Course: Calculus (3)

Lecture No: [2]

Chapter: [11]

THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.1]

RECTANGULAR COORDINATES IN 3-SPACE; SPHERES; CYLINDRICAL SURFACES

In this chapter:

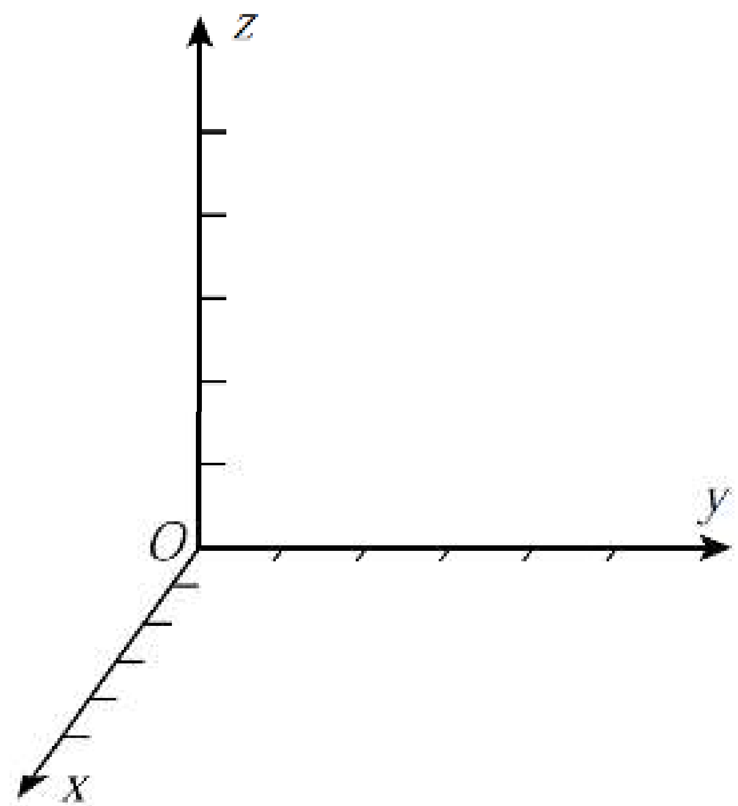
- We will discuss rectangular coordinate systems in three dimensions.
- We will study the analytic geometry of lines, planes, and other basic surfaces.
- We will study vectors. We will introduce various algebraic operations on vectors.
- Finally, we will discuss cylindrical and spherical coordinate systems.

RECTANGULAR COORDINATE SYSTEMS

In the remainder of this slides, we will call:

- three-dimensional space: **3-space**
- two-dimensional space (a plane): **2-space**
- one-dimensional space (a line): **1-space**

Points in 3-space can be placed in one-to-one correspondence with triples of real numbers by using three mutually perpendicular coordinate lines, called the x $-$axis, the y $-$axis, and the z $-$axis, positioned so that their origins coincide.



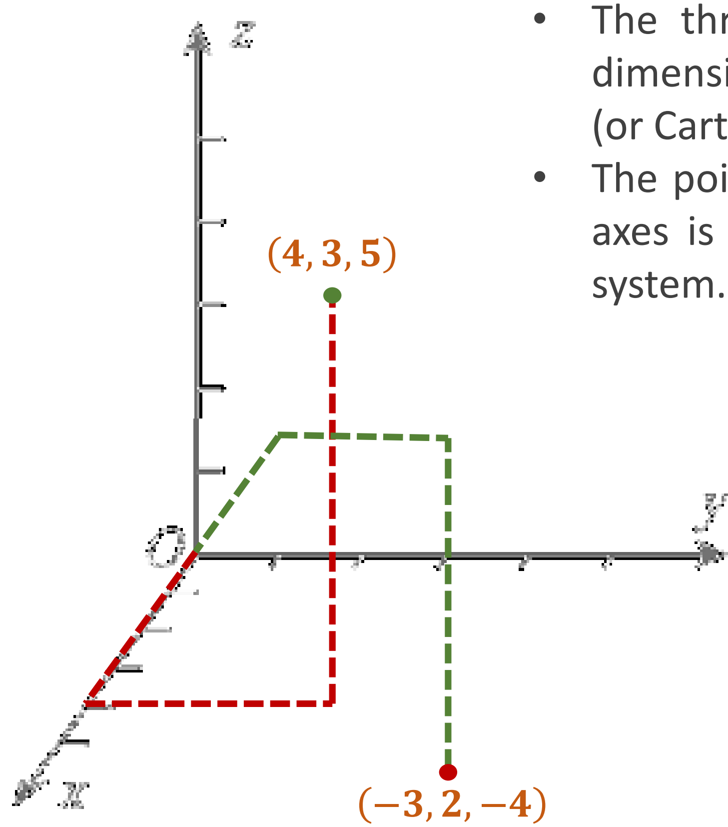
RECTANGULAR COORDINATE SYSTEMS

Example

Draw the point $(4,3,5)$

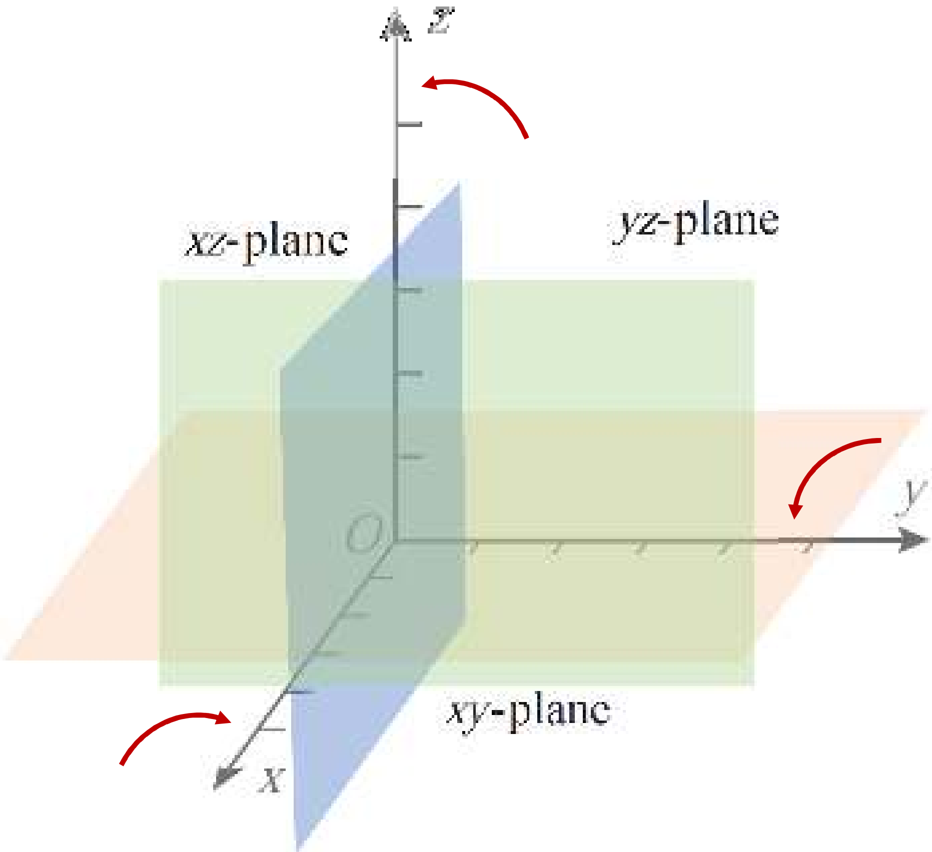
Example

Draw the point $(-3,2,-4)$



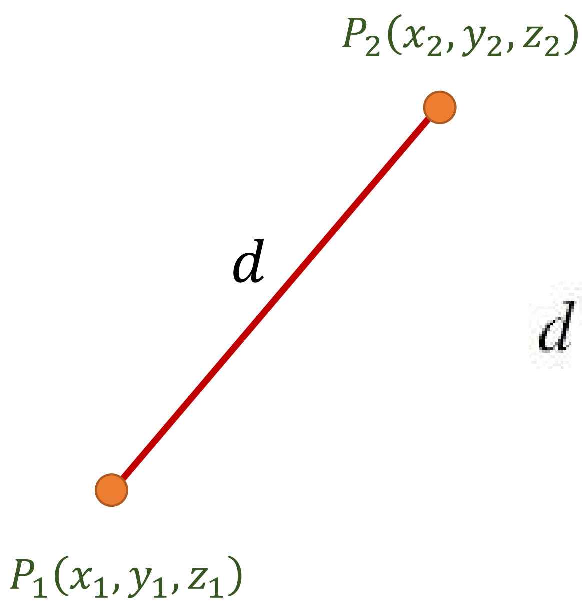
- The three coordinate axes form a three-dimensional rectangular coordinate system (or Cartesian coordinate system).
- The point of intersection of the coordinate axes is called the origin of the coordinate system.

RECTANGULAR COORDINATE SYSTEMS



REGION	DESCRIPTION

DISTANCE IN 3-SPACE; SPHERES



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

DISTANCE IN 3-SPACE; SPHERES

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example

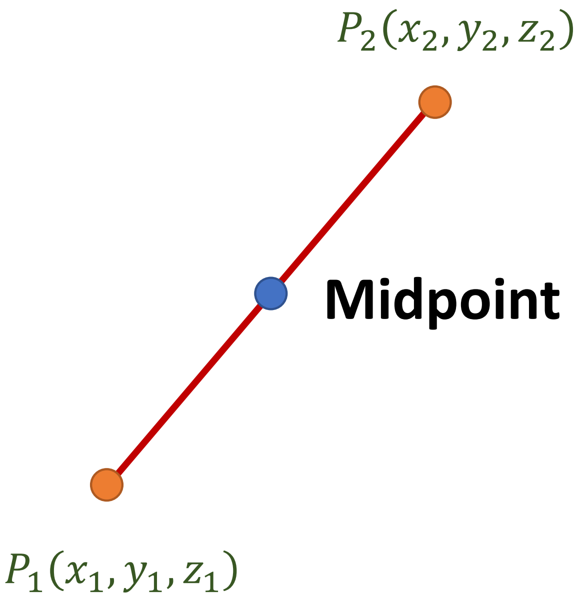
Find the distance d between the points $(\overset{x_1}{2}, \overset{y_1}{3}, \overset{z_1}{-1})$ and $(\overset{x_2}{4}, \overset{y_2}{-1}, \overset{z_2}{3})$.

$$d = \sqrt{(4-2)^2 + (-1-3)^2 + \big(3-(-1)\big)^2}$$

$$= \sqrt{4 + 16 + 16}$$

$$= 6$$

DISTANCE IN 3-SPACE; SPHERES



$$\textbf{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

DISTANCE IN 3-SPACE; SPHERES

Midpoint $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

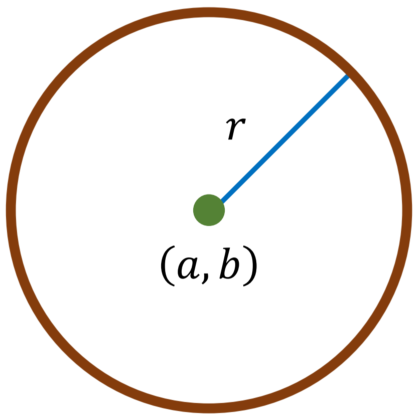
Example

Find the midpoint between the points $(\overset{x_1}{2}, \overset{y_1}{3}, \overset{z_1}{-1})$ and $(\overset{x_2}{4}, \overset{y_2}{-1}, \overset{z_2}{3})$.

$$midpoint = \left(\frac{2+4}{2}, \frac{3+(-1)}{2}, \frac{-1+3}{2}\right)$$

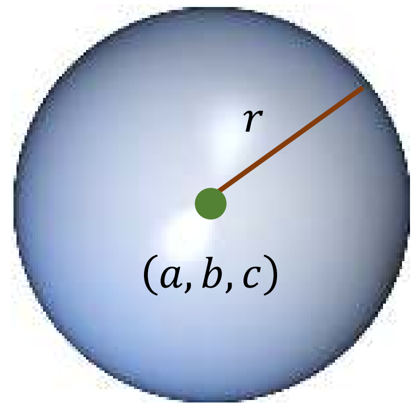
$$= (3,1,1)$$

DISTANCE IN 3-SPACE; SPHERES



$$(x-a)^2+(y-b)^2=r^2$$

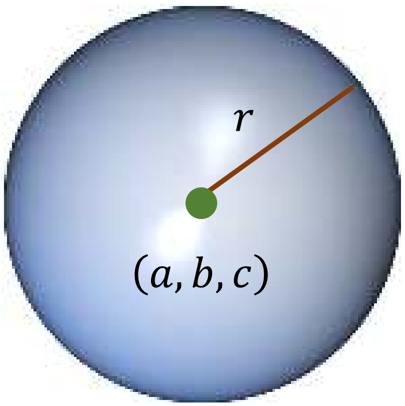
Circle in 2-space



$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Sphere in 3-space

DISTANCE IN 3-SPACE; SPHERES



$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Sphere in 3-space

Example

Find the equation of the sphere with center $(1, -2, -4)$ and radius 3.

