^x) = e^x$, then

$$\int \frac{e^x}{9+e^x} dx = \ln|9+e^x| + C$$

□

Example 5.9. Find the general indefinite integral

$$\int \frac{x}{1+x^2} dx$$

Solution 5.9. Note $\frac{d}{dx}(1+x^2) = 2x \neq x$. So, multiply the integral by $\frac{2}{2}$ to obtain:

$$\begin{aligned} \int \frac{x}{1+x^2} dx &= \frac{2}{2} \int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{2x}{1+x^2} dx \\ &= \frac{1}{2} \ln\left|1+x^2\right| + C \end{aligned}$$

□

Example 5.10. Find the general indefinite integral

$$\int \tan x dx$$

Solution 5.10. To evaluate this integral, write $\tan x$ as $\sin x/\cos x$. Thus,

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx \\ &= -\ln|\cos x| + C = \ln|\sec x| + C \end{aligned}$$

□

Rule 6 The rules of integration of exponential functions are as follows.

$$\int e^x dx = e^x + C \text{ and } \int a^x dx = \frac{a^x}{\ln a} + C$$

In general,

$$\int e^{\alpha x+\beta} dx = \frac{1}{\alpha} e^{\alpha x+\beta} + C \text{ and } \int a^{\alpha x+\beta} dx = \frac{a^{\alpha x+\beta}}{\alpha \ln a} + C$$

Example 5.11. Find the general indefinite integral

$$\int 2^{8-6x} dx$$

Solution 5.11.

$$\int 2^{8-6x} dx = -\frac{2^{8-6x}}{6 \times \ln 2} + C$$

□

Example 5.12. Find the general indefinite integral

$$\int e^{2\ln x} dx$$

Solution 5.12.

$$\int e^{2\ln x} dx = \int e^{\ln\left(x^2\right)} dx = \int x^2 dx = \frac{1}{3} x^3 + C$$

□

Rule 7 The following rules represent the rules of integrations of some trigonometric functions.

(1) $$\int \sin x dx = -\cos x + C.$$

$$\int \sin(\alpha x + \beta) dx = -\frac{\cos(\alpha x + \beta)}{\alpha} + C$$

(2) $$\int \cos x dx = \sin x + C$$

$$\int \cos(\alpha x + \beta) dx = \frac{\sin(\alpha x + \beta)}{\alpha} + C$$

(3) $$\int \sec^2 x dx = \tan x + C$$

$$\int \sec^2(\alpha x + \beta) dx = \frac{\tan(\alpha x + \beta)}{\alpha} + C$$

(4) $$\int \csc^2 x dx = -\cot x + C$$

$$\int \csc^2(\alpha x + \beta) dx = -\frac{\cot(\alpha x + \beta)}{\alpha} + C$$

(5) $$\int \sec x \tan x dx = \sec x + C$$

$$\int \sec(\alpha x + \beta) \tan(\alpha x + \beta) dx = \frac{\sec(\alpha x + \beta)}{\alpha} + C$$

(6) $$\int \csc x \cot x dx = -\csc x + C$$

$$\int \csc(\alpha x + \beta) \cot(\alpha x + \beta) dx = -\frac{\csc(\alpha x + \beta)}{\alpha} + C$$

Example 5.13. Find the general indefinite integral

$$\int \sec x(\tan x + \cos x) dx$$

Solution 5.13.

$$\begin{aligned}\int \sec x(\tan x + \cos x)dx &= \int \sec x \tan x dx + \int \sec x \cos x dx \\ &= \int \sec x \tan x dx + \int \frac{1}{\cos x} \cdot \cos x dx \\ &= \int \sec x \tan x dx + \int 1 dx \\ &= \sec x + x + C\end{aligned}$$

□

Example 5.14. Find the general indefinite integral

$$\int \sec x dx$$

Solution 5.14.

$$\begin{aligned}\int \sec x dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

□

Example 5.15. Find the general indefinite integral

$$\int \sin(3x - 10)dx$$

Solution 5.15.

$$\int \sin(3x - 10)dx = -\frac{1}{3}\cos(3x - 10) + C$$

□

Example 5.16. Find the general indefinite integral

$$\int \tan^2 x dx$$

Solution 5.16.

$$\begin{aligned}
\int \tan^2 x dx &= \int \left(\sec^2 x - 1\right) dx \\
&= \int \sec^2 x dx - \int 1 dx \\
&= \tan x - x + C
\end{aligned}$$

□

Example 5.17. Find the general indefinite integral

$$\int \frac{1}{1+\sin x} dx$$

Solution 5.17.

$$\begin{aligned}
\int \frac{1}{1+\sin x} dx &= \int \frac{1}{1+\sin x} \cdot \frac{1-\sin x}{1-\sin x} dx \\
&= \int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx \\
&= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx \\
&= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx \\
&= \int \sec^2 x dx - \int \tan x \sec dx \\
&= \tan x - \sec x + C
\end{aligned}$$

□

Rule 8 The integration rules of the main inverse trigonometric functions are:

$$(1)\qquad \int \frac{1}{1+x^2}dx = \tan^{-1} x + C$$

$$\int \frac{1}{a+bx^2}dx = \frac{1}{\sqrt{ab}}\tan^{-1}\left(\sqrt{\frac{b}{a}}x\right) + C$$

$$(2)\qquad \int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$$

$$\int \frac{1}{\sqrt{a-bx^2}}dx = \frac{1}{\sqrt{b}}\sin^{-1}\left(\sqrt{\frac{b}{a}}x\right) + C$$

$$(3)\qquad \int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1} x + C$$

Example 5.18. Find the general indefinite integral

$$\int \frac{1}{25+4x^2}dx$$

Solution 5.18.

$$\begin{aligned}\int \frac{1}{25+4x^2}dx &= \frac{1}{\sqrt{25\times 4}}\tan^{-1}\left(\sqrt{\frac{4}{25}}x\right) + C\\ &= \frac{1}{10}\tan^{-1}\left(\frac{2}{5}x\right) + C\end{aligned}$$

Example 5.19. Find the general indefinite integral

$$\int \frac{1}{\sqrt{16-9x^2}}dx$$

Solution 5.19.

$$\begin{aligned}\int \frac{1}{\sqrt{16-9x^2}} &= \frac{1}{\sqrt{9}}\sin^{-1}\left(\sqrt{\frac{9}{16}}x\right) + C\\ &= \frac{1}{3}\sin^{-1}\left(\frac{3}{4}x\right) + C\end{aligned}$$

Exercise 5.2. Evaluate the general indefinite integral.

$$(1) \quad \int t^{\frac{1}{3}}\left(1+t^2\right)^2 dt$$

$$(2) \quad \int \frac{t+1}{t^2+2t+5} dt$$

$$(3) \quad \int \frac{e^t-e^{-t}}{e^t+e^{-t}} dt$$

$$(4) \quad \int \csc t dt$$

$$(5) \quad \int \cot t dt$$

$$(6) \quad \int \frac{\sin 2t}{\sin t} dt$$

$$(7) \quad \int \frac{1}{1+\cos 2t} dt$$

$$(8) \quad \int \left[\frac{1}{2t}+5^t\right] dt$$

$$(9) \quad \int \frac{1+t+t^2}{1+t^2} dt$$

5.3 Integration by Substitution

In this section we will study a technique, called **substitution**, that can often be used to transform complicated integration problems into simpler ones. The method of substitution can be motivated by examining the chain rule from the viewpoint of anti-differentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \times g'(x)$$

which we can write in integral form as

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

or since F is an antiderivative of f,

$$\int f(g(x))g'(x)dx = F(g(x)) + C \tag{5.3.1}$$

For our purposes it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x)dx$. With this notation, equation 5.3.1 can be expressed as

$$\int f(u)du = F(u) + C \tag{5.3.2}$$

The process of evaluating an integral of form 5.3.1 by converting it into form 5.3.2 with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)dx$$

is called the **method of $u-$substitution**. Here our emphasis is not on the interpretation of the expression $du = g'(x)dx$. Rather, the differential notation serves primarily as a useful *bookkeeping* device for the method of $u-$substitution. The following example illustrates how the method works.

Example 5.20. Evaluate

$$\int 2x\left(x^2+1\right)^{50} dx$$

Solution 5.20. If we let $u = x^2 + 1$, then $du/dx = 2x$, which implies that $du = 2xdx$. Thus, the given integral can be written as

$$\int 2x\left(x^2+1\right)^{50} dx = \int u^{50}du = \frac{u^{51}}{51} + C = \frac{\left(x^2+1\right)^{51}}{51} + C$$

□

Example 5.21. Evaluate

$$\int \sin^2 x \cos x dx$$

Solution 5.21. If we let $u = \sin x$, then $du/dx = \cos x$, so $du = \cos x dx$. Thus,

$$\int \sin^2 x \cos x dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C$$

Example 5.22. Evaluate

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$$

Solution 5.22. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}} dx \quad \text{or} \quad 2du = \frac{1}{\sqrt{x}} dx$$

Thus.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int 2e^u du = 2e^u + C = 2e^{\sqrt{x}} + C$$

□

Example 5.23. Evaluate

$$\int x^4 \sqrt[3]{3 - 5x^5} dx$$

Solution 5.23. If we let $u = 3 - 5x^5$, then

$$\frac{du}{dx} = -25x^4, \quad \text{so} \quad du = -25x^4 dx \quad \text{or} \quad -\frac{1}{25} du = x^4 dx$$

Thus.

$$\begin{aligned} \int x^4 \sqrt[3]{3 - 5x^5} dx &= \int -\frac{1}{25} \sqrt[3]{u} du = -\frac{1}{25} \int u^{\frac{1}{3}} du \\ &= -\frac{1}{25} \times \frac{3}{4} u^{\frac{4}{3}} + C \\ &= -\frac{3}{100} \left(3 - 5x^5\right)^{\frac{4}{3}} + C \end{aligned}$$

□

Example 5.24. Evaluate

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$$

Solution 5.24. Substituting $u = e^x$ and $du = e^x dx$ yields

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1}(e^x) + C$$

□

The method of substitution is relatively straightforward, provided the integrand contains an easily recognized composition $f(g(x))$ and the remainder of the integrand is a constant multiple of $g'(x)$. If this is not the case, the method may still apply but may require more computation.