$$(x-1)^2 + (y+2)^2 + (z+4)^2 = 9$$

$$x^2 + y^2 + z^2 - 2x + 4y + 8z = -12$$

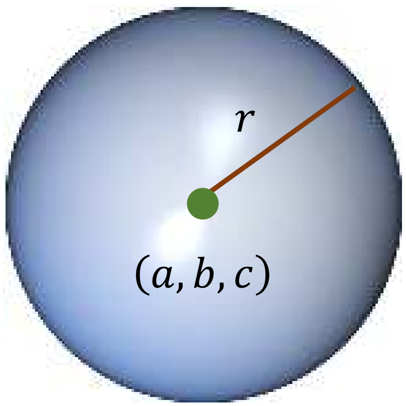
Example

Find the center and radius of the sphere

$$(x-5)^2 + y^2 + (z+3)^2 = 5$$

Center	(, ,)
Radius	$\sqrt{5}$

DISTANCE IN 3-SPACE; SPHERES



$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Standard equation of the sphere

If the terms in the equation of **SPHERE** are *expanded* and *like terms are collected*, then the resulting equation has the form

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

The following example shows how the center and radius of a sphere that is expressed in this form can be obtained by **completing the squares.**

DISTANCE IN 3-SPACE; SPHERES

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Example Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

DISTANCE IN 3-SPACE; SPHERES

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Example Find the center and radius of the sphere

$$x^2 + y^2 + z^2 - 2x - 4y + 8z + 17 = 0$$

$$(x^2 - 2x) + (y^2 - 4y) + (z^2 + 8z) = -17$$

DISTANCE IN 3-SPACE; SPHERES

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Example Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

$$(x^2-2x+\quad-\quad)+(y^2-4y+\quad-\quad)+(z^2+8z+\quad-\quad)=-17$$

$$\left(\frac{-2}{2}\right)^2=1 \qquad \left(\frac{-4}{2}\right)^2=4 \qquad \left(\frac{8}{2}\right)^2=16$$

DISTANCE IN 3-SPACE; SPHERES

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Example Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

$$(x^2-2x+1-1)+(y^2-4y+4-4)+(z^2+8z+16-16)=-17$$

DISTANCE IN 3-SPACE; SPHERES

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Example Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

$$(x^2-2x+1)-1+(y^2-4y+4)-4+(z^2+8z+16)-16=-17$$

DISTANCE IN 3-SPACE; SPHERES

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Example Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

$$(x^2-2x+1)+(y^2-4y+4)+(z^2+8z+16)=1+4+16-17$$

DISTANCE IN 3-SPACE; SPHERES

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Example Find the center and radius of the sphere

$$x^2+y^2+z^2-2x-4y+8z+17=0$$

$$(x-1)^2+(y-2)^2+(z+4)^2=4$$

Center $= \quad (1,2,-4)$

Radius $= \quad \sqrt{4}=2$

DISTANCE IN 3-SPACE; SPHERES

NOTE: In general, completing the squares produces an equation of the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = k$$

If $k > 0$ the graph of this equation is a sphere

If $k = 0$ the graph of this equation is the point (x_0, y_0, z_0)

If $k < 0$ no graph !!

11.1.1 THEOREM *An equation of the form*

$$x^2 + y^2 + z^2 + Gx + Hy + Iz + J = 0$$

represents a sphere, a point, or has no graph.

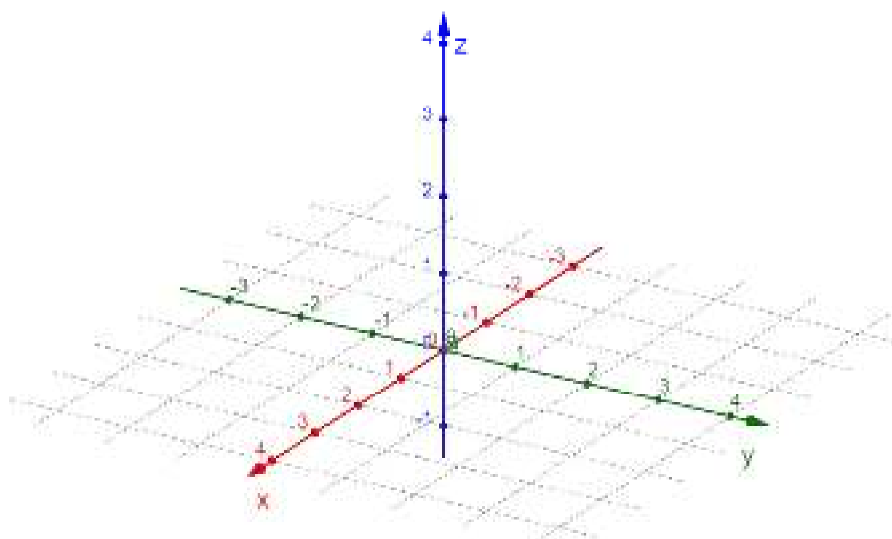
CYLINDRICAL SURFACES

It is possible to graph equations in two variables in 3 $-$space.

Example: $x^2 + y^2 = 1$

Observe that the equation does not impose any restrictions on z.

This means that we can obtain the graph of $x^2 + y^2 = 1$ in an xyz $-$coordinate system by first graphing the equation in the xy $-$plane.



CYLINDRICAL SURFACES

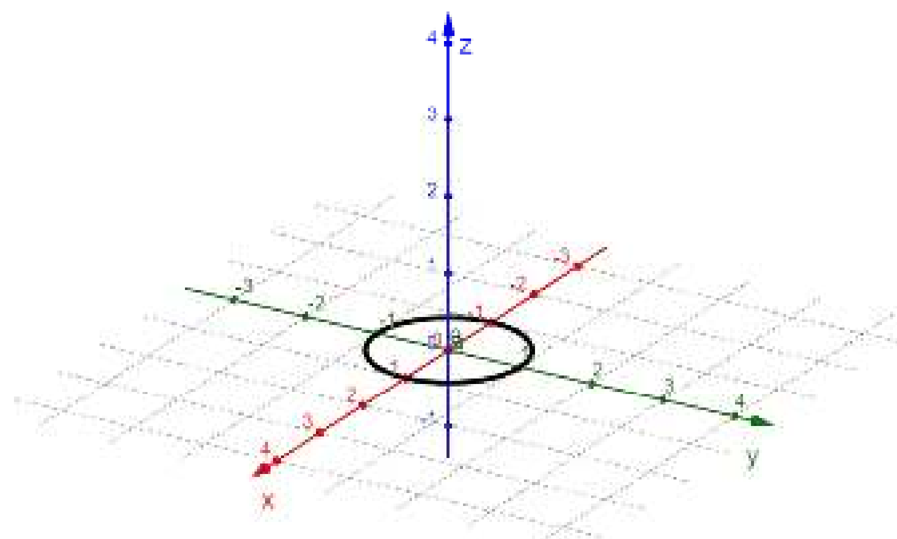
It is possible to graph equations in two variables in 3 $-$space.

Example: $x^2 + y^2 = 1$

Observe that the equation does not impose any restrictions on z.

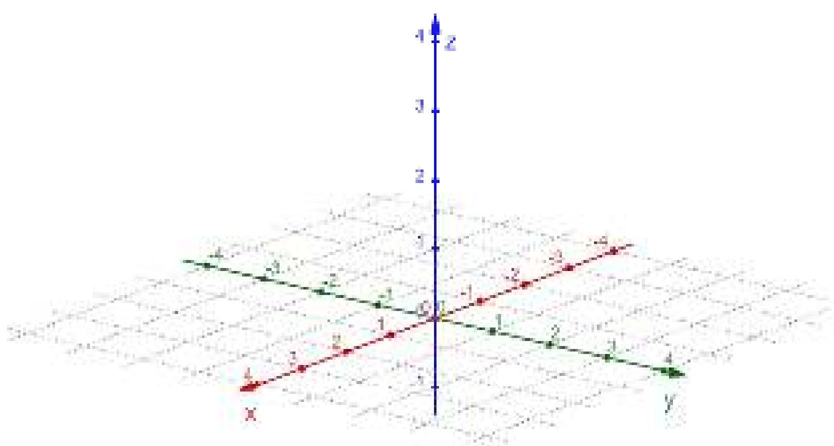
This means that we can obtain the graph of $x^2 + y^2 = 1$ in an xyz $-$coordinate system by first graphing the equation in the xy $-$plane.

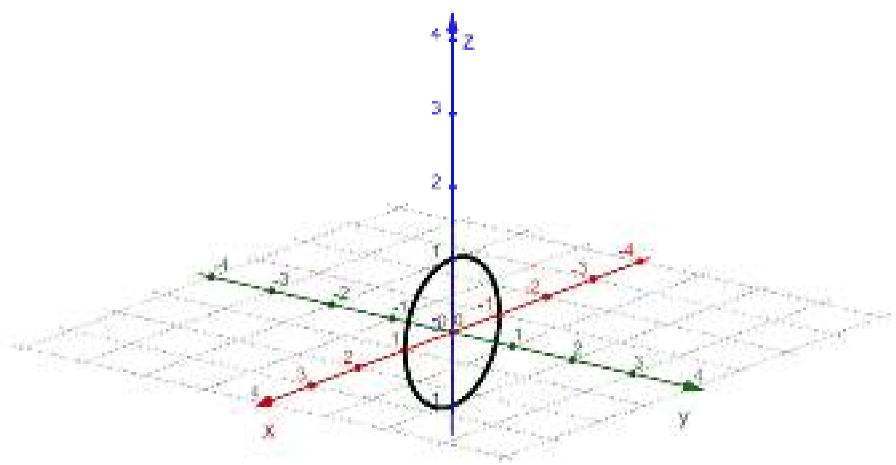
And then translating that graph parallel to the z $-$axis to generate the entire graph.



CYLINDRICAL SURFACES

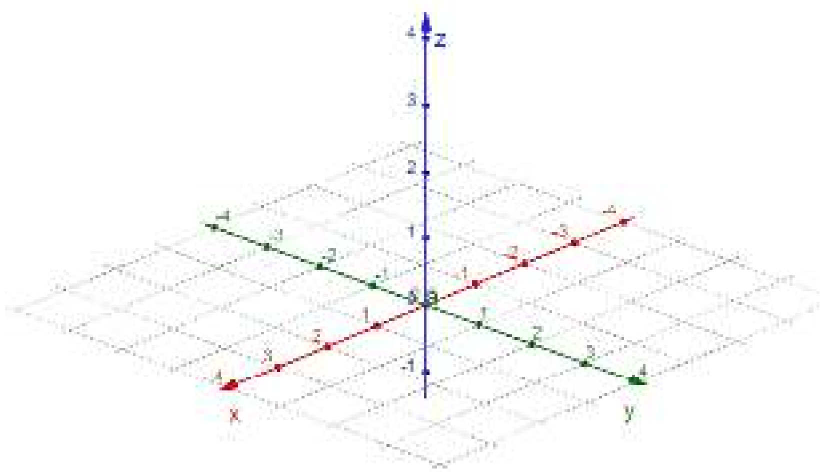
Example $x^2 + z^2 = 1$

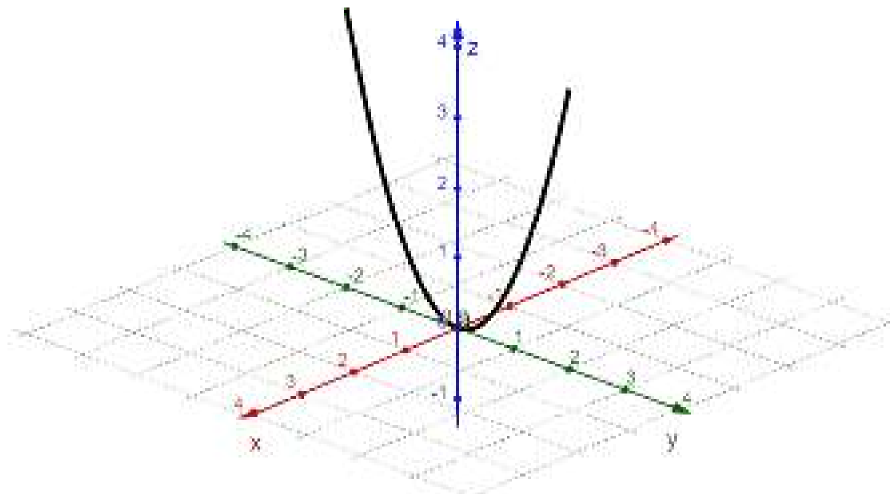




CYLINDRICAL SURFACES

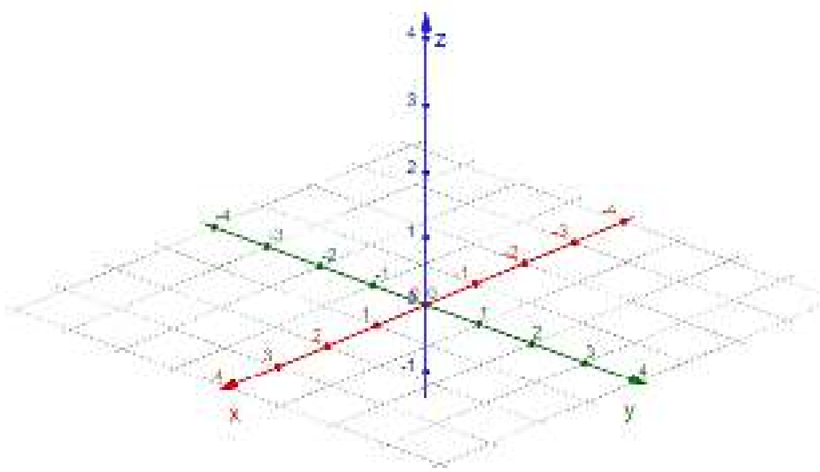
Example $z = y^2$

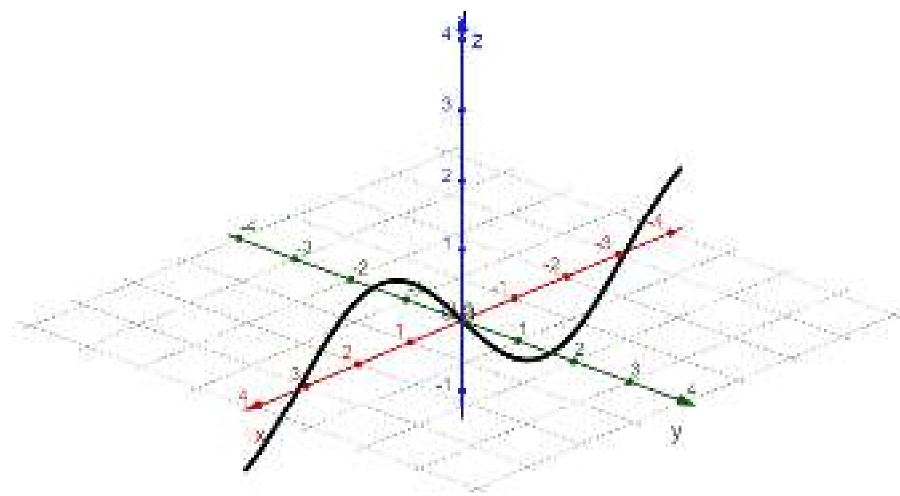




CYLINDRICAL SURFACES

Example $z = \sin x$





CYLINDRICAL SURFACES

11.1.2 THEOREM *An equation that contains only two of the variables x, y, and z represents a cylindrical surface in an xyz-coordinate system. The surface can be obtained by graphing the equation in the coordinate plane of the two variables that appear in the equation and then translating that graph parallel to the axis of the missing variable.*

Course: Calculus (3)

Lecture No: [4]

Chapter: [11]

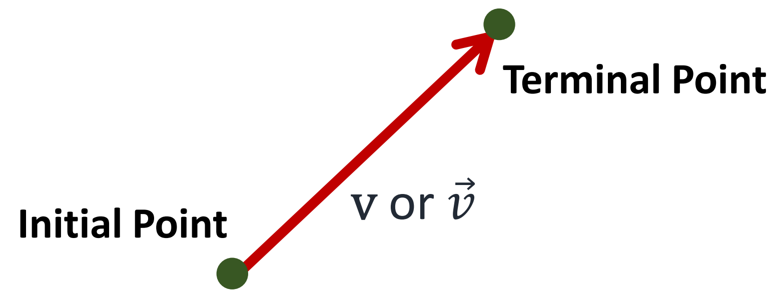
THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.2]

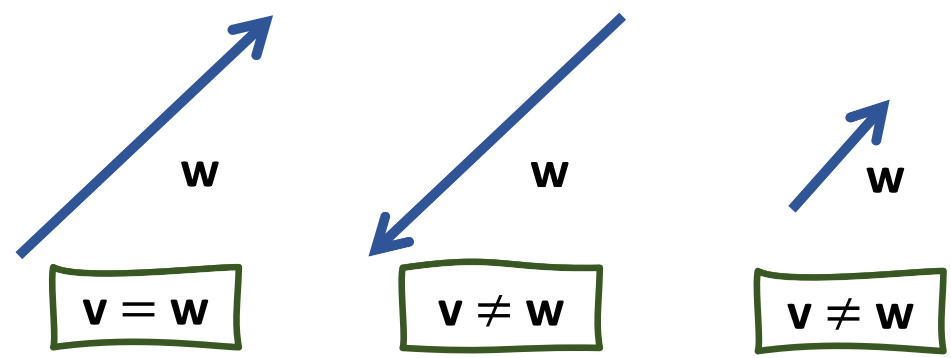
VECTORS

VECTORS VIEWED GEOMETRICALLY

A **Vector** in 2-space or 3-space; is an **arrow** with *direction* and *length* (magnitude).

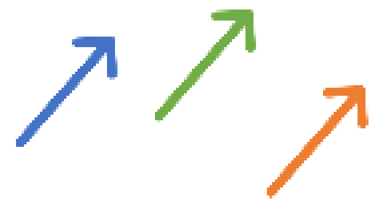


Two vectors **v** and **w** are *equal* if they have the same **length** and **same direction**, and we write $\mathbf{v} = \mathbf{w}$.

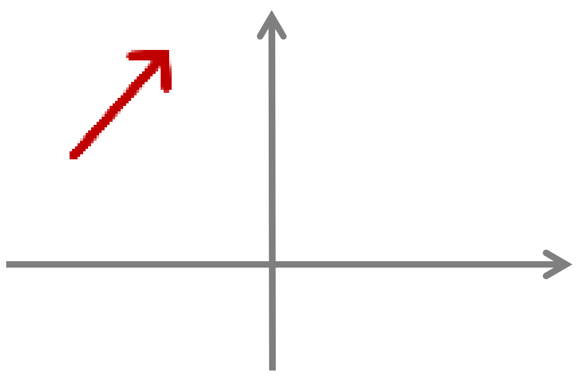


VECTORS VIEWED GEOMETRICALLY

Two vectors are equal if they are translations of one another.

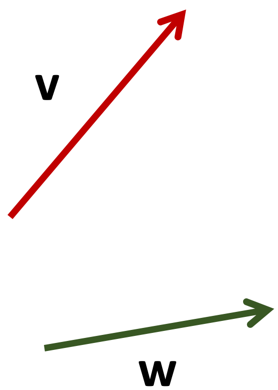


Because vectors are **not** *affected* by translation, the initial point of a vector **v** can be *moved* to any convenient point A by making an appropriate *translation*.



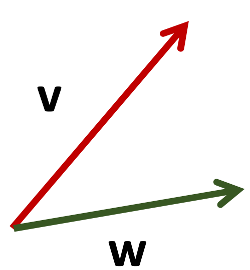
VECTORS VIEWED GEOMETRICALLY

11.2.1 DEFINITION If **v** and **w** are vectors, then the *sum* $\mathbf{v} + \mathbf{w}$ is the vector from the initial point of **v** to the terminal point of **w** when the vectors are positioned so the initial point of **w** is at the terminal point of **v**



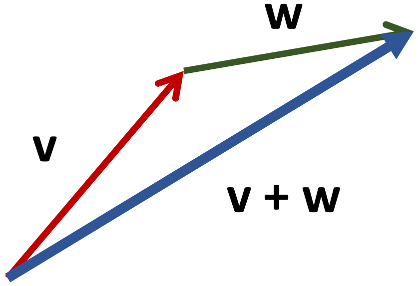
VECTORS VIEWED GEOMETRICALLY

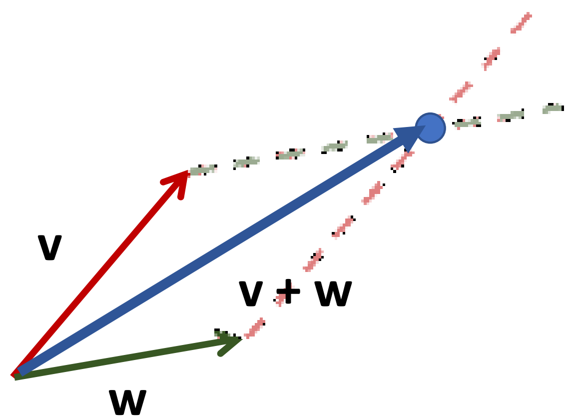
11.2.1 DEFINITION If $\mathbf{v}$ and $\mathbf{w}$ are vectors, then the *sum* $\mathbf{v}+\mathbf{w}$ is the vector from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$ when the vectors are positioned so the initial point of $\mathbf{w}$ is at the terminal point of $\mathbf{v}$



VECTORS VIEWED GEOMETRICALLY

11.2.1 DEFINITION If $\mathbf{v}$ and $\mathbf{w}$ are vectors, then the *sum* $\mathbf{v} + \mathbf{w}$ is the vector from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$ when the vectors are positioned so the initial point of $\mathbf{w}$ is at the terminal point of $\mathbf{v}$





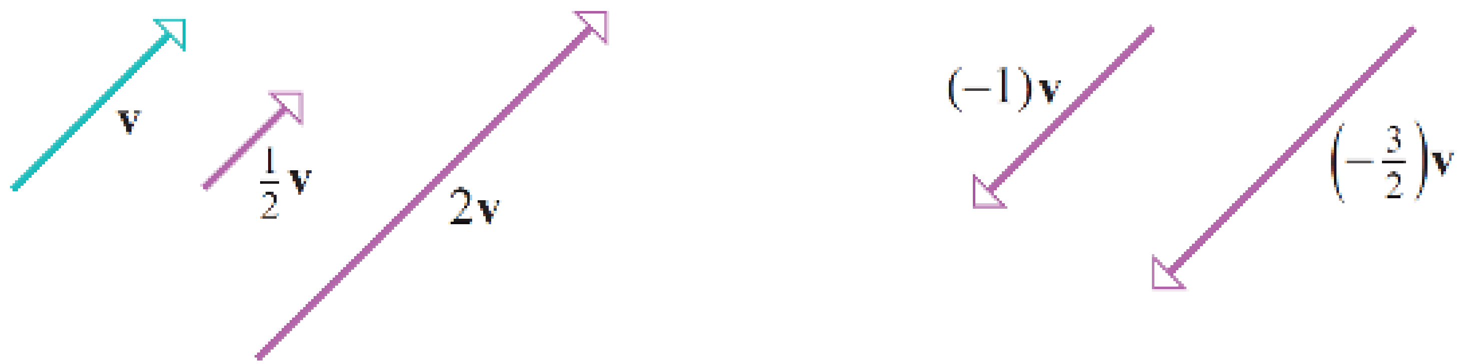
VECTORS VIEWED GEOMETRICALLY

NOTE:

- If the *initial* and *terminal* points of a vector *coincide*, then the vector has **length zero**; we call this the **zero vector** and denote it by **0**.
- The zero vector does not have a specific direction
- $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ and $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$.

VECTORS VIEWED GEOMETRICALLY

11.2.2 DEFINITION If $\mathbf{v}$ is a nonzero vector and k is a nonzero real number (a scalar), then the ***scalar multiple*** $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if $k > 0$ and opposite to that of $\mathbf{v}$ if $k < 0$. We define $k\mathbf{v} = \mathbf{0}$ if $k = 0$ or $\mathbf{v} = \mathbf{0}$.

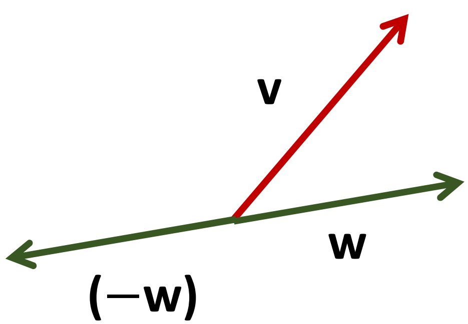


NOTE: The vectors $\mathbf{v}$ and $k\mathbf{v}$ are parallel vectors.