Example 5.25. Evaluate

$$\int x^2 \sqrt{x-1} dx$$

Solution 5.25. The composition $\sqrt{x-1}$ suggests the substitution $u = x-1$ so that $du = dx$. From the equality $u = x - 1$ we have

$$x^2 = (u+1)^2 = u^2 + 2u + 1$$

so that

$$\begin{aligned}
\int x^2 \sqrt{x-1} &= \int \left(u^2 + 2u + 1\right) \sqrt{u} du = \int \left(u^{5/2} + 2u^{3/2} + u^{1/2}\right) du \\
&= \frac{2}{7} u^{7/2} + 2 \times \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C \\
&= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C
\end{aligned}$$

□

Exercise 5.3. Evaluate the integrals using appropriate substitutions.

(1) $\int t^3\sqrt{5+t^4}dt$

(2) $\int \frac{\sin(1/t)}{3t^2}dt$

(3) $\int e^{\sin t}\cos t dt$

(4) $\int \frac{t}{1+t^4}dt$

(5) $\int \sqrt{e^t}dt$

(6) $\int \frac{(\ln t)^2}{t}dt$

(7) $\int t\sqrt{t-1}dt$

(8) $\int \frac{1}{t\ln t}dt$

(9) $\int \frac{(\tan^{-1} t)^2}{1+t^2}dt$

(10) $\int \frac{1}{t^2+2t+5}dt$

(11) $\int \frac{1}{t+\sqrt{t}}dt$

(12) $\int \frac{\sin(2t)}{1+\cos^2 t}dt$

(13) $\int \frac{(t-1)^5}{t^7}dt$

(14) $\int \tan t \ln(\cos t)dt$

(15) $\int \frac{1}{t}\sin\left(\ln\sqrt{t}\right)dt$

5.4 The Definite Integral

In this section we define the definite integral of a function f on an interval I. Let us assume, for the time being, that $f(x)$ is defined and continuous on the closed, finite interval $[a, b]$. We no longer assume that the values of f are nonnegative. The symbol for the definite integral of $f(x)$ over the interval $[a, b]$ is

$$\int_a^b f(x)dx$$

which is read as **the integral from** a **to** b **of** f **of** x **with respect to** x. The component parts in the integral symbol also have names as shown in Figure 5.1 below.

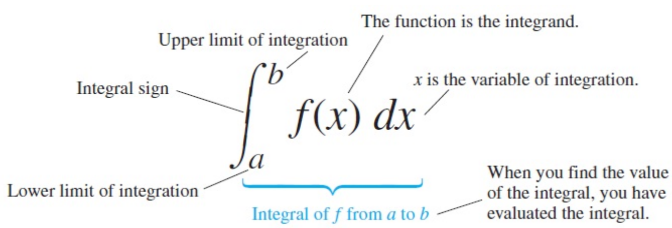


Figure 5.1:

The following theorem says that if a function is continuous on a finite closed interval, then it is integrable on that interval, and its definite integral is the net signed area between the graph of the function and the interval.

Theorem 5.4.1. *If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$, and the net signed area A between the graph of f and the interval $[a, b]$ is*

$$A = \int_a^b f(x)dx$$

Example 5.26. Sketch the region whose area is represented by the definite integral

$$\int_1^4 2dx$$

and evaluate the integral using an appropriate formula from geometry.

Solution 5.26. The graph of the integrand is the horizontal line $y = 2$, so the region is a rectangle of height 2 extending over the interval from 1 to 4 (Figure 5.2). Thus,

$$\int_1^4 2dx = (\text{area of rectangle}) = 2 \times 3 = 6$$

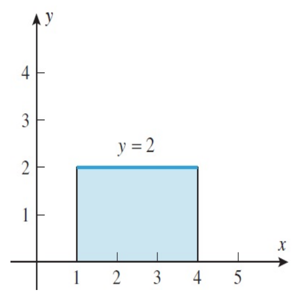


Figure 5.2:

Example 5.27. Sketch the region whose area is represented by the definite integral

$$\int_0^1 \sqrt{1 - x^2}dx$$

and evaluate the integral using an appropriate formula from geometry.

Solution 5.27. The graph of $\sqrt{1 - x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = 0$ to $x = 1$ (Figure 5.3). Thus,

$$\int_0^1 \sqrt{1 - x^2}dx = (\text{area of quarter-circle}) = \frac{1}{4} \times (1)^2\pi = \frac{\pi}{4}$$

Exercise 5.4. Sketch the region whose area is represented by the definite integral

$$\int_{-1}^2 (x + 2)dx$$

and evaluate the integral using an appropriate formula from geometry.

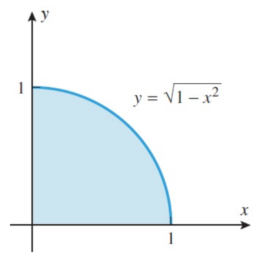


Figure 5.3:

Example 5.28. Evaluate

$$(1) \qquad \int_0^2 (x-1)dx$$
$$(2) \qquad \int_0^1 (x-1)dx$$

Solution 5.28. The graph of $y = x - 1$ is shown in Figure 5.4, and we leave it for you to verify that the shaded triangular regions both have area $\frac{1}{2}$. Over the interval $[0, 2]$ the net signed area is $A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = 0$, and over the interval $[0, 1]$ the net signed area is $-A_2 = -\frac{1}{2}$. Thus,

$$\int_0^2 (x-1)dx = 0 \quad \text{and} \quad \int_0^1 (x-1)dx = -\frac{1}{2}$$

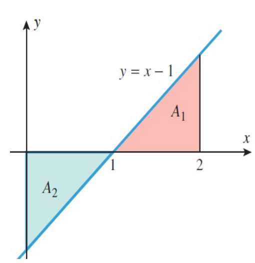


Figure 5.4:

Exercise 5.5. Use the areas shown in Figure 5.5 to find:

(1) $\int_a^b f(x)dx$

(2) $\int_b^c f(x)dx$

(3) $\int_a^c f(x)dx$

(4) $\int_a^d f(x)dx$

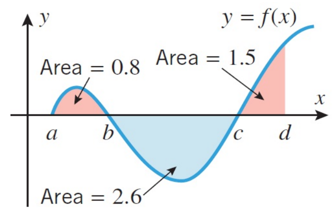


Figure 5.5:

Properties of the Definite Integral

- If a is in the domain of f, we define

$$\int_a^a f(x)dx = 0$$

- If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

- If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f+g$, and $f-g$ are integrable on $[a, b]$ and

(1) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(2) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

- If f is integrable on a closed interval containing the three points a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

no matter how the points are ordered.

- If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq 0$$

and if $f(x) \leq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \leq 0$$

- If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

Example 5.29. Find

$$\int_{-1}^2 [f(x) + 2g(x)]dx$$

if

$$\int_{-1}^2 f(x)dx = 5 \quad \text{and} \quad \int_{-1}^2 g(x)dx = -3$$

Solution 5.29.

$$\begin{aligned} \int_{-1}^2 [f(x) + 2g(x)]dx &= \int_{-1}^2 f(x)dx + 2\int_{-1}^2 g(x)dx \\ &= 5 + 2 \times (-3) \\ &= -1 \end{aligned}$$

□

Exercise 5.6. Find

$$\int_1^4 [3f(t) - g(t)]dt$$

if

$$\int_1^4 f(t)dt = 2 \quad \text{and} \quad \int_1^4 g(t)dt = 10$$

Example 5.30. Find

$$\int_1^5 f(x)dx$$

if

$$\int_0^1 f(x)dx = -2 \quad \text{and} \quad \int_0^5 f(x)dx = 1$$

Solution 5.30.

$$\begin{aligned} \int_0^5 f(x)dx &= \int_0^1 f(x)dx + \int_1^5 f(x)dx \\ 1 &= -2 + \int_1^5 f(x)dx \\ \int_1^5 f(x)dx &= 3 \end{aligned}$$

□

Exercise 5.7. Find

$$\int_3^{-2} f(t)dt$$

if

$$\int_{-2}^1 f(t)dt = 2 \quad \text{and} \quad \int_1^3 f(t)dt = -6$$

Example 5.31. Determine whether the value of the integral

$$\int_2^3 \frac{\sqrt{x}}{1-x}dx$$

is positive or negative.