VECTORS VIEWED GEOMETRICALLY

Vector subtraction is defined in terms of addition and scalar multiplication by

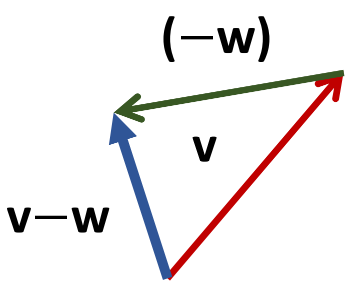
$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$



VECTORS VIEWED GEOMETRICALLY

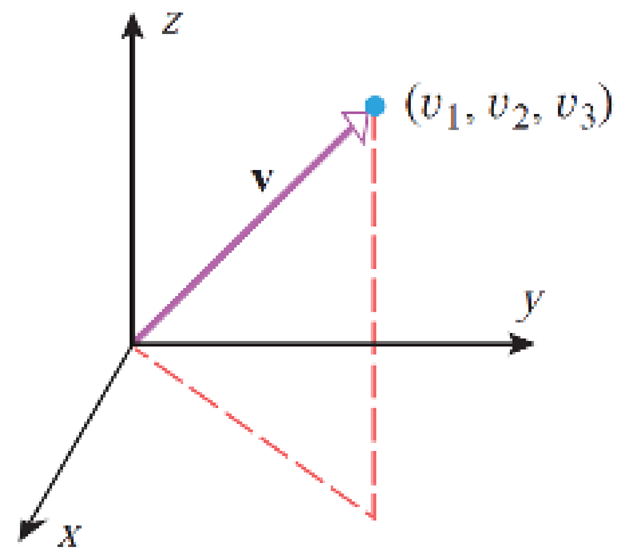
Vector subtraction is defined in terms of addition and scalar multiplication by

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$



NOTE: $\mathbf{v} + (-\mathbf{v}) = \mathbf{v} - \mathbf{v} = \mathbf{0}$

VECTORS IN COORDINATE SYSTEMS



If a vector **v** is positioned with its initial point at the *origin* of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2, v_3).

We call these coordinates the *components of* **v**, and we write **v** in component form using the *bracket* notation

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

VECTORS IN COORDINATE SYSTEMS

NOTE: $\mathbf{0} = \langle 0, 0, 0 \rangle$

11.2.3 THEOREM *Two vectors are equivalent if and only if their corresponding components are equal.*

Example: Find the values of a, b, c if $\langle -2, b, 3 \rangle = \langle a, 0, c \rangle$.

ARITHMETIC OPERATIONS ON VECTORS

11.2.4 THEOREM *If $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{w} = \langle w_1, w_2 \rangle$ are vectors in 2-space and k is any scalar, then*

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2 \rangle \tag{1}$$

$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2 \rangle \tag{2}$$

$$k\mathbf{v} = \langle kv_1, kv_2 \rangle \tag{3}$$

Similarly, if $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space and k is any scalar, then

$$\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle \tag{4}$$

$$\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle \tag{5}$$

$$k\mathbf{v} = \langle kv_1, kv_2, kv_3 \rangle \tag{6}$$

ARITHMETIC OPERATIONS ON VECTORS

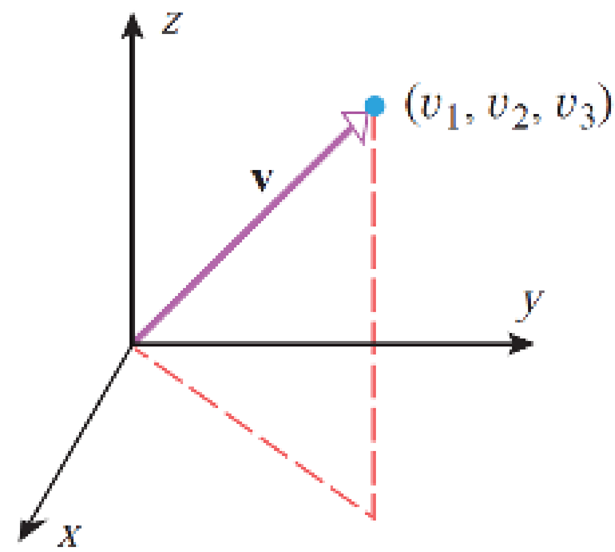
Example: If $\mathrm{v} = \langle 2,0,1 \rangle$ and $\mathrm{w} = \langle 3,5,-4 \rangle$, then

$$1.\ \mathrm{v} + \mathrm{w} = \langle 2,0,1 \rangle + \langle 3,5,-4 \rangle = \langle 5,5,-3 \rangle$$

$$\begin{aligned} 2.\ \mathrm{v} - 2\mathrm{w} &= \langle 2,0,1 \rangle - 2\langle 3,5,-4 \rangle \\ &= \langle 2,0,1 \rangle - \langle 6,10,-8 \rangle \\ &= \langle -4,-10,9 \rangle \end{aligned}$$

VECTORS IN COORDINATE SYSTEMS

If a vector **v** is positioned with its initial point at the *origin* of a rectangular coordinate system, then its terminal point will have coordinates of the form (v_1, v_2, v_3).



We call these coordinates the *components of* **v**, and we write **v** in component form using the *bracket* notation

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle$$

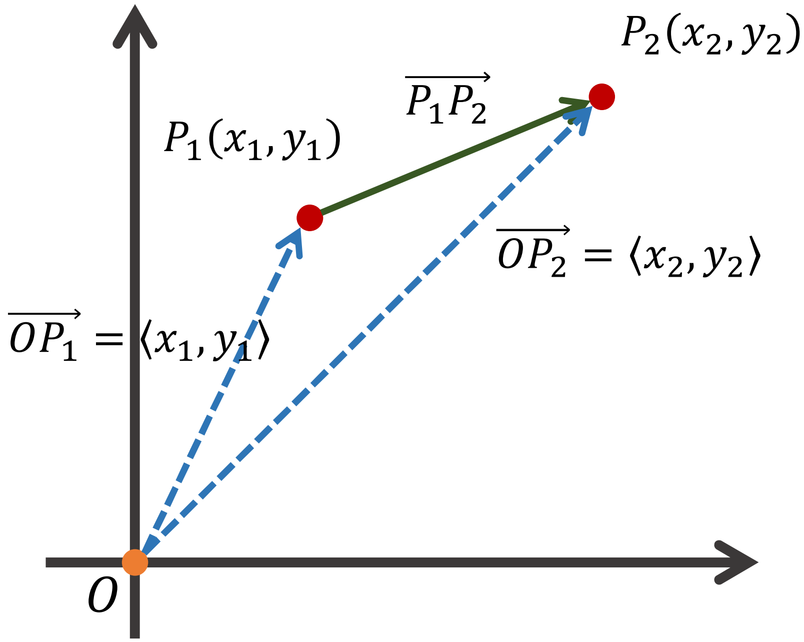
VECTORS WITH INITIAL POINT NOT AT THE ORIGIN

$$\overrightarrow{P_1P_2} + \overrightarrow{OP_1} = \overrightarrow{OP_2}$$

$$\overrightarrow{P_1P_2} + \langle x_1, y_1 \rangle = \langle x_2, y_2 \rangle$$

$$\overrightarrow{P_1P_2} = \langle x_2, y_2 \rangle - \langle x_1, y_1 \rangle$$

$$= \langle x_2 - x_1, y_2 - y_1 \rangle$$

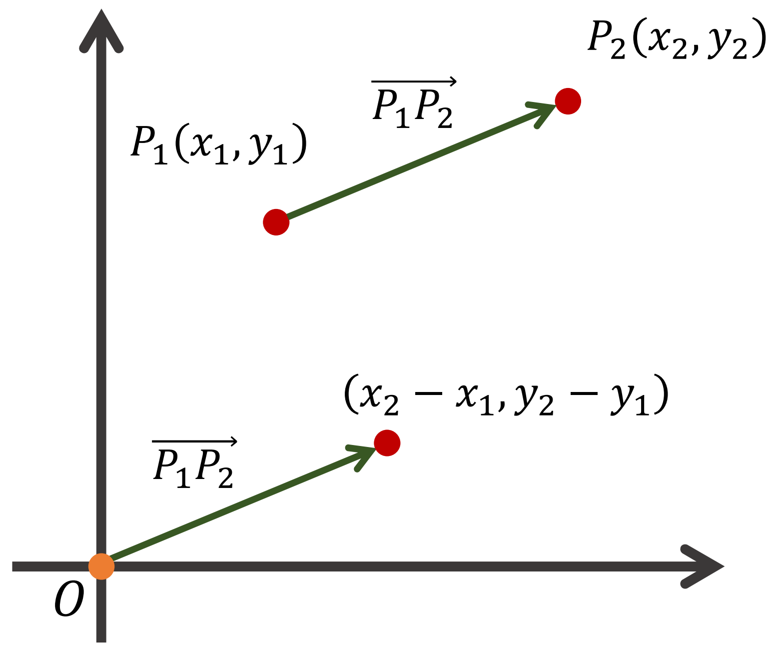


VECTORS WITH INITIAL POINT NOT AT THE ORIGIN

Example:

The vector from the point $A(0,-2,5)$ to the point $B(3,4,-1)$ is

$$\begin{aligned}\overrightarrow{AB} &= \langle 3-0, 4-(-2), -1-5\rangle \\ &= \langle 3,6,-6\rangle\end{aligned}$$



$$\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1\rangle$$

RULES OF VECTOR ARITHMETIC

11.2.6 THEOREM *For any vectors* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *and any scalars* k *and* l, *the following relationships hold:*

(*a*) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$

(*b*) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

(*c*) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(*d*) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$

(*e*) $k(l\mathbf{u})=(kl)\mathbf{u}$

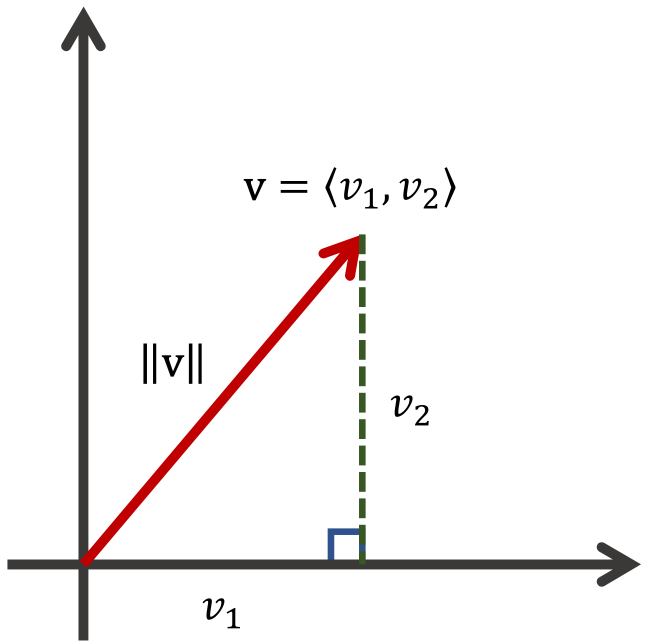
(*f*) $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$

(*g*) $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$

(*h*) $1\mathbf{u}=\mathbf{u}$

NORM OF A VECTOR

- The distance between the initial and terminal points of a vector **v** is called the **length**, the **norm**, or the **magnitude** of **v** and is denoted by $\|\mathrm{v}\|$.
- This *distance does not change* if the vector is *translated*, so for purposes of calculating the norm we can assume that the vector is positioned with its initial point at the origin.



$$\|\mathrm{v}\| = \sqrt{v_1^2 + v_2^2}$$

NORM OF A VECTOR

NOTE $\|k\mathbf{v}\| = |k|\,\|\mathbf{v}\|$

Example: If $w = \langle 2,3,6 \rangle$ then find the norm of

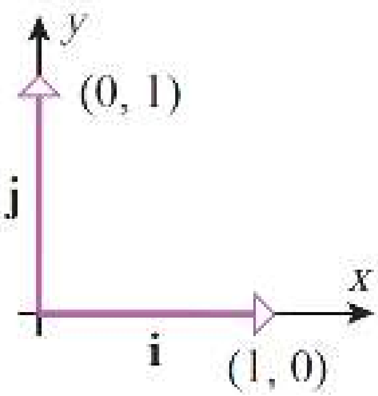
1. w $\quad \|w\| = \sqrt{(2)^2 + (3)^2 + (6)^2} = \sqrt{49} = 7$

2. $-3w$ $\quad \|-3w\| = |-3| \times \|w\| = 3 \times 7 = 21$

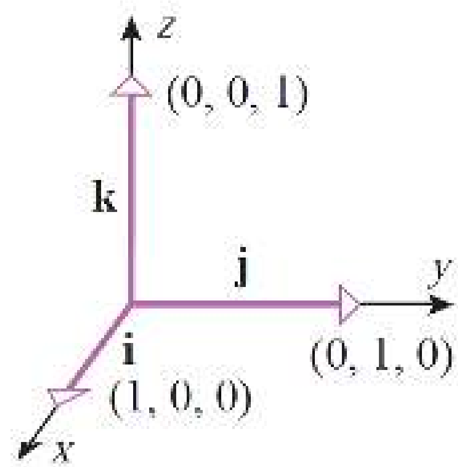
UNIT VECTORS

- A vector of length **1** is called a *unit vector*.
- In an xy −coordinate system the unit vectors along the x − and y −axes are denoted by $\mathbf{i}$ and $\mathbf{j}$, respectively.

$$\mathbf{i} = \langle 1, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1 \rangle$$



- In an xyz −coordinate system the unit vectors along the x −, y − and z −axes are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively.