Solution 5.31. Since $\sqrt{x} > 0$ and $1 - x < 0$ on $[2, 3]$ then

$$\frac{\sqrt{x}}{1-x} < 0 \quad \text{and} \quad \int_2^3 \frac{\sqrt{x}}{1-x} dx < 0$$

□

Exercise 5.8. Determine whether the value of the integral

$$\int_0^4 \frac{t^2}{3 - \cos t} dt$$

is positive or negative.

5.5 The Fundamental Theorem of Calculus

In this section we will establish two basic relationships between definite and indefinite integrals that together constitute a result called the **Fundamental Theorem of Calculus**. One part of this theorem will relate the antiderivative method for calculating areas, and the second part will provide a powerful method for evaluating definite integrals using antiderivatives.

The first part of the Fundamental Theorem of Calculus says that: The definite integral can be evaluated by finding any antiderivative of the integrand and then subtracting the value of this antiderivative at the lower limit of integration from its value at the upper limit of integration.

Theorem 5.5.1. *If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a)$$

Observe that in the preceding theorem we did not include a constant of integration in the antiderivatives. In general, when applying the Fundamental Theorem of Calculus there is no need to include a constant of integration because it will drop out anyway.

Example 5.32. Evaluate

$$(1) \qquad \int_1^4 3dx$$

$$(2) \qquad \int_1^9 xdx$$

$$(3) \qquad \int_0^{\pi/3} \sec^2 xdx$$

$$(4) \qquad \int_0^{\ln 3} 5e^x dx$$

$$(5) \qquad \int_{-e}^{-1} \frac{1}{x} dx$$

$$(6) \qquad \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} dx$$

Solution 5.32.

$$(1) \qquad \int_1^4 3dx = 3x\Big]_1^4 = 3(4) - 3(1) = 9$$

$$(2) \qquad \int_1^9 xdx = \frac{1}{2}x^2\Big]_1^9 = \frac{1}{2}(9)^2 - \frac{1}{2}(1)^2 = 40$$

$$(3) \qquad \int_0^{\pi/3} \sec^2 xdx = \tan x\Big]_0^{\pi/3} = \tan\frac{\pi}{3} - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$(4) \qquad \int_0^{\ln 3} 5e^x dx = 5e^x\Big]_0^{\ln 3} = 5e\ln 3 - 5e^0 = 5(3) - 5(1) = 10$$

$$(5) \qquad \int_{-e}^{-1} \frac{1}{x} dx = \ln|x|\Big]_{-e}^{-1} = \ln|-1| - \ln|-e| = 0 - 1 = -1$$

$$(6) \qquad \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x\Big]_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{6} - \frac{-\pi}{6} = \frac{\pi}{3}$$

□

Example 5.33. Evaluate

$$\int_0^3 f(x)dx \quad \text{if} \quad f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x \geq 2 \end{cases}$$

Solution 5.33. From the properties of definite integral we can integrate from 0 to 2 and from 2 to 3 separately and add the results. This yields

$$\begin{aligned}\int_0^3 f(x)dx &= \int_0^2 f(x)dx + \int_2^3 f(x)dx = \int_0^2 x^2dx + \int_2^3 (3x-2)dx \\ &= \frac{x^3}{3}\Big]_0^2 + \left(\frac{3x^2}{2} - 2x\right)\Big]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}\end{aligned}$$

□

Exercise 5.9. Evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus.

$$(1) \qquad \int_{-1}^{2} 4t\left(1-t^2\right)dt$$

$$(2) \qquad \int_{-\pi/2}^{\pi/2} \sin t dt$$

$$(3) \qquad \int_{-1}^{1} |2t-1|dt$$

The next theorem describes the method for evaluating definite integrals in which a substitution is required.

Theorem 5.5.2. *If g' is continuous on $[a,b]$ and f is continuous on an interval containing the values of $g(x)$ for $a \le x \le b$, then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 5.34. Evaluate

$$\int_0^2 x\left(x^2+1\right)^3 dx$$

Solution 5.34. If we let $u = x^2 + 1$ so that $du = 2xdx$ or $\frac{1}{2}du = xdx$, then

$$\begin{aligned}&\text{if} \quad x = 0 \quad , \quad u = 1 \\ &\text{if} \quad x = 2 \quad , \quad u = 5\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^2 x\left(x^2+1\right)^3 dx &= \frac{1}{2}\int_1^5 u^3 du\\ &= \left.\frac{u^4}{8}\right]_1^5 = \frac{625}{8}-\frac{1}{8}=78\end{aligned}$$

□

Example 5.35. Evaluate

$$\int_0^{\pi/8} \sin^5 2x \cos 2x dx$$

Solution 5.35. If we let $u = \sin 2x$ so that $du = 2\cos 2x dx$ or $\frac{1}{2}du = \cos 2x dx$, then

$$\begin{aligned}&\text{if} \quad x = 0 \quad , \quad u = \sin(0) = 0\\ &\text{if} \quad x = \frac{\pi}{8} \quad , \quad u = \sin(\pi/8) = \frac{1}{\sqrt{2}}\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^{\pi/8} \sin^5 2x \cos 2x dx &= \frac{1}{2}\int_0^{1/\sqrt{2}} u^5 du\\ &= \left.\frac{u^6}{12}\right]_0^{1/\sqrt{2}} = \frac{1}{12\left(\sqrt{2}\right)^6} - 0 = \frac{1}{96}\end{aligned}$$

□

Exercise 5.10. Evaluate the integrals.

$$(1) \qquad \int_0^{\pi/4} \tan^2 t \sec^2 t dt$$

$$(2) \qquad \int_0^1 t^3\sqrt{t^2+3}dt$$

$$(3) \qquad \int_1^{\sqrt{e}} \frac{1}{t\sqrt{1-(\ln t)^2}}dt$$

Example 5.36. Find

$$\int_{-2}^{0} xf\left(x^2\right)dx \quad \text{if} \quad \int_0^4 f(x)dx = 1$$

Solution 5.36. If we let $u = x^2$ so that $du = 2xdx$ or $\frac{1}{2}du = xdx$, then

$$\begin{aligned} \text{if} \quad x = -2 \quad &, \quad u = 4 \\ \text{if} \quad x = 0 \quad &, \quad u = 0 \end{aligned}$$

Thus,

$$\int_{-2}^{0} xf\left(x^2\right)dx = \frac{1}{2}\int_0^4 f(u)du = \frac{1}{2} \times 1 = \frac{1}{2}$$

□

Exercise 5.11. Find

$$\int_0^1 f(3t+1)dt \quad \text{if} \quad \int_1^4 f(t)dt = 5$$

The second part of the Fundamental Theorem of Calculus says: If a definite integral has a variable upper limit of integration, a constant lower limit of integration, and a continuous integrand, then the derivative of the integral with respect to its upper limit is equal to the integrand evaluated at the upper limit.

Theorem 5.5.3. *If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by*

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Corollary 5.5.4.

$$(1) \qquad \frac{d}{dx}\left[\int_a^{h(x)} f(t)dt\right] = f(h(x)) \cdot h'(x)$$

$$(2) \qquad \frac{d}{dx}\left[\int_{l(x)}^{h(x)} f(t)dt\right] = f(h(x)) \cdot h'(x) - f(l(x)) \cdot l'(x)$$

Example 5.37. Find

$$\frac{d}{dx}\left[\int_1^x t^3 dt\right]$$

Solution 5.37. The integrand is a continuous function, so

$$\frac{d}{dx}\left[\int_1^x t^3 dt\right] = x^3$$

□

Example 5.38. Find

$$\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t} dt\right]$$

Solution 5.38. The integrand is a continuous on any interval that does not contain the origin, so on the interval $(0, \infty)$ we have

$$\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t} dt\right] = \frac{\sin x}{x}$$

□

Example 5.39. Use the part 2 of the Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\left[\int_1^{x^2} e^t dt\right]$$

Solution 5.39. The integrand is a continuous function, so

$$\frac{d}{dx}\left[\int_1^{x^2} e^t dt\right] = e^{x^2} \frac{d}{dx}\left(x^2\right) = 2xe^{x^2}$$

Example 5.40. Use the part 2 of the Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\left[\int_{\sin x}^{\cos x} t^2 dt\right]$$

Solution 5.40. The integrand is a continuous function, so

$$\begin{aligned}\frac{d}{dx}\left[\int_{\sin x}^{\cos x} t^2 dt\right] &= \cos^2 x \frac{d}{dx}(\cos x) - \sin^2 x \frac{d}{dx}(\sin x) \\ &= -\sin x \cos^2 x - \cos x \sin^2 x\end{aligned}$$

□

Exercise 5.12. Use the part 2 of the Fundamental Theorem of Calculus to find

$$(1) \qquad \frac{d}{dx}\left[x\int_{2}^{x^2} \sin\left(t^3\right) dt\right]$$

$$(2) \qquad \frac{d}{dx}\left[\int_{2^x}^{1} \sqrt[3]{t}dt\right]$$

$$(3) \qquad \frac{d}{dx}\left[\int_{-x}^{x^2} \sin^{-1} t dt\right]$$

The two parts of the Fundamental Theorem of Calculus, when taken together, tell us that differentiation and integration are inverse processes in the sense that each undoes the effect of the other. To see why this is so, note that Part 1 of the Fundamental Theorem of Calculus implies that

$$\int_a^x f'(t)dt = f(x) - f(a)$$

which tells us that if the value of $f(a)$ is known, then the function f can be recovered from its derivative f' by integrating. Conversely, Part 2 of the Fundamental Theorem of Calculus states that

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

which tells us that the function f can be recovered from its integral by differentiating. Thus, differentiation and integration can be viewed as inverse processes.

Example 5.41. Find a function f and a number a such that

$$6+\int_a^x \frac{f(t)}{t^2}dt = 2\sqrt{x} \quad \text{for} \quad x \geq 0$$

Solution 5.41. If we let $x = a \geq 0$, we have

$$\begin{aligned} 6+\int_a^a \frac{f(t)}{t^2}dt &= 2\sqrt{a} \\ 6+0 &= 2\sqrt{a} \\ a &= 9 \end{aligned}$$

To find f, differentiate both sides of the equation with respect to x:

$$\begin{aligned} \frac{d}{dx}\left[6+\int_a^x \frac{f(t)}{t^2}dt\right] &= \frac{d}{dx}\left[2\sqrt{x}\right] \\ 0+\frac{f(x)}{x^2} &= \frac{1}{\sqrt{x}} \\ f(x) &= \frac{x^2}{\sqrt{x}} = \sqrt{x^3} \quad \text{where} \quad x \geq 0 \end{aligned}$$

□

Exercise 5.13. If

$$f(x) = \int_0^x \left(1-t^2\right) dt,$$

on what intervals is f increasing, decreasing, concave up or down, and what are the extremum and inflection points, if any, of the function f?

5.6 Area Between Two Curves

In this section we review and extend the use of definite integrals to represent plane areas. Recall that the integral

$$\int_a^b f(x)dx$$

measures the area between the graph of f and the $x-$axis from $x = a$ to $x = b$, but treats as negative any part of this area that lies below the

$x-$axis. (We are assuming that $a < b$.) In order to express the total area bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$, counting all of the area positively, we should integrate the absolute value of f (see Figure 5.6):

$$\int_a^b f(x)dx = A_1 - A_2 \quad \text{and} \quad \int_a^b |f(x)|dx = A_1 + A_2$$

There is no rule for integrating

$$\int_a^b |f(x)|dx,$$

one must break the integral into a sum of integrals over intervals where $f(x) > 0$ so $|f(x)| = f(x)$, and intervals where $f(x) < 0$ so $|f(x)| = -f(x)$.

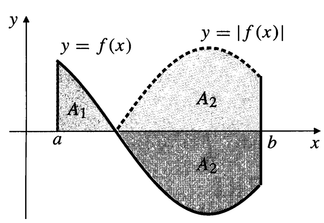


Figure 5.6:

To find the area between the graph of f and the $x-$axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Example 5.42. Figure 5.7 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute the area between the graph of $f(x)$ and the $x-$axis over $[0, 2\pi]$.

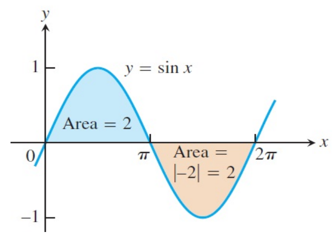


Figure 5.7:

Solution 5.42. First find the zeros of f. On $[0, 2\pi]$, the zeros of $f(x) = \sin x$ are $x = 0,\ \pi,\ 2\pi$. So, the area between the graph of $f(x)$ and the $x-$axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\begin{aligned}\int_0^{\pi} \sin x dx &= -\cos x]_0^{\pi} = -(\cos\pi - \cos 0) = 2\\ \int_{\pi}^{2\pi} \sin x dx &= -\cos x]_{\pi}^{2\pi} = -(\cos 2\pi - \cos\pi) = -2\end{aligned}$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$\text{Area } = |2| + |-2| = 4.$$

□

Example 5.43. Find the area of the region between the $x-$axis and the graph of $f(x) = x^3 - x^2 - 2x$ over $[-1, 2]$.

Solution 5.43. First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x+1)(x-2),$$

the zeros are $x = 0,\ -1$ and 2 (Figure 5.8). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$ on which $f \geq 0$ and $[0, 2]$, on which $f \leq 0$.

We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0}\left(x^3-x^2-2x\right)dx = \left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{-1}^{0} = \frac{5}{12}$$
$$\int_{0}^{2}\left(x^3-x^2-2x\right)dx = \left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{0}^{2} = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals.

$$\text{Area} = \left|\frac{5}{12}\right| + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

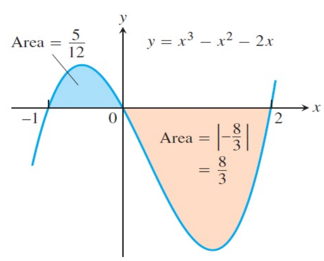


Figure 5.8:

□

Exercise 5.14. Find the total area between the region and the $x-$axis.

(1) $f(x) = -x^2 - 2x$ over $[-3, 2]$

(2) $f(x) = x^3 - x$

(3) $f(x) = x^{1/3} - x$ over $[-1, 8]$

Areas Between Curves Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve

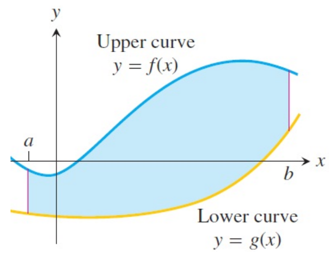


Figure 5.9:

$y = g(x)$ and on the left and right by the lines $x = a$ and $x = b$ (Figure 5.9). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area with an integral.

Definition 5.6.1. If f and g are continuous throughout $[a, b]$, then the area of the region between the curves $f(x)$ and $g(x)$ from a to b is the integral of $|f - g|$ from a to b:

$$A = \int_a^b |f(x) - g(x)| dx.$$

When applying this definition it is helpful to graph the curves. It helps you find the limits of integration if they are not given. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x. Then you can integrate the function for the area between the intersections.

Example 5.44. Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = \frac{1}{2}e^x$, on the left by $x = 0$, and on the right by $x = 1$.

Solution 5.44. Figure 5.10 displays the graphs of the curves and the region whose area we want to find. The area between the curves over the interval is given by

$$\begin{aligned}\int_0^1 \left|\left(2e^{-x}+x\right)-\frac{1}{2}e^x\right| &= \left|\left[-2e^{-x}+\frac{x^2}{2}-\frac{1}{2}e^x\right]_0^1\right| \\ &= 3-\frac{2}{e}-\frac{e}{2} \approx 0.9051\end{aligned}$$

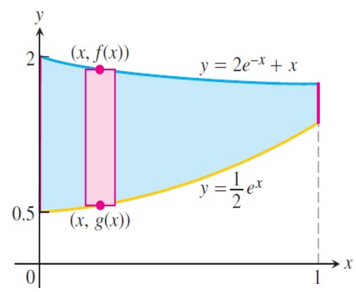


Figure 5.10:

Example 5.45. Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution 5.45. First we sketch the two curves (Figure 5.11). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$\begin{aligned}2-x^2 &= -x \\ x^2-x-2 &= 0 \\ (x+1)(x-2) &= 0 \\ x=-1 &\text{ or } x=2\end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$ and $b = 2$. The area between the curves is

$$\begin{aligned}\int_{-1}^2 \left|\left(2-x^2\right)-(-x)\right| &= \int_{-1}^2 \left|2+x-x^2\right| \\ &= \left|\left[2x+\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^2\right| = \frac{9}{2}\end{aligned}$$

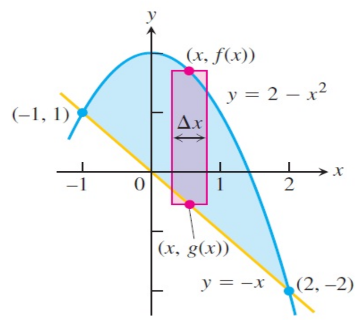


Figure 5.11:

Example 5.46. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \pi/2$.

Solution 5.46. The points of intersection occur when $\sin x = \cos x$, that is, when $x = \pi/4$ since $0 \leq x \leq \pi/2$. The intersection point $x = \pi/4$ subdivide $[0, \pi/2]$ into two subintervals: $[0, \pi/4]$ on which $\cos x \geq \sin x$ and $[\pi/4, \pi/2]$, on which $\sin x \geq \cos x$. The region is sketched in Figure 5.12. We integrate $\sin x - \cos x$ over each subinterval and add the absolute values of the calculated integrals.

$$\begin{aligned}\int_0^{\pi/4} (\sin x - \cos x) &= [-\cos x - \sin x]_0^{\pi/4} = 1 - \sqrt{2}\\ \int_{\pi/4}^{\pi/2} (\sin x - \cos x) &= [-\cos x - \sin x]_{\pi/4}^{\pi/2} = -1 + \sqrt{2}\end{aligned}$$

Therefore the required area is: $|1 - \sqrt{2}| + |-1 + \sqrt{2}| = 2\sqrt{2} - 2$.

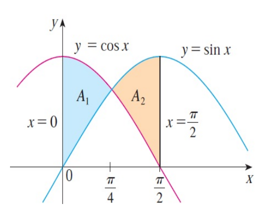


Figure 5.12:

Appendix A

Solving Equations and Inequalities

The Cartesian Plane: Just as you can represent real numbers by points on a real number line, you can represent ordered pairs of real numbers by points in a plane called the rectangular coordinate system, or the Cartesian plane.