$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

UNIT VECTORS

NOTE Every vector in $2-$space is expressible uniquely in terms of i and j as follows:

$$\mathrm{v} = \langle v_1, v_2 \rangle = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle$$
$$= v_1\langle 1,0 \rangle + v_2\langle 0,1 \rangle = v_1\mathrm{i} + v_2\mathrm{j}$$

Also, every vector in $3-$space is expressible uniquely in terms of i, j and k as follows:

$$\mathrm{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k}$$

UNIT VECTORS

Example:

$$\langle 2, 3\rangle = 2\mathbf{i} + 3\mathbf{j}$$

$$\langle 2, -3, 4\rangle = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$\langle -4, 0\rangle = -4\mathbf{i} + 0\mathbf{j} = -4\mathbf{i}$$

$$\langle 0, 3, 0\rangle = 3\mathbf{j}$$

$$\langle 0, 0, 0\rangle = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

$$5(6\mathbf{i} - 2\mathbf{j}) = 30\mathbf{i} - 10\mathbf{j}$$

$$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = -\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$$

$$\|\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

NORMALIZING A VECTOR

The *unit vector* $\mathbf{u}$ that has the *same direction* as some given nonzero vector $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

The process of obtaining a unit vector with the same direction of $\mathbf{v}$ is called *normalizing* $\mathbf{v}$.

Example: Find the unit vector that has the same direction as $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

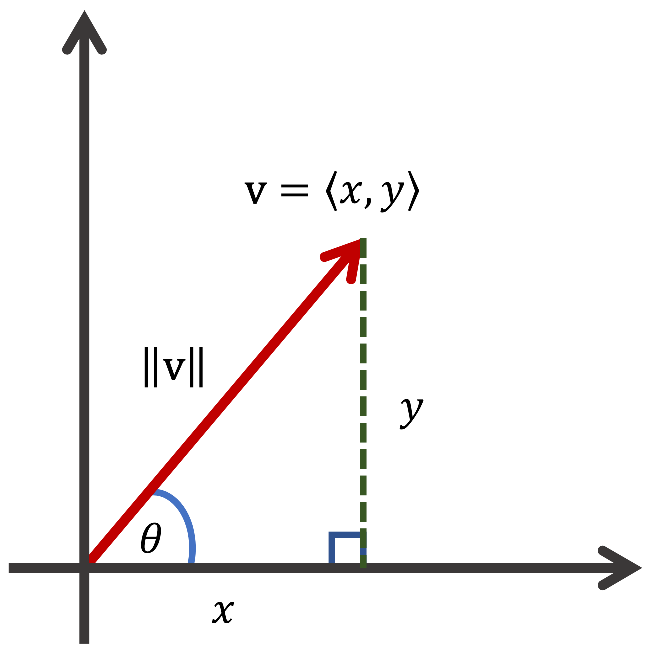
$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3 \qquad \therefore\ \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \right\rangle$$

VECTORS DETERMINED BY LENGTH AND ANGLE

$$\cos\theta = \frac{x}{\|\mathrm{v}\|} \quad \Rightarrow \quad x = \|\mathrm{v}\|\cos\theta$$

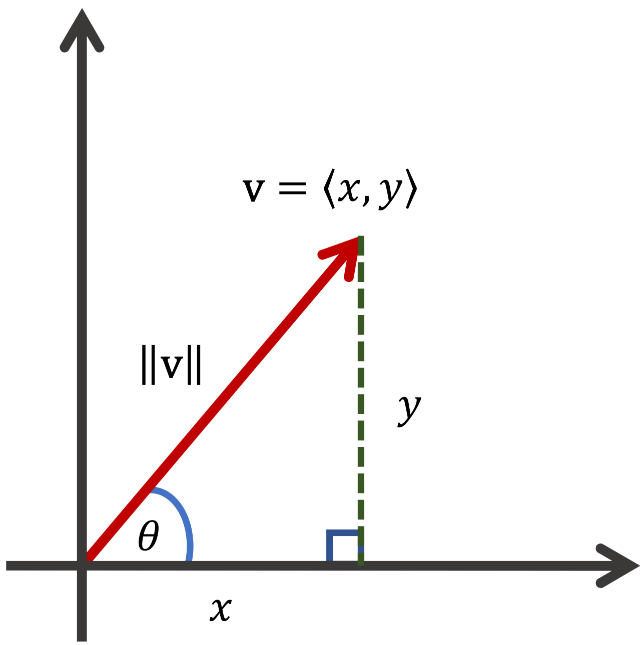
$$\sin\theta = \frac{y}{\|\mathrm{v}\|} \quad \Rightarrow \quad y = \|\mathrm{v}\|\sin\theta$$

$$\therefore\ \mathrm{v} = \langle \|\mathrm{v}\|\cos\theta\ ,\ \|\mathrm{v}\|\sin\theta \rangle$$



VECTORS DETERMINED BY LENGTH AND ANGLE

$$\therefore \mathrm{v} = \langle \|\mathrm{v}\| \cos\theta \, , \|\mathrm{v}\| \sin\theta \rangle$$



Example:

Find the vector of length 2 that makes an angle of $\frac{\pi}{4}$ with the positive x −axis.

$$\mathrm{v} = \left\langle 2\cos\frac{\pi}{4} \, , 2\sin\frac{\pi}{4} \right\rangle \;\; = \left\langle \frac{2}{\sqrt{2}} \, , \frac{2}{\sqrt{2}} \right\rangle$$

$$= \left\langle \sqrt{2}, \sqrt{2} \right\rangle$$

VECTORS DETERMINED BY LENGTH AND ANGLE

Example:

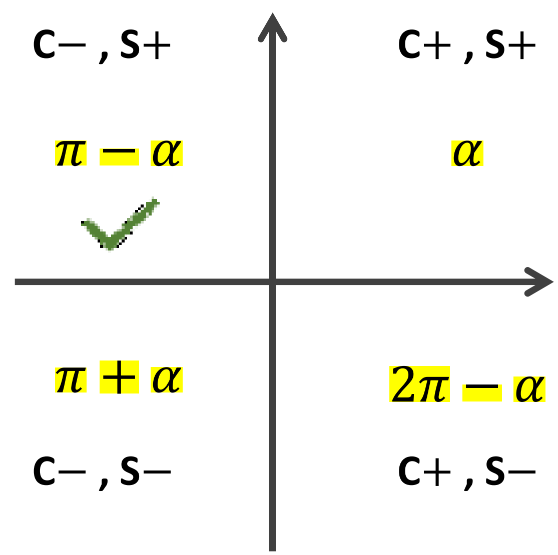
$$\therefore \mathrm{v} = \langle \|\mathrm{v}\| \cos\theta \,, \|\mathrm{v}\| \sin\theta \rangle$$

Find the angle that the vector $\mathrm{v} = -\sqrt{3}\mathrm{i} + \mathrm{j}$ makes with the positive x −axis.

$$\|\mathrm{v}\| = \sqrt{\left(-\sqrt{3}\right)^2 + 1^2} = 2$$

$$\cos\theta = \frac{x}{\|\mathrm{v}\|} = \frac{-\sqrt{3}}{2}$$

$$\sin\theta = \frac{y}{\|\mathrm{v}\|} = \frac{1}{2}$$



Reference Angle

$$\alpha = \frac{\pi}{6}$$

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Course: Calculus (3)

Lecture No: [6]

Chapter: [11]

THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.3]

DOT PRODUCT; PROJECTIONS

DEFINITION OF THE DOT PRODUCT

In this section we will define a *new kind of multiplication* in which two vectors are multiplied to produce a scalar.

11.3.1 DEFINITION If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are vectors in 2-space, then the ***dot product*** of $\mathbf{u}$ and $\mathbf{v}$ is written as $\mathbf{u} \cdot \mathbf{v}$ and is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$$

Similarly, if $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then their dot product is defined as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Example:

$$\langle 3, 5 \rangle \cdot \langle -1, 2 \rangle = 3(-1) + 5(2) = 7$$

$$\langle 2, 3 \rangle \cdot \langle -3, 2 \rangle = 2(-3) + 3(2) = 0$$

$$(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) = 1(1) + (-3)(5) + 4(2) = -6$$

ALGEBRAIC PROPERTIES OF THE DOT PRODUCT

11.3.2 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in 2- or 3-space and* k *is a scalar, then:*

(*a*) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(*b*) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

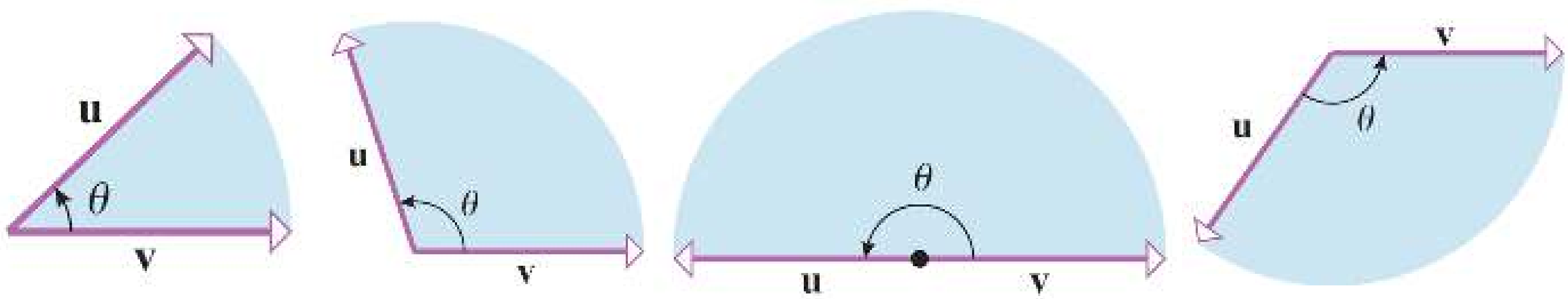
(*c*) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(*d*) $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ $\qquad \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

(*e*) $\mathbf{0} \cdot \mathbf{v} = 0$

ANGLE BETWEEN VECTORS

- Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $2-$space or $3-$space that *are positioned so their initial points coincide*.
- We define the angle between $\mathbf{u}$ and $\mathbf{v}$ to be the angle θ determined by the vectors that satisfies the condition $\theta \in [0, \pi]$.



ANGLE BETWEEN VECTORS

11.3.3 THEOREM *If* **u** *and* **v** *are nonzero vectors in 2-space or 3-space, and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

Example: Find the angle between the vector $\mathrm{u} = \mathrm{i} - 2\mathrm{j} + 2\mathrm{k}$ and

(a) $\mathrm{v} = -3\mathrm{i} + 6\mathrm{j} + 2\mathrm{k}$

$$\mathrm{u}\cdot\mathrm{v} = (1)(-3) + (-2)(6) + (2)(2) = -11$$

$$\|\mathrm{u}\| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

$$\|\mathrm{v}\| = \sqrt{(-3)^2 + 6^2 + 2^2} = 7$$

$$\therefore \cos\theta = \frac{-11}{(3)(7)}$$

$$\theta = \cos^{-1}\left(\frac{-11}{21}\right)$$

ANGLE BETWEEN VECTORS

11.3.3 THEOREM *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are nonzero vectors in 2-space or 3-space, and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

Example: Find the angle between the vector $\mathrm{u} = \mathrm{i} - 2\mathrm{j} + 2\mathrm{k}$ and

(b) $\mathrm{w} = 2\mathrm{i} + 7\mathrm{j} + 6\mathrm{k}$

$$\mathrm{u}\cdot\mathrm{w} = (1)(2) + (-2)(7) + (2)(6) = 0$$

$$\therefore \cos\theta = 0$$

$$\theta = \frac{\pi}{2}$$

ANGLE BETWEEN VECTORS

11.3.3 THEOREM *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are nonzero vectors in 2-space or 3-space, and if* θ *is the angle between them, then*

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$$

Example: Find the angle between the vector $\mathrm{u} = \mathrm{i} - 2\mathrm{j} + 2\mathrm{k}$ and