The Cartesian plane is formed by using two real number lines intersecting at right angles, as shown in Figure A.1. The horizontal real number line is usually called the $x-$axis, and the vertical real number line is usually called the $y-$axis. The point of intersection of these two axes is the origin, and the two axes divide the plane into four parts called quadrants. Each point in the plane corresponds to an ordered

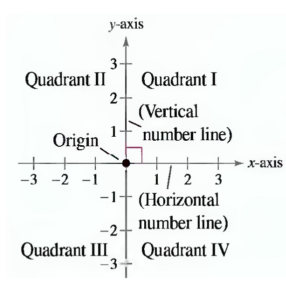


Figure A.1:

pair (x, y) of real numbers x and y, called coordinates of the point.

The $x-$coordinate represents the directed distance from the $y-$axis to the point, and the $y-$coordinate represents the directed distance from the x-axis to the point, as shown in Figure A.2.

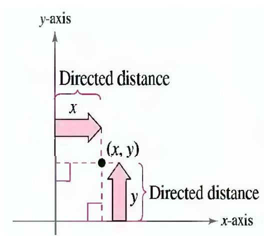


Figure A.2:

Example A.1. Plot the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$.

Solution A.1. See Figure A.3 below.

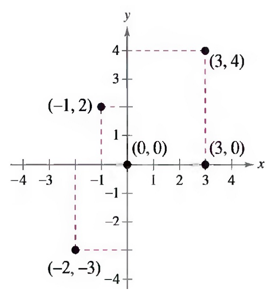


Figure A.3:

□

Suppose you want to determine the distance d between two points (x_1, y_1) and (x_2, y_2) in the plane, use the distance formula.

Theorem A.0.1. *The distance d between two points (x_1, y_1) and (x_2, y_2) in the plane is*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example A.2. Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Solution A.2. Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$. Then

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - (-2))^2 + (4 - 1)^2} \\ &= \sqrt{5^2 + 3^2} \\ &= \sqrt{34} \end{aligned}$$

□

To find the midpoint of the line segment that joins two points (x_1, y_1) and (x_2, y_2) in a coordinate plane, find the average values of the respective coordinates of the two endpoints using the **Midpoint Formula**.

$$\textbf{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example A.3. Find the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$.

Solution A.3. Let $(x_1, y_1) = (-5, -3)$ and $(x_2, y_2) = (9, 3)$. Then

$$\begin{aligned} \textbf{Midpoint} &= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \\ &= \left(\frac{-5 + 9}{2}, \frac{-3 + 3}{2}\right) \\ &= (2, 0) \end{aligned}$$

□

Solving Equations Algebraically An equation in x is a statement that two algebraic expressions are equal. For example, $3x - 5 = 7$, $x^2 - x - 6 = 0$, and $\sqrt{2x} = 4$ are equations. To solve an equation in x means to find all values of x for which the equation is true. Such values are solutions. For instance, $x = 4$ is a solution of the equation $3x - 5 = 7$ because $3(4) - 5 = 7$ is a true statement. The solutions of an equation depend on the kinds of numbers being considered. For instance, in the set of rational numbers, $x^2 = 10$ has no solution because there is no rational number whose square is 10. However, in the set of real numbers, the equation has the two solutions $\sqrt{10}$ and $-\sqrt{10}$.

An equation that is true for every real number in the domain (the domain is the set of all real numbers for which the equation is defined) of the variable is called an identity. For example, $x^2-9 = (x+3)(x-3)$ is an identity because it is a true statement for any real value of x, and $x/(3x^2) = 1/(3x)$, where $x \neq 0$, is an identity because it is true for any nonzero real value of x. An equation that is true for just some (or even none) of the real numbers in the domain of the variable is called a conditional equation. The equation $x^2 - 9 = 0$ is conditional because $x = 3$ and $x = -3$ are the only va1ues in the domain that satisfy the equation. The equation $2x + 1 = 2x - 3$ is also conditional because there are no real values of x for which the equation is true.

Polynomial equations can be classified by their degree. The degree of a polynomial equation is the highest degree of its terms. In general, the higher the degree, the more difficult it is to solve the equation either algebraically or graphically. You should be familiar with the following four methods for solving quadratic equations algebraically:

- **Factoring:** if $a \times b = 0$ then $a = 0$ or $b = 0$. The following are some important factorizations.
  - $a^2 - b^2 = (a - b)(a + b)$.
  - $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.
  - $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
  - $a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$.
- **Extracting Square Roots:** if $x^2 = c$, where $c > 0$, then $x = \pm\sqrt{c}$.
- **Completing the Square:** if $x^2+bx = c$, then $\left(x + \frac{b}{2}\right)^2 = c+\frac{b^2}{4}$.
- **Quadratic Formula:** if $ax^2 + bx + c = 0$ then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Example A.4. Solve $x^2 - 5x + 4 = 0$.

Solution A.4.

$$\begin{aligned} x^2 - 5x + 4 &= 0 \\ (x-1)(x-4) &= 0 \\ (x-1) &= 0 \hookrightarrow x = 1 \\ \text{or } (x-4) &= 0 \hookrightarrow x = 4 \end{aligned}$$

□

Example A.5. Solve $(x+3)^2 = 16$.

Solution A.5.

$$\begin{aligned} (x+3)^2 &= 16 \\ (x+3) &= \pm 4 \\ x &= -3 \pm 4 \Rightarrow x = 1 \text{ or } x = -7 \end{aligned}$$

□

Example A.6. Solve $x^2 + 6x = 5$.

Solution A.6.

$$\begin{aligned} x^2 + 6x &= 5 \\ \left(x + \frac{6}{2}\right)^2 &= 5 + \frac{6^2}{4} \\ (x+3)^2 &= 14 \\ x + 3 &= \pm\sqrt{14} \\ x &= -3 \pm \sqrt{14} \end{aligned}$$

□

Example A.7. Solve $2x^2 + 3x - 1 = 0$.

Solution A.7.

$$\begin{aligned} 2x^2 + 3x - 1 &= 0 \\ x &= \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)} \\ x &= \frac{-3 \pm \sqrt{17}}{4} \end{aligned}$$

The methods used to solve quadratic equations can sometimes be extended to polynomial equations of higher degree, as shown in the next two examples.

Example A.8. Solve $x^4 - 3x^2 + 2 = 0$

Solution A.8. The expression $x^4 - 3x^2 + 2$ is said to be in quadratic form because it is written in the form $au^2 + bu + c$, where u is any expression in x, namely x^2. You can use factoring to solve the equation as follows.

$$\begin{array}{rcll} x^4 - 3x^2 + 2 & = & 0 & \\ (x^2)^2 - 3(x^2) + 2 & = & 0 & \\ (x^2 - 1)(x^2 - 2) & = & 0 & \\ (x-1)(x+1)(x^2 - 2) & = & 0 & \\ x - 1 & = & 0 & \Rightarrow x = 1 \\ \text{or} \quad x + 1 & = & 0 & \Rightarrow x = -1 \\ \text{or} \quad x^2 - 2 & = & 0 & \Rightarrow x = \pm\sqrt{2} \end{array}$$

□

Example A.9. Solve $2x^3 - 6x^2 - 6x + 18 = 0$.

Solution A.9. This equation has a common factor of 2. You can simplify the equation by first dividing each side of the equation by 2.

$$\begin{array}{rcll} 2x^3 - 6x^2 - 6x + 18 & = & 0 & \\ x^3 - 3x^2 - 3x + 9 & = & 0 & \\ x^2(x-3) - 3(x-3) & = & 0 & \\ (x-3)(x^2 - 3) & = & 0 & \\ x - 3 & = & 0 & \Rightarrow x = 3 \\ \text{or} \quad x^2 - 3 & = & 0 & \Rightarrow x = \pm\sqrt{3} \end{array}$$

□

An equation involving a radical expression can often be cleared of radicals by raising each side of the equation to an appropriate power.

When using this procedure, it is crucial to check for extraneous solutions because of the restricted domain of a radical equation.

Example A.10. Solve $\sqrt{2x+7} - x = 2$

Solution A.10.

$$\begin{aligned} \sqrt{2x+7} - x &= 2 \\ \sqrt{2x+7} &= x+2 \\ 2x+7 &= x^2+4x+4 \\ x^2+2x-3 &= 0 \\ (x+3)(x-1) &= 0 \\ x+3 &= 0 \quad \Rightarrow x=-3 \\ x-1 &= 0 \quad \Rightarrow x=1 \end{aligned}$$

By substituting into the original equation, you can determine that $x = -3$ is extraneous, whereas $x = 1$ is valid. So, the equation has only one real solution: $x = 1$.

□

Example A.11. Solve $\sqrt{2x+6} - \sqrt{x+4} = 1$

Solution A.11.

$$\begin{aligned} \sqrt{2x+6} - \sqrt{x+4} &= 1 \\ \sqrt{2x+6} &= 1+\sqrt{x+4} \\ 2x+6 &= 1+2\sqrt{x+4}+(x+4) \\ x+1 &= 2\sqrt{x+4} \\ x^2+2x+1 &= 4(x+4) \\ x^2-2x-15 &= 0 \\ (x-5)(x+3) &= 0 \\ x-5 &= 0 \quad \Rightarrow x=5 \\ x+3 &= 0 \quad \Rightarrow x=-3 \end{aligned}$$

By substituting into the original equation, you can determine that $x = -3$ is extraneous, whereas $x = 5$ is valid. So that $x = 5$ is the only solution.