(c) $\mathrm{v} = -3\mathrm{i} + 6\mathrm{j} - 6\mathrm{k}$

$$\mathrm{u}\cdot\mathrm{v} = (1)(-3) + (-2)(6) + (2)(-6) = -27$$

$$\|\mathrm{u}\| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

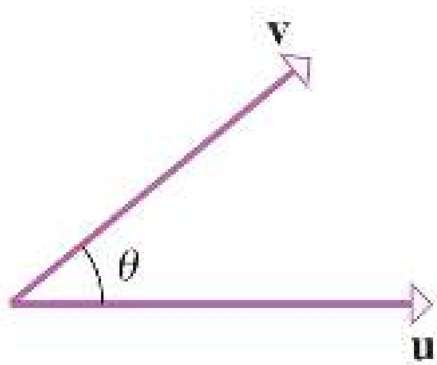
$$\|\mathrm{v}\| = \sqrt{(-3)^2 + 6^2 + (-6)^2} = 9$$

$$\therefore \cos\theta = \frac{-27}{(3)(9)} = -1$$

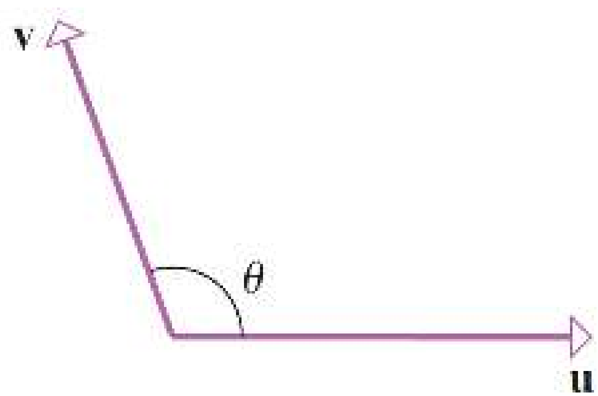
$$\theta = \pi$$

ANGLE BETWEEN VECTORS

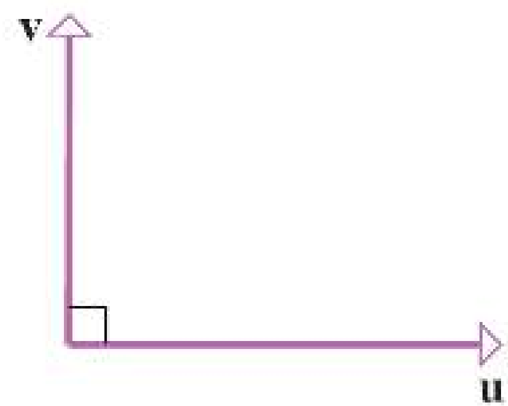
$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$$



$\mathbf{u} \cdot \mathbf{v} > 0$



$\mathbf{u} \cdot \mathbf{v} < 0$

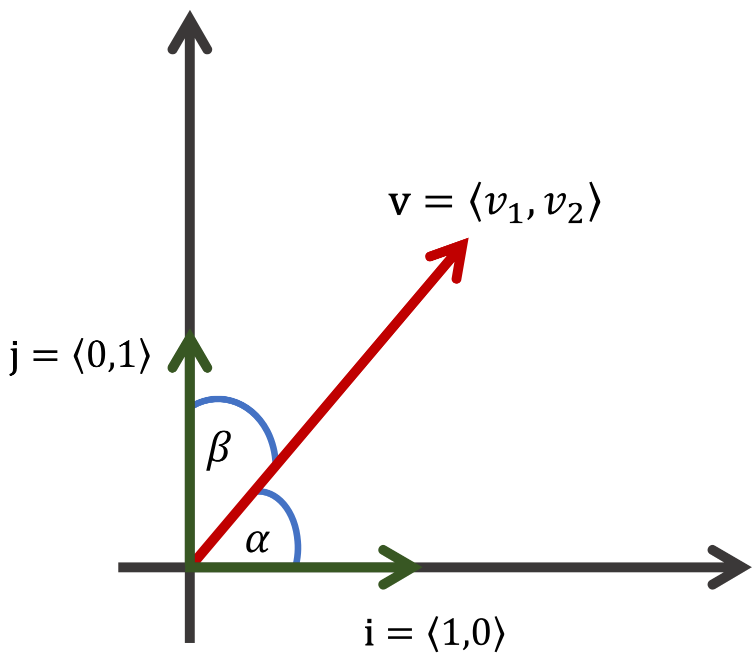


$\mathbf{u} \cdot \mathbf{v} = 0$

DIRECTION ANGLES

$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\,\|\mathbf{i}\|} = \frac{v_1}{\|\mathbf{v}\|}$$

$$\cos\beta = \frac{\mathbf{v}\cdot\mathbf{j}}{\|\mathbf{v}\|\,\|\mathbf{j}\|} = \frac{v_2}{\|\mathbf{v}\|}$$



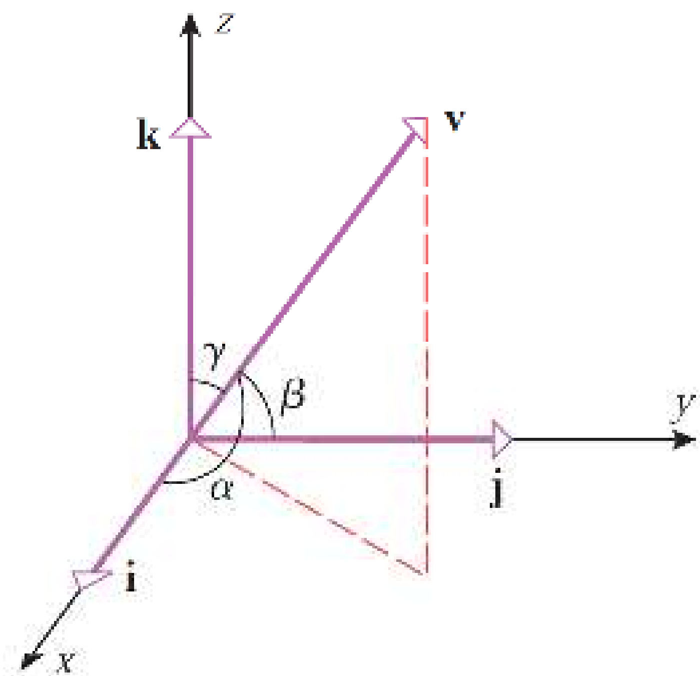
DIRECTION ANGLES

11.3.4 THEOREM *The direction cosines of a nonzero vector* $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ *are*

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|}, \quad \cos\beta = \frac{v_2}{\|\mathbf{v}\|}, \quad \cos\gamma = \frac{v_3}{\|\mathbf{v}\|}$$

NOTE:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$



DIRECTION ANGLES

Example: Find the direction cosines of the vector $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

$$\|\mathbf{v}\| = \sqrt{4 + 16 + 16} = 6$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \tfrac{1}{3}\mathbf{i} - \tfrac{2}{3}\mathbf{j} + \tfrac{2}{3}\mathbf{k}$$

$$\cos\alpha = \tfrac{1}{3}$$
$$\cos\beta = -\tfrac{2}{3}$$
$$\cos\gamma = \tfrac{2}{3}$$

$$\alpha = \cos^{-1}\left(\tfrac{1}{3}\right) \approx 71^\circ$$
$$\beta = \cos^{-1}\left(-\tfrac{2}{3}\right) \approx 132^\circ$$
$$\gamma = \cos^{-1}\left(\tfrac{2}{3}\right) \approx 48^\circ$$

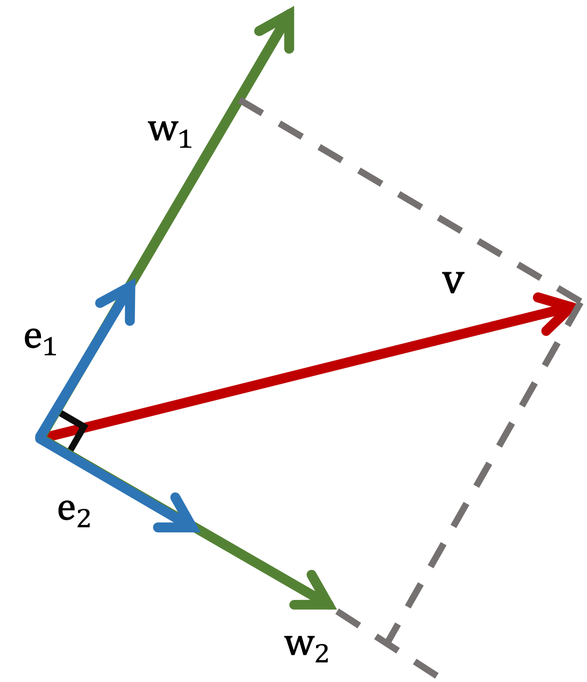
DECOMPOSING VECTORS INTO ORTHOGONAL COMPONENTS

In many applications it is desirable to **decompose** a vector into a sum of two orthogonal vectors with convenient specified directions.

$$\mathrm{v} = a\mathrm{w}_1 + b\mathrm{w}_2 \quad \boxed{\mathrm{v} = k_1\mathrm{e}_1 + k_2\mathrm{e}_2}$$

$$\begin{aligned} \mathrm{v}\cdot\mathrm{e}_1 &= (k_1\mathrm{e}_1 + k_2\mathrm{e}_2)\cdot\mathrm{e}_1 \\ &= k_1(\mathrm{e}_1\cdot\mathrm{e}_1) + k_2(\mathrm{e}_2\cdot\mathrm{e}_1) \\ &= k_1\|\mathrm{e}_1\|^2 + 0 = k_1 \end{aligned}$$

$$\therefore k_1 = \mathrm{v}\cdot\mathrm{e}_1$$



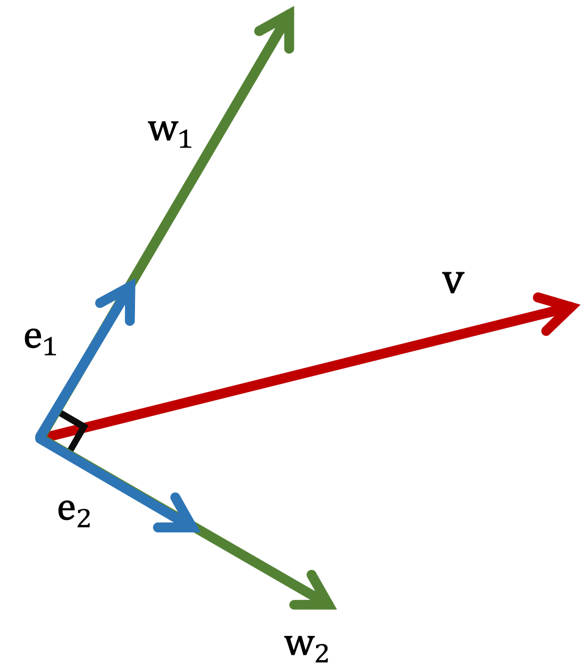
DECOMPOSING VECTORS INTO ORTHOGONAL COMPONENTS

In many applications it is desirable to **decompose** a vector into a sum of two orthogonal vectors with convenient specified directions.

$$\mathrm{v} = a\mathrm{w}_1 + b\mathrm{w}_2 \quad \boxed{\mathrm{v} = k_1\mathrm{e}_1 + k_2\mathrm{e}_2} \quad \therefore k_1 = \mathrm{v}\cdot\mathrm{e}_1$$

$$\begin{aligned}\mathrm{v}\cdot\mathrm{e}_2 &= (k_1\mathrm{e}_1 + k_2\mathrm{e}_2)\cdot\mathrm{e}_2 \\ &= k_1(\mathrm{e}_1\cdot\mathrm{e}_2) + k_2(\mathrm{e}_2\cdot\mathrm{e}_2) \\ &= 0 + k_2\|\mathrm{e}_2\|^2 = k_2\end{aligned}$$

$$\therefore k_2 = \mathrm{v}\cdot\mathrm{e}_2$$



DECOMPOSING VECTORS INTO ORTHOGONAL COMPONENTS

In many applications it is desirable to **decompose** a vector into a sum of two orthogonal vectors with convenient specified directions.