Example A.12. Solve $(x+1)^{2/3} = 4$

Solution A.12.

$$\begin{aligned} (x+1)^{2/3} &= 4 \\ \sqrt[3]{(x+1)^2} &= 4 \\ (x+1)^2 &= 64 \\ x+1 &= \pm 8 \\ x &= 7 \text{ or } -9 \end{aligned}$$

□

Example A.13. Solve $\frac{2}{x} = \frac{3}{x-2} - 1$

Solution A.13. For this equation, the least common denominator of the three terms is $x(x-2)$, so you can begin by multiplying each term of the equation by this expression.

$$\begin{aligned} \frac{2}{x} &= \frac{3}{x-2} - 1 \\ x(x-2)\frac{2}{x} &= x(x-2)\frac{3}{x-2} - x(x-2) \\ 2(x-2) &= 3x - x(x-2); \quad x \neq 0, 2 \\ x^2 - 3x - 4 &= 0 \\ (x-4)(x+1) &= 0 \\ x-4 &= 0 \quad \Rightarrow x = 4 \\ x+1 &= 0 \quad \Rightarrow x = -1 \end{aligned}$$

□

Solving Inequalities Algebraically The inequality symbols $<, \leq, >$, and $\geq$ are used to compare two numbers and to denote subsets of real numbers. For instance, the simple inequality $x \geq 3$ denotes all real numbers x that are greater than or equal to 3.

As with an equation, you solve an inequality in the variable x by finding all values of x for which the inequality is true. These values are solutions of the inequality and are said to satisfy the inequality. For instance, the number 9 is a solution of the inequality $5x - 7 > 3x + 9$ because

$$\begin{aligned} 5(9) - 7 &> 3(9) + 9 \\ 38 &> 36 \end{aligned}$$

On the other hand, the number 7 is not a solution because

$$\begin{aligned} 5(7) - 7 &\not> 3(7) + 9 \\ 28 &\not> 30 \end{aligned}$$

The set of all real numbers that are solutions of an inequality is the solution set of the inequality. The procedures for solving linear inequalities in one variable are much like those for solving linear equations. To isolate the variable, you can make use of the properties of inequalities, see Theorem A.0.2. These properties are similar to the properties of equality, but there are two important exceptions. When each side of an inequality is multiplied or divided by a negative number, the direction of the inequality symbol must be reversed in order to maintain a true statement.

Theorem A.0.2. *Let a, b, c and d be real numbers.*

1. *Transitive Property: if $a < b$ and $b < c$, then $a < c$*
2. *Addition of Inequalities: if $a < b$ and $c < d$, then $a + c < b + d$*
3. *Addition of a Constant: if $a < b$, then $a + c < b + c$*
4. *Multiplying by a Constant: we have two cases*
   - *a) for $c > 0$, if $a < b$, then $ac < bc$*
   - *b) for $c < 0$, if $a < b$, then $ac > bc$*

Each of the properties above is true if the symbol $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$.

Example A.14. Solve $2x + 1 \leq 5x - 8$.

Solution A.14.

$$\begin{aligned} 2x + 1 &\leq 5x - 8 \\ -3x &\leq -9 \\ x &\geq 3 \end{aligned}$$

So, the solution set is all real numbers that are greater than or equal to 3. The interval notation for this solution set is $[3, \infty)$.

□

Example A.15. Solve $1 - \frac{3}{2}x \geq x - 4$.

Solution A.15.

$$\begin{aligned} 1 - \frac{3}{2}x &\geq x - 4 \\ 2 - 3x &\geq 2x - 8 \\ -5x &\geq -10 \\ x &\leq 2 \end{aligned}$$

So, the solution set is all real numbers that are less than or equal to 2. The interval notation for this solution set is $(-\infty, 2]$.

□

Sometimes it is possible to write two inequalities as a double inequality, as demonstrated in the following example.

Example A.16. Solve $1 < 1 - 3x \leq 6$.

Solution A.16.

$$\begin{aligned} 1 &< 1 - 3x \leq 6 \\ 0 &< -3x \leq 5 \\ 0 &> x \geq -\frac{5}{3} \\ -\frac{5}{3} &\leq x < 0 \end{aligned}$$

The solution set is all real numbers that are greater than or equal to $-\frac{5}{3}$ and less than 0. The interval notation for this solution set is $\left[-\frac{5}{3}, 0\right)$.

□

Exercise A.1. Solve $4x < 2x + 1 \leq 3x + 2$.

Example A.17. Solve $x^2 + x > 6$.

Solution A.17. First, write $x^2+x > 6$ as $x^2+x-6 > 0$. The solution of $x^2 + x - 6 > 0$ is the set of all values of x that makes $x^2 + x - 6$ has positive sign. So,

$$\begin{aligned} x^2 + x - 6 > 0 &\hookrightarrow (x+3)(x-2) > 0 \\ &\hookrightarrow (x < -3) \cup (x > 2) \\ &\hookrightarrow x \in (-\infty, -3) \cup (2, \infty) \end{aligned}$$

The solution set is all real numbers that are less than -3 **or** greater than 2.

□

Example A.18. Solve $x^2 - 4 \leq 0$.

Solution A.18. The solution of $x^2 - 4 \leq 0$ is the set of all values of x that makes $x^2 - 4$ has non positive sign. So,

$$\begin{aligned} x^2 - 4 \leq 0 &\hookrightarrow (x+2)(x-2) \leq 0 \\ &\hookrightarrow -2 \leq x \leq 2 \\ &\hookrightarrow x \in [-2, 2] \end{aligned}$$

The solution set is all real numbers that are greater than or equal to -2 and less than or equal to 2.

□

Example A.19. Solve $\frac{1}{x-1} \geq 2$.

Solution A.19. First, rewrite $\frac{1}{x-1} \geq 2$ as follows.

$$\begin{aligned} \frac{1}{x-1} \geq 2 &\hookrightarrow \frac{1}{x-1} - 2 \geq 0 \\ &\hookrightarrow \frac{1}{x-1} - \frac{2(x-1)}{x-1} \geq 0 \\ &\hookrightarrow \frac{3-2x}{x-1} \geq 0 \end{aligned}$$

The solution of $\frac{3-2x}{x-1} \geq 0;\ x \neq 1$ is the set of all values of x that makes $\frac{3-2x}{x-1}$ has non negative sign. So,

$$\begin{aligned} \frac{3-2x}{x-1} \geq 0 \quad &\hookrightarrow \quad 1 < x \leq \frac{3}{2} \\ &\hookrightarrow \quad x \in \left(1, \frac{3}{2}\right] \end{aligned}$$

□

Exercise A.2. Solve the following inequalities.

(a) $\frac{1}{x^2+1} > 0$

(b) $x^2 + 4 < 0$

(c) $-3 < \frac{1}{x} \leq 1$

Appendix B

Absolute Value

Definition B.0.2. The absolute value of a real number x is

$$|x| = \begin{cases} x & : \ x \geq 0 \\ -x & : \ x < 0 \end{cases}$$

For example, $|4| = 4$, $|-\frac{1}{2}| = \frac{1}{2}$, and $|0| = 0$. The absolute value of a number a is the distance from a to 0 on the real number line. Distances are always positive or 0, so we have $|a| \geq 0$ for every number a.

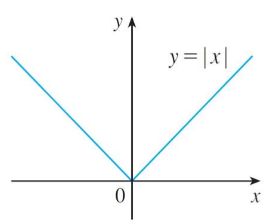


Figure B.1:

Remark B.0.1. Let a and b be real numbers, then:

1. $|a| \geq 0$ for all real number a.
2. $\sqrt{a^2} = |a|$.
3. If $a \in \mathbb{R}$, then $|-a| = |a|$.

4. $|x| = a$ if and only if $x = a$ or $x = -a$. In general, $|f(x)| = a$ if and only if $f(x) = a$ or $f(x) = -a$.

5. $|x| \leq a$ if and only if $-a \leq x \leq a$. In general, $|f(x)| \leq a$ if and only if $-a \leq f(x) \leq a$.

6. $|x| \geq a$ if and only if $x \geq a$ or $x \leq -a$. In general, $|f(x)| \geq a$ if and only if $f(x) \geq a$ or $f(x) \leq -a$.

7. $|a \cdot b| = |a| \cdot |b|$ and $|\frac{a}{b}| = \frac{|a|}{|b|}$.

8. If $a \in \mathbb{R}$ and $n = 1, 2, 3, \cdots$, then $|a^n| = |a|^n$.

9. **Triangle Inequality:** $|a + b| \leq |a| + |b|$. If a and b have the same sign, then $|a + b| = |a| + |b|$.

Example B.1. Solve $|x - 2| = 5$.

Solution B.1.

$$\begin{aligned} |x-2| = 5 \quad &\hookrightarrow \quad x - 2 = 5 \text{ or } x - 2 = -5 \\ &\hookrightarrow \quad x = 7 \text{ or } x = -3. \end{aligned}$$

□

Example B.2. Solve $|3x - 6| \leq 9$.

Solution B.2.

$$\begin{aligned} |3x-6| \leq 9 \quad &\hookrightarrow \quad -9 \leq 3x - 6 \leq 9 \\ &\hookrightarrow \quad -3 \leq 3x \leq 15 \\ &\hookrightarrow \quad -1 \leq x \leq 5 \\ &\hookrightarrow \quad x \in [-1, 5] \end{aligned}$$

□

Example B.3. Solve $|x^2 - 1| < 1$.

Solution B.3.

$$\begin{aligned}
|x^2 - 1| < 1 \quad &\hookrightarrow \quad -1 < x^2 - 1 < 1 \\
&\hookrightarrow \quad 0 < x^2 < 2 \\
&\hookrightarrow \quad 0 < \sqrt{x^2} < \sqrt{2} \\
&\hookrightarrow \quad 0 < |x| < \sqrt{2} \\
&\hookrightarrow \quad (|x| > 0) \cap (|x| < \sqrt{2}) \\
&\hookrightarrow \quad (\mathbb{R} - \{0\}) \cap (-\sqrt{2} < x < \sqrt{2}) \\
&\hookrightarrow \quad x \in (\mathbb{R} - \{0\}) \cap (-\sqrt{2}, \sqrt{2}) \\
&\hookrightarrow \quad x \in (-\sqrt{2}, \sqrt{2}) - \{0\}
\end{aligned}$$

□

Example B.4. Solve $|x + 4| > 7$.

Solution B.4.

$$\begin{aligned}
|x + 4| > 7 \quad &\hookrightarrow \quad x + 4 > 7 \text{ or } x + 4 < -7 \\
&\hookrightarrow \quad x > 3 \text{ or } x < -11 \\
&\hookrightarrow \quad x \in (-\infty, -11) \cup (3, \infty)
\end{aligned}$$

□

Exercise B.1. Solve the following equations and inequalities.

(a) $1 \leq |x| \leq 4$

(b) $|x + 3| = |1 + 2x|$

(c) $|x^2 - 9| = 9 - x^2$

Appendix C

Equation of Line

Definition C.0.3. For $x_1 \neq x_2$, the slope of the straight line through the points (x_1, y_1) and (x_2, y_2) is the number $m = \frac{y_2 - y_1}{x_2 - x_1}$.

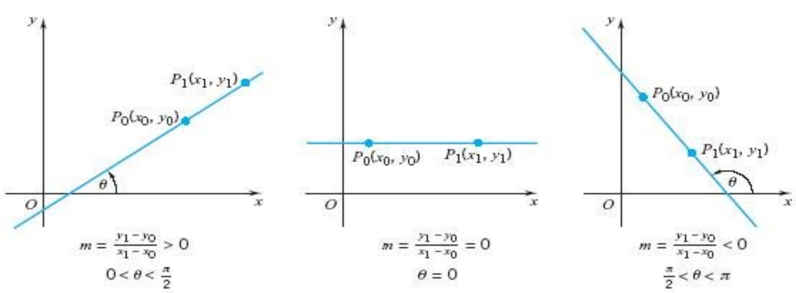


Figure C.1:

Example C.1. Find the slope of the line through the points $(-3, 5)$ and $(2, -4)$.

Solution C.1. The slope is $m = \frac{-4-5}{2-(-3)} = \frac{-9}{5}$.

□

Remark C.0.2. (Vertical and Horizontal Lines)

- When $x_1 = x_2$, the line is vertical and the slope is undefined. The equation of the line in this case is $x = x_1$.
- When $y_1 = y_2$, the line is horizontal and the slope is zero. The equation of the line in this case is $y = y_1$.

Remark C.0.3. The point-slope form of a line is $y - y_0 = m\,(x - x_0)$, where the line passes through (x_0, y_0) with slope m.

Example C.2. Find the equation of the line that passes through the point $(2, 1)$ with slope $\frac{2}{3}$.

Solution C.2.

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \frac{2}{3}(x - 2) \\ y - 1 &= \frac{2}{3}x - \frac{4}{3} \\ y &= \frac{2}{3}x - \frac{1}{3} \end{aligned}$$

□

Example C.3. Find an equation of the line through the points $(3, 1)$ and $(4, -1)$.

Solution C.3. The slope is $m = \frac{-1-1}{4-3} = -2$. So, the equation is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= -2(x - 3) \\ y - 1 &= -2x + 6 \\ y &= -2x + 7 \end{aligned}$$

□

Remark C.0.4. The equation $y = mx + b$ is called the slope-intercept form, where m is the slope and b is the $y-$intercept. The $x-$intercept is $x = -\frac{b}{m}$.

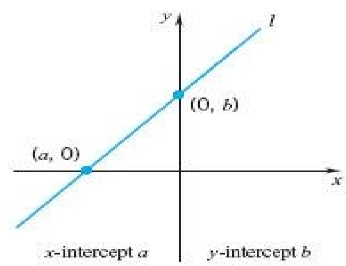


Figure C.2:

Example C.4. If a line has an equation $y + 2x - 7 = 0$, find the $x-$ and $y-$intercepts of the line.

Solution C.4.

1. To find the $x-$intercept, let $y = 0$ in the equation of the line.
$$0 + 2x - 7 = 0 \hookrightarrow x = \frac{7}{2}.$$
2. To find the $y-$intercept, let $x = 0$ in the equation of the line.
$$y + 0 - 7 = 0 \hookrightarrow y = 7.$$

□

Example C.5. Find the slope and $y-$intercept of the line $y = -2$.

Solution C.5. The slope is $m = 0$ (see remark C.0.2), and the $y-$intercept is -2.

□

Example C.6. If the point $(2, k)$ lies on the line with slope 3 and passing through the point $(1, 6)$, find the value of k.

Solution C.6.

$$\begin{aligned} \text{The slope of the line is } m &= \frac{k-6}{2-1} \\ 3 &= k-6 \\ 9 &= k \end{aligned}$$

□

Theorem C.0.3. *(Parallel and Perpendicular Lines) Two lines of slope m_1 and m_2 are parallel if $m_1 = m_2$, and are perpendicular if $m_1 \cdot m_2 = -1$.*

Example C.7. Find an equation for the line parallel to $y = 3x - 2$ and passes through the point $(-1, 3)$.

Solution C.7. The slope of our line is the same as of $y = 3x - 2$ since they are parallel. Hence the slope is $m = 3$. The equation of line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 3 &= 3(x - (-1)) \\ y - 3 &= 3x + 3 \\ y &= 3x + 6 \end{aligned}$$

□

Example C.8. Find an equation for the line perpendicular to $y = -2x + 4$ at the point $(1, 2)$.

Solution C.8. The slope of our line equals $-1/($ slope of $y = -2x + 4)$ since they are perpendicular. Hence the slope is $m = \frac{-1}{-2} = 0.5$. The equation of line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 2 &= 0.5(x - 1) \\ y - 2 &= 0.5x - 0.5 \\ y &= 0.5x + 1.5 \end{aligned}$$

Example C.9. For what values of k will be the line $kx + 5y = 2k$ be perpendicular to the line $2x + 3y = 1$.

Solution C.9. First, write the equations of lines in standard form as follows.

$$\begin{aligned} kx + 5y = 2k \quad &\hookrightarrow \quad y = -\frac{1}{5}kx + \frac{2}{5}k \text{ (its slope is } m_1 = -\tfrac{1}{5}k) \\ 2x + 3y = 1 \quad &\hookrightarrow \quad y = -\frac{2}{3}x + \frac{1}{3} \text{ (its slope is } m_2 = -\tfrac{2}{3}) \end{aligned}$$

The lines are perpendicular if

$$\begin{aligned} m_1 \times m_2 = -1 \quad &\hookrightarrow \quad -\frac{1}{5}k \times -\frac{2}{3} = -1 \\ &\hookrightarrow \quad k = -\frac{15}{2} \end{aligned}$$

□

Exercise C.1. Find an equation for the line passes through the point $(2, 7)$ and perpendicular to the $x-$axis.

Appendix D

Final Answers of Exercises

Chapter One

1. **(a)** $\mathbb{R}-\{1\}$ **(b)** $\mathbb{R}$
2. **(a)** $[0,\infty)$ **(b)** $(\infty,0]$ **(c)** $(0,1]$
3. **(a)** $x = 3\pi/2 \pm 2n\pi$ **(b)** $x = \pm n\pi$ **(c)** $x = \pi/4 \pm n\pi$ **(d)** $x = \pm\pi/3 \pm 2n\pi$ **(e)** $\pi/4 \pm n\pi/2$
4. **(a)** $\mathbb{R}$ **(b)** $[0 \pm 2n\pi, \pi/2 \pm 2n\pi)\bigcup[\pi \pm 2n\pi, 3\pi/2 \pm 2n\pi)$ **(c)** $x = \pm n\pi$
5. $(f-g)(x) = 2-1/x$ and its domain is $\mathbb{R}-\{0\}$. Also, $(f-g)(2) = 3/2$
6. $f^{-1}(x) = x/(1-x)$
7. $[-\sqrt{5},-\sqrt{3}]\bigcup[\sqrt{3},\sqrt{5}]$
8. **(1)** $\pi/16$ **(2)** $-\pi/2$ **(3)** $2\pi/3$ **(4)** $5/12$ **(5)** $4/\sqrt{7}$ **(6)** $1/\sqrt{5}$ **(7)** $24/25$
9. 5
10. **(1)** $x = \pm e^2$ **(2)** $x = 2$ **(3)** $x = 4$ **(4)** $x = \sqrt{3/2}$ **(5)** $x = \ln 3$ **(6)** $x = e^e$
11. **(1)** $\mathbb{R}$ **(2)** $[0 \pm 2n\pi, \pi/2 \pm 2n\pi)\bigcup(3\pi/2 \pm 2n\pi, 2\pi \pm 2n\pi]$ **(3)** $(-\infty,0)$ **(4)** $\mathbb{R}-\{0\}$ **(5)** $(1/e,\infty)$ **(6)** $(-\infty,1)$ **(7)** $(-\infty,4)$
12. **(1)** $g^{-1}(t) = (1+\log_3 t)/2$ **(2)** $g^{-1}(t) = 10^t - 3$
13. **(1)** 1 **(2)** 3/4 **(3)** 1

14. $\sinh x = 12$, $\cosh x = 13$. Now the other values are easy!!
15. $\sinh x = 4/3$ only since $x > 0$. Now the other values are easy!!