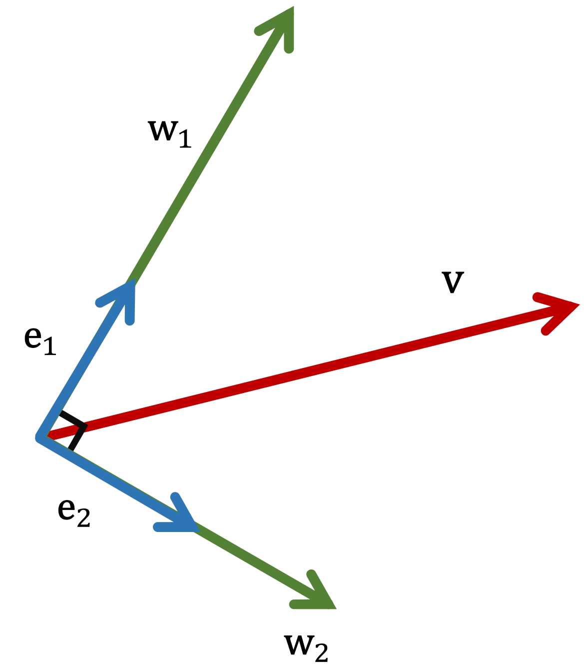
$$\mathrm{v} = a\mathrm{w}_1 + b\mathrm{w}_2 \quad \boxed{\mathrm{v} = k_1\mathrm{e}_1 + k_2\mathrm{e}_2} \quad \therefore k_1 = \mathrm{v}\cdot\mathrm{e}_1$$

$$\therefore k_2 = \mathrm{v}\cdot\mathrm{e}_2$$

$$\mathrm{v} = (\mathrm{v}\cdot\mathrm{e}_1)\mathrm{e}_1 + (\mathrm{v}\cdot\mathrm{e}_2)\mathrm{e}_2$$

$$= \left(\frac{\mathrm{v}\cdot\mathrm{w}_1}{\|\mathrm{w}_1\|}\right)\frac{\mathrm{w}_1}{\|\mathrm{w}_1\|} + \left(\frac{\mathrm{v}\cdot\mathrm{w}_2}{\|\mathrm{w}_2\|}\right)\frac{\mathrm{w}_2}{\|\mathrm{w}_2\|}$$

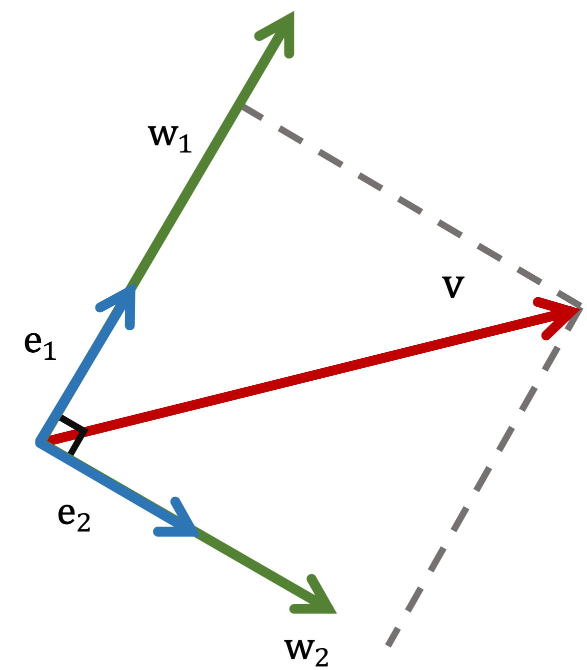
$$\therefore \mathrm{v} = \left(\frac{\mathrm{v}\cdot\mathrm{w}_1}{\|\mathrm{w}_1\|^2}\right)\mathrm{w}_1 + \left(\frac{\mathrm{v}\cdot\mathrm{w}_2}{\|\mathrm{w}_2\|^2}\right)\mathrm{w}_2$$



DECOMPOSING VECTORS INTO ORTHOGONAL COMPONENTS

$$\therefore \mathrm{v} = \underbrace{\left(\frac{\mathrm{v} \cdot \mathrm{w}_1}{\|\mathrm{w}_1\|^2}\right)\mathrm{w}_1}_{\text{the } \textit{orthogonal projection} \text{ of } \mathrm{v} \text{ on } \mathrm{w}_1} + \underbrace{\left(\frac{\mathrm{v} \cdot \mathrm{w}_2}{\|\mathrm{w}_2\|^2}\right)\mathrm{w}_2}_{\text{the } \textit{orthogonal projection} \text{ of } \mathrm{v} \text{ on } \mathrm{w}_2}$$

$\mathrm{proj}_{\mathrm{w}_1}\mathrm{v}$ $\qquad$ $\mathrm{proj}_{\mathrm{w}_2}\mathrm{v}$



ORTHOGONAL PROJECTIONS

The orthogonal projection of v on an arbitrary nonzero vector b is

$$\mathrm{proj}_{\mathrm{b}}\mathrm{v} = \frac{\mathrm{v} \cdot \mathrm{b}}{||\mathrm{b}||^2}\ \mathrm{b}$$

Moreover, if we subtract $\mathrm{proj}_{\mathrm{b}}\mathrm{v}$ from v, then the resulting vector

$$\mathrm{v}\ - \mathrm{proj}_{\mathrm{b}}\mathrm{v}$$

will be orthogonal to b; and we call this *the vector component of* v *orthogonal to* b*.*

ORTHOGONAL PROJECTIONS

Example: Find the orthogonal projection of $v = i + j + k$ on $b = 2i + 2j$, and then find the vector component of v orthogonal to b.

$$\mathbf{v} \cdot \mathbf{b} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j}) = 2 + 2 + 0 = 4$$

$$\|\mathbf{b}\|^2 = 2^2 + 2^2 = 8$$

Thus, the orthogonal projection of $\mathbf{v}$ on $\mathbf{b}$ is

$$\operatorname{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{4}{8}(2\mathbf{i} + 2\mathbf{j}) = \mathbf{i} + \mathbf{j}$$

and the vector component of $\mathbf{v}$ orthogonal to $\mathbf{b}$ is

$$\mathbf{v} - \operatorname{proj}_{\mathbf{b}}\mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathbf{i} + \mathbf{j}) = \mathbf{k}$$

Course: Calculus (3) Lecture No: [8]

Chapter: [11]

THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.4]

CROSS PRODUCT

DETERMINANTS

- A **matrix** is a rectangular *array* (*table*) of numbers arranged in rows and columns.
- For example, $\begin{bmatrix} -1 & 0 & 3 \\ 2 & 5 & -7 \end{bmatrix}$.
- The **determinant** is a function that assigns numerical value to square matrix (number of rows = number of columns) of numbers.
- We define a 2×2 determinant by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- For example, $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = (3)(5) - (-2)(4) = 15 + 8 = 23$

DETERMINANTS

A 3×3 determinant is defined in terms of 2×2 determinants by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example

$$\begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 3(20) + 2(2) - 5(3) = 49$$

DETERMINANTS

11.4.1 THEOREM

(*a*) *If two rows in the array of a determinant are the same, then the value of the determinant is* 0.

(*b*) *Interchanging two rows in the array of a determinant multiplies its value by* -1.

PROOF (*a*)

$$\begin{vmatrix} a_1 & a_2 \\ a_1 & a_2 \end{vmatrix} = a_1a_2 - a_2a_1 = 0$$

PROOF (*b*)

$$\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = b_1a_2 - b_2a_1 = -(a_1b_2 - a_2b_1) = -\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad \blacksquare$$

CROSS PRODUCT

11.4.2 DEFINITION If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ are vectors in 3-space, then the ***cross product*** $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Example Let $\mathbf{u} = \langle 1, 2, -2 \rangle$ and $\mathbf{v} = \langle 3, 0, 1 \rangle$. Find $\mathbf{u} \times \mathbf{v}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \mathbf{k} = 2\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}$$

ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

- Keep in mind the essential differences between the *cross product* and the *dot product*:
  - ✓ The cross product is defined only for vectors in $3-$space, whereas the dot product is defined for vectors in $2-$space and $3-$space.
  - ✓ The cross product of two vectors is a vector, whereas the dot product of two vectors is a scalar.

ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

11.4.3 THEOREM *If* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are any vectors in 3-space and* k *is any scalar, then:*

(*a*) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(*b*) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(*c*) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(*d*) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

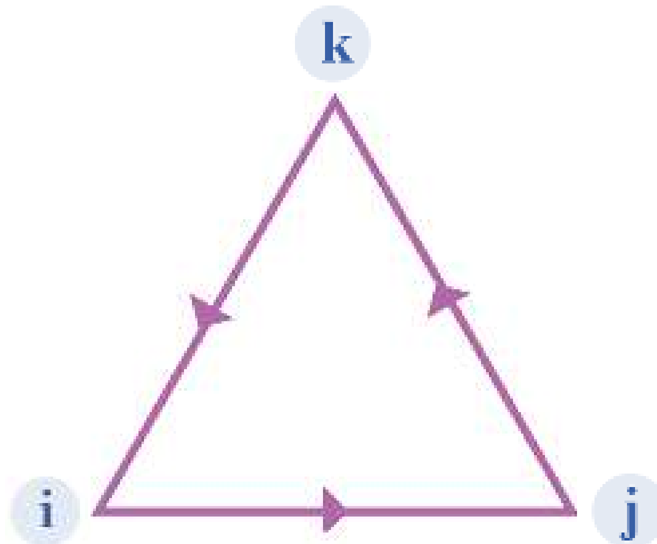
(*e*) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(*f*) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

The following cross products occur so frequently that it is helpful to be familiar with them:

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$



ALGEBRAIC PROPERTIES OF THE CROSS PRODUCT

WARNING

- We can write a product of three real numbers as abc since the associative law $(ab)c = a(bc)$ ensures that the same value for the product results no matter how the factors are grouped.
- The *associative* law does not hold for cross products. For example,

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}$$

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

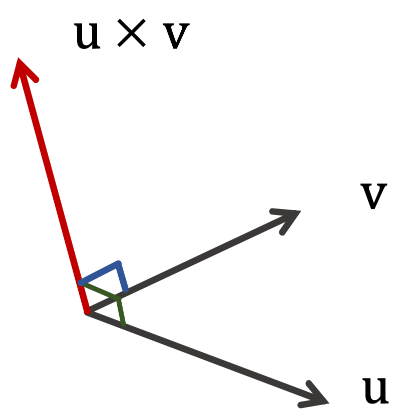
- Thus, we cannot write a cross product with three vectors as $\mathbf{u} \times \mathbf{v} \times \mathbf{w}$, since this expression is ambiguous (مُبهم) without parentheses.

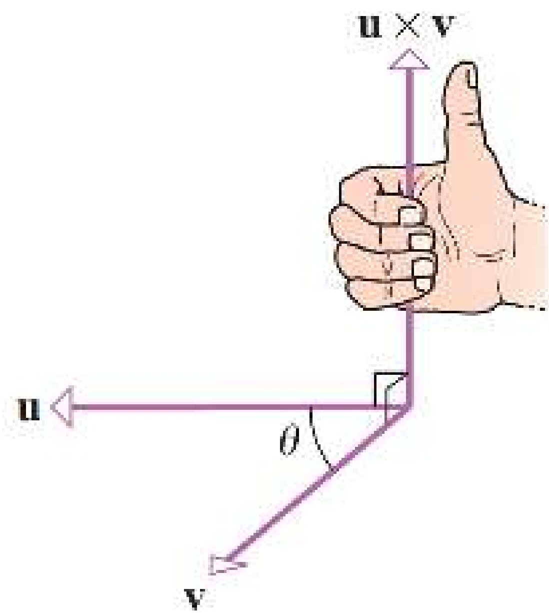
GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

11.4.4 THEOREM *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in 3-space, then:*

(*a*) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{u}$)

(*b*) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ *is orthogonal to* $\mathbf{v}$)





GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

Example Find a vector that is orthogonal to both of the vectors $\mathbf{u} = \langle 2, -1, 3 \rangle$ and $\mathbf{v} = \langle -7, 2, -1 \rangle$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -7 & 2 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ -7 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ -7 & 2 \end{vmatrix} \mathbf{k} = -5\mathbf{i} - 19\mathbf{j} - 3\mathbf{k}$$

GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

11.4.5 THEOREM *Let* **u** *and* **v** *be nonzero vectors in 3-space, and let* θ *be the angle between these vectors when they are positioned so their initial points coincide.*

(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$

GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

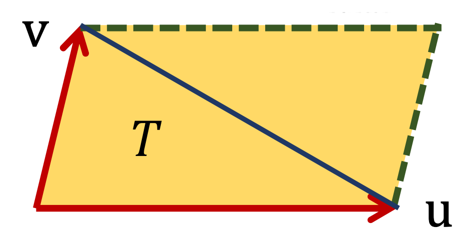
11.4.5 THEOREM *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be nonzero vectors in 3-space, and let* θ *be the angle between these vectors when they are positioned so their initial points coincide.*

(*a*) $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$

(*b*) *The area* A *of the parallelogram that has* $\mathbf{u}$ *and* $\mathbf{v}$ *as adjacent sides is*

$$A = \|\mathbf{u} \times \mathbf{v}\|$$

$$T = \frac{1}{2}\|\mathrm{u} \times \mathrm{v}\|$$



(*c*) $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel vectors, that is, if and only if they are scalar multiples of one another.*