Chapter Two

1. **(1)** 3 **(2)** 4 **(3)** 2 **(4)** does not exist **(5)** 2
2. **(1)** $-1/12$ **(2)** 0
3. **(1)** $1/128$ **(2)** 12
4. 2
5. $-1/2$
6. 2, -2, the limit does not exist.
7. Does not exist.
8. 7
9. Both limits are 0
10. Does not exist.
11. **(1)** -1 **(2)** ∞ **(3)** 0
12. **(1)** Vertical: $x = 3$, Horizontal: $y = 1$ **(2)** Vertical: $x = 0$, Oblique: $y = x$
13. Easy!!
14. $4/5$
15. $-\pi$ and the second limit does not exist.
16. g continuous on $\mathbb{R} - \{-2, 2, 4, 6, 8\}$
17. $1 = \lim_{x\to 0} f(x) \neq f(0) = 0$
18. $(0, \infty) - \{1\}$
19. **(a)** $a = 10$ **(b)** $a = b = 1/2$
20. 0

Chapter Three

1. $-1/25$
2. $f'(x) = 1/2\sqrt{x}$, so $f'(0)$ does not exist.
3. $df/dx = 2$ at $x = -1$, and $df^{-1}/dx = 1/2$ at $x = f(-1) = 1$

4. **(1)** $(1+3x)/(2\sqrt{x})$ **(2)** $2^x \ln 2 - 2x$ **(3)** $e^x/(1+x^2) + e^x \tan^{-1} x$ **(4)** $2x(7x+10)/(7x+5)^2$ **(5)** $x \cosh x$
5. $m = 4$, $b = -4$
6. **(1)** $f = x^9$, $c = 1$, limit$= 9$ **(2)** $f = \sqrt[4]{x}$, $c = 16$, limit$= 1/32$
7. $f'' = 2/x^3$, $f'''(2) = -3/8$
8. $-\cos x$
9. **(1)** $n!$ **(2)** $(-1)^n \frac{1}{x^{n+1}}$
10. **(1)** $3\left(4x^3 - 6x\right)\left(x^4 - 3x^2 + 5\right)^2$ **(2)** $\sec^{-1} x/\sqrt{1 - \tan^2 x}$ **(3)** $\left(2+4x^2\right)/\sqrt{1+x^2}$ **(4)** $2\cos(2x)e^{\sin(2x)}$ **(5)** $3^{x \ln x} \ln 3(1 + \ln x)$ **(6)** $(2+x)/(x \ln 10)$ **(7)** $2\left(1+2e^x\right)\ln\left[2e^x + x\right]/(2e^x + x)$ **(8)** $3e^{\cosh(3x)} \sinh(3x)$ **(9)** $1/\sqrt{1-e^{-2x}}$ **(10)** $x/\sqrt{1+x^2} - x/(1-x^2) + \sinh^{-1} x$
11. $(6 - 162x^4)/(1+9x^4)^2$
12. $\frac{1}{2}\sqrt{1-4y^2}$
13. 0
14. $(x \cot x + \ln \sin x)(\sin x)^x$
15. $y = 2t + 2$
16. $y = 20x + 1$
17. $2y = 9x - 5$

Chapter Four

1. **(1)** does not exist **(2)** 0 **(3)** 1 **(3)** does not exist
2. **(1)** π **(2)** 0 **(3)** 0
3. **(1)** 0 **(2)** 1 **(3)** does not exist
4. **(1)** $1/e^2$ **(2)** e^{ab}
5. $c = \pi/2$
6. $c = \ln(6e^6/(e^6 - 1))$
7. 16
8. Easy!!
9. $g(-2) = 4$ is absolute maximum and $g(0) = 0$ is absolute minimum

10. $a = \sqrt{e}/2$, $b = -1/8$
11. **(1)** $g(-2) = 1/2$ is abs. min, $g(-1) = 1$ is abs. max **(2)** $g(-1) = -1$ is abs. min, $g(8) = 2$ is abs. max **(3)** $g(-\pi/2) = -1$ is abs. min, $g(\pi/2) = 1$ is abs. max, $g(5\pi/6) = 1/2$ is local min
12. **(1)** Increases on $[10, \infty)$, Decreases on $[1, 10]$, $g(1) = 1$ local max, $g(10) = -8$ abs. min **(2)** $g(1/e) = -1/e$ is abs. min
13. $g' = 1 - 1/t < 0$ when $t > 1$
14. **(1)** Increases on $\mathbb{R}$, Concave up on $(-\infty, 0)$, Concave down on $(0, \infty)$, (0,0) is inflection point **(2)** Increases on $(-2, 2)$, Decreases on $(-\infty, -2) \bigcup (2, \infty)$, Concave up $(0, \infty)$, Concave down $(-\infty, 0)$.
15. $P : f' < 0, f'' > 0$, $Q : f' > 0, f'' = 0$, $R : f' > 0, f'' < 0$, $S : f' = 0, f'' < 0$, $T : f' < 0, f'' < 0$,
16. Decreases on $(1, 3) \bigcup (5, 6)$, Increases on $(0, 1) \bigcup (3, 5)$, f has min at $x = 0, 3, 6$ and has max at $x = 1, 5$. Concave down on $(0, 2) \bigcup (4, 6)$, Concave up $(2, 4)$. The inflection points are $(2, f(2))$ and $(4, f(4))$.

Chapter Five

1. Show $\frac{d}{dx}\left[\ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\tan^{-1} x}{x} + C\right] = \frac{\tan^{-1} x}{x^2}$
2. **(1)** $\frac{3}{80}t^{4/3}(20 + 16t^2 + 5t^4) + C$ **(2)** $\frac{1}{2}\ln|5 + 2t + t^2| + C$ **(3)** $\ln|e^t + e^{-t}| + C$ **(4)** $-\ln|\csc t + \cot t| + C$ **(5)** $\ln|\sin t| + C$ **(6)** $2\sin t + C$ **(7)** $\frac{1}{2}\tan t + C$ **(8)** $\frac{1}{2}\ln|t| + \frac{5^t}{\ln 5} + C$ **(9)** $t + \frac{1}{2}\ln(1 + t^2) + C$
3. **(1)** $\frac{1}{6}(5 + t^4)^{3/2} + C$ **(2)** $\frac{1}{3}\cos\left[\frac{1}{t}\right] + C$ **(3)** $e^{\sin t} + C$ **(4)** $\frac{1}{2}\tan^{-1}(t^2)$ **(5)** $2\sqrt{e^t} + C$ **(6)** $\frac{(\ln t)^3}{3} + C$ **(7)** $\frac{2}{15}(-1 + t)^{3/2}(2 + 3t) + C$ **(8)** $\ln|\ln t| + C$ **(9)** $\frac{(\tan^{-1} t)^3}{3} + C$ **(10)** $\frac{1}{2}\tan^{-1}\left(\frac{t+1}{2}\right)$ **(11)** $2\ln(1 + \sqrt{t}) + C$ **(12)** $-\ln|1 + \cos^2 t|$ **(13)** $(t - 1)^6/6t^6 + C$ **(14)** $-(\ln\cos t)^2/2$ **(15)** $-2\cos(\ln t/2)$
4. $15/2$
5. **(1)** 0.8 **(2)** -2.6 **(3)** -1.8 **(4)** -0.3
6. -4
7. 4

8. Since $t^2/(3-\cos t) \geq 0$ then the integral > 0.
9. **(1)** -9 **(2)** 0 **(3)** $5/2$
10. **(1)** $1/3$ **(2)** $2(3\sqrt{3}-4)/5$ **(3)** $\pi/6$
11. $5/3$
12. **(1)** $2x^2\sin(x^6)+\int_2^{x^2}\sin(t^3)\,dt$ **(2)** $-2^{4x/3}\ln 2$ **(3)** $2x\sin^{-1}x^2 - \sin^{-1}x$
13. Increasing: $(-1,1)$, Decreasing: $(-\infty,-1)\bigcup(1,\infty)$, $f(-1)$ local min, $f(1)$ local max, Concave up: $(-\infty,0)$, Concave down: $(0,\infty)$
14. **(1)** $28/3$ **(2)** $1/2$ **(3)** $(82+\sqrt{19})/4$

Appendix A

1. $[-1,1/2)$
2. **(1)** $\mathbb{R}$ **(2)** $\{\} = \varnothing$ **(3)** $(-\infty,-3)\bigcup[1,\infty)$

Appendix B

1. **(1)** $[-4,-1]\bigcup[1,4]$ **(2)** $x=2$, $x=-4/3$ **(3)** $[-3,3]$

Appendix C

1. $x=2$