GEOMETRIC PROPERTIES OF THE CROSS PRODUCT

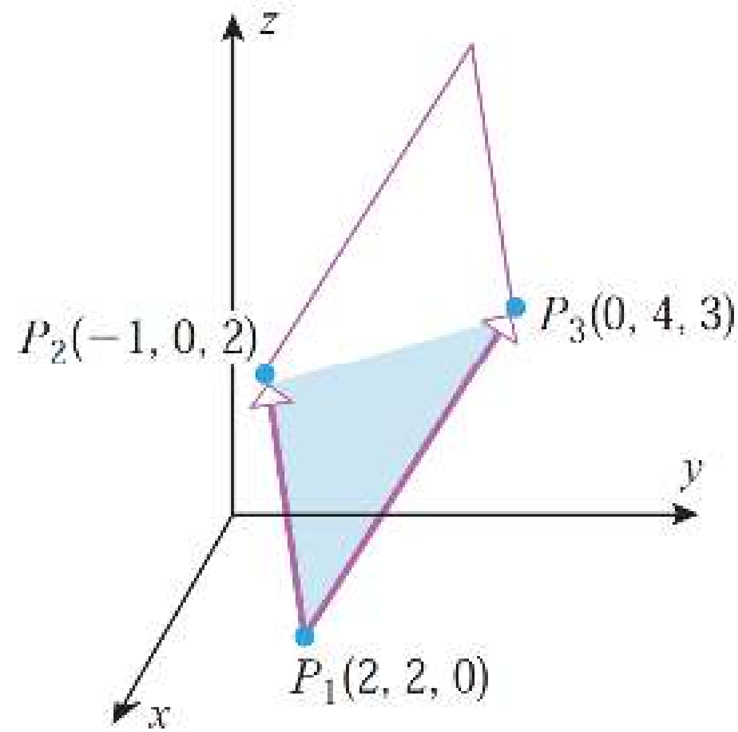
Example Find the area of the triangle that is determined by the points $P_1(2,2,0)$, $P_2(-1,0,2)$, and $P_3(0,4,3)$.

$$\overrightarrow{P_1P_2} = \langle -3, -2, 2\rangle$$

$$\overrightarrow{P_1P_3} = \langle -2, 2, 3\rangle$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \langle -10, 5, -10\rangle$$

$$A = \tfrac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \tfrac{15}{2}$$



SCALAR TRIPLE PRODUCTS

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ are vectors in 3-space, then the number

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

is called the ***scalar triple product*** of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Example Calculate the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ of the vectors

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \quad \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

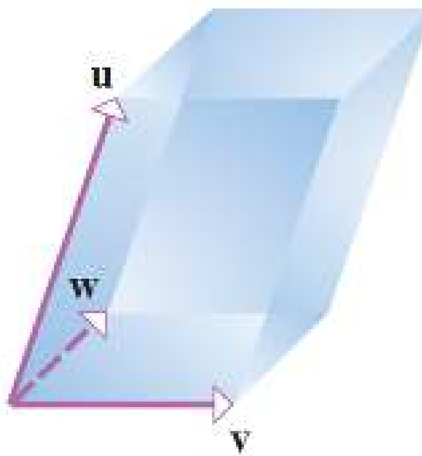
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 49$$

GEOMETRIC PROPERTIES OF THE SCALAR TRIPLE PRODUCT

11.4.6 THEOREM *Let* **u**, **v**, *and* **w** *be nonzero vectors in 3-space.*

(*a*) *The volume* V *of the parallelepiped that has* **u**, **v**, *and* **w** *as adjacent edges is*

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$



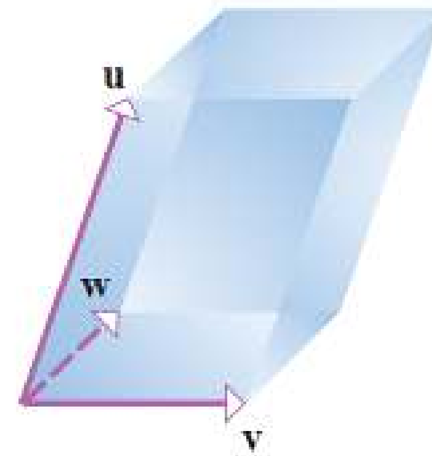
GEOMETRIC PROPERTIES OF THE SCALAR TRIPLE PRODUCT

11.4.6 THEOREM *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be nonzero vectors in 3-space.*

(*a*) *The volume* V *of the parallelepiped that has* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *as adjacent edges is*

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

(*b*) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ *if and only if* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *lie in the same plane.*



ALGEBRAIC PROPERTIES OF THE SCALAR TRIPLE PRODUCT

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$$

Course: Calculus (3)

Lecture No: [9]

Chapter: [11]

THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.*]

REVIEW OF PARAMETRIC EQUATIONS

PARAMETRIC EQUATIONS

- Until now, you have been representing a graph by a single equation involving *two* variables.
- You will study situations in which *three* variables are used to represent a curve in the plane.
- Suppose that a particle moves along a curve C (*trajectory*) in the xy −plane.
- The rectangular equation of the curve C, does not tell the whole story.
- Although it does tell you *where* the object has been, it doesn't tell you *when* the object was at a given point (x, y).
- To determine this time, you can introduce a third variable t, called a **parameter**.

PARAMETRIC EQUATIONS

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the equations

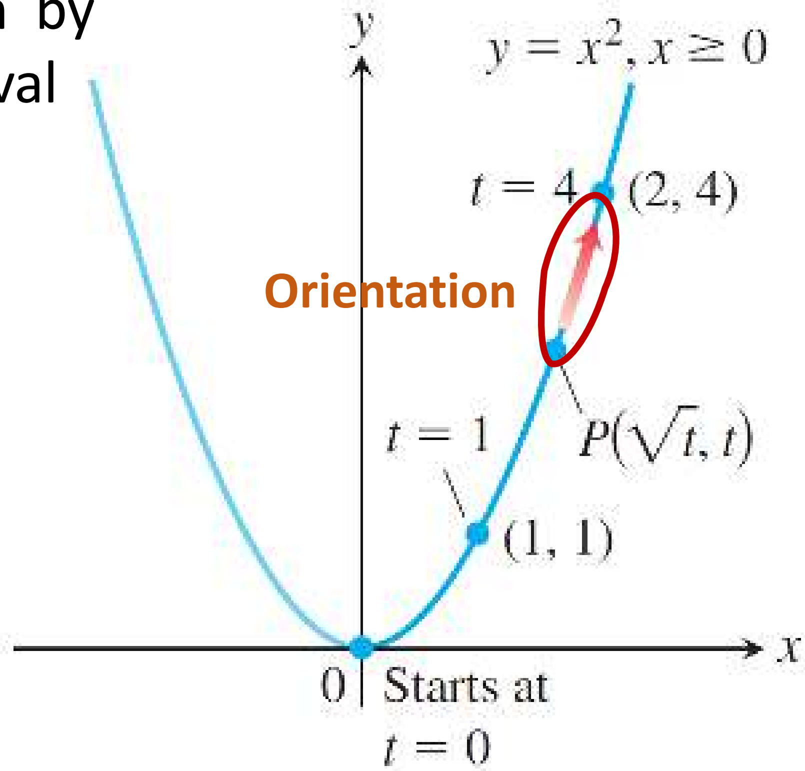
$$x = f(t) \quad \text{and} \quad y = g(t)$$

are **parametric equations** and t is the **parameter.** The set of points (x, y) obtained as t varies over the interval I is the **graph** of the parametric equations. Taken together, the parametric equations and the graph are a **plane curve,** denoted by C.

PARAMETRIC EQUATIONS

Example The position $P(x, y)$ of a particle moving in the xy −plane is given by the equations and parameter interval

$$x = \sqrt{t} \quad , \quad y = t \quad , \quad t \geq 0$$



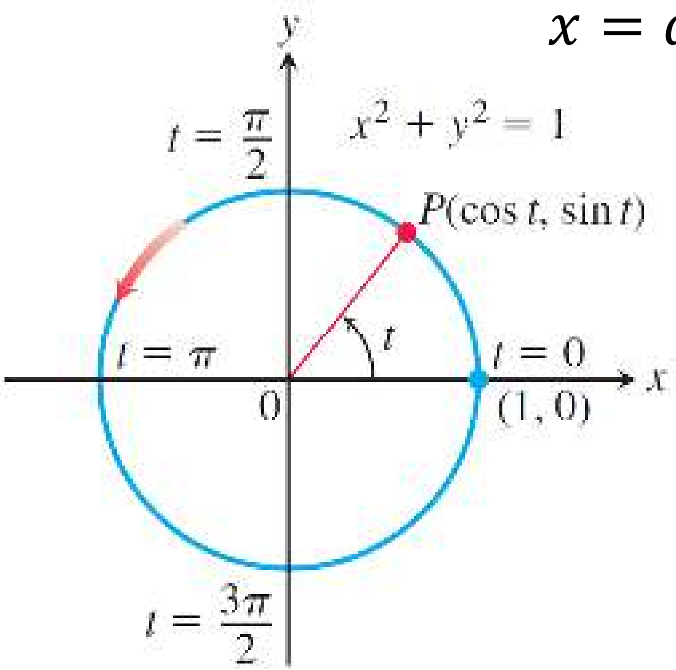
We try to *identify the path* by eliminating t between the equations:

$$y = t = \left(\sqrt{t}\right)^2 = x^2$$

PARAMETRIC EQUATIONS

Example The *counter-clockwise* orientation parametric equations of the circle $x^2 + y^2 = a^2$ are

$$x = a\cos t \quad , \quad y = a\sin t \quad , \quad 0 \le t \le 2\pi$$



$$x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2.$$

Course: Calculus (3) Lecture No: [10]

Chapter: [11]

THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.5]

PARAMETRIC EQUATIONS OF LINES

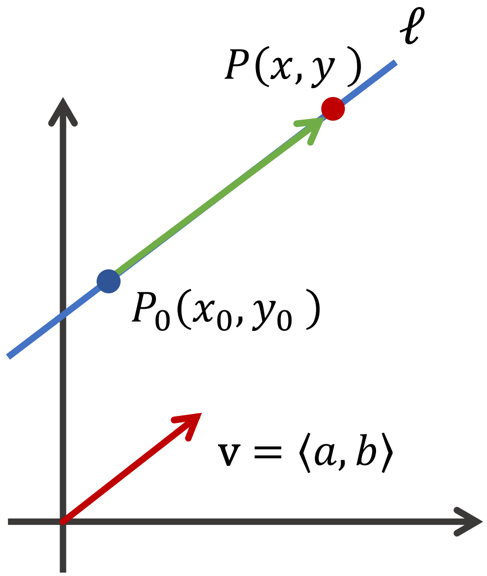
LINES DETERMINED BY A POINT AND A VECTOR

$$\langle x - x_0, y - y_0 \rangle = \langle ta, tb \rangle$$

$$\begin{aligned} x - x_0 &= ta \\ y - y_0 &= tb \end{aligned} \quad \Rightarrow$$

The parametric equations of the line in $2-$space that passes through the point $P_0(x_0, y_0)$ and is parallel to the nonzero vector $\mathbf{v} = \langle a, b \rangle = a\mathbf{i} + b\mathbf{j}$ are

$$x = x_0 + at \quad , \quad y = y_0 + bt$$



$$\ell \parallel \mathbf{v}$$

$$\overrightarrow{P_0P} \parallel \mathbf{v}$$

$$\overrightarrow{P_0P} = t\,\mathbf{v}$$

LINES DETERMINED BY A POINT AND A VECTOR

The parametric equations of the line in $3-$space that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the nonzero vector $\mathrm{v} = \langle a, b, c\rangle = a\mathrm{i} + b\mathrm{j} + c\mathrm{k}$ are

$$x = x_0 + at \quad , \quad y = y_0 + bt \quad , \quad z = z_0 + ct$$

Example Find parametric equations of the line passing through $(1, 2, -3)$ and parallel to $\mathrm{v} = 4\mathrm{i} + 5\mathrm{j} - 7\mathrm{k}$.

$$x = 1 + 4t \quad , \quad y = 2 + 5t \quad , \quad z = -3 - 7t$$

LINES DETERMINED BY A POINT AND A VECTOR

Example

1. Find parametric equations of the line ℓ passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

The vector $\overrightarrow{P_1P_2} = \langle 5-2, 0-4, 7-(-1)\rangle = \langle 3, -4, 8\rangle$ is parallel to ℓ.

If P_1 is chosen:

$$\boxed{x} = 2 + 3t_1$$
$$\boxed{y} = 4 - 4t_1$$
$$\boxed{z} = -1 + 8t_1$$

$$4 - 4t_1 = -4t_2$$
$$-1 + t_1 = t_2$$

If P_2 is chosen:

$$\boxed{x} = 5 + 3t_2$$
$$\boxed{y} = -4t_2$$
$$\boxed{z} = 7 + 8t_2$$

LINES DETERMINED BY A POINT AND A VECTOR

Example

1. Find parametric equations of the line ℓ passing through the points $P_1(2, 4, -1)$ and $P_2(5, 0, 7)$.

2. Where does the line intersect the xy −plane?

$$x = 5 + 3t_2$$
$$y = -4t_2$$
$$z = 7 + 8t_2$$

$$z = 0 \qquad 7 + 8t_2 = 0 \qquad t_2 = \frac{-7}{8}$$

The point is $\left(\frac{19}{8}, \frac{7}{2}, 0\right)$

LINES DETERMINED BY A POINT AND A VECTOR

Example Let ℓ_1 and ℓ_2 be the lines

$$\ell_1: \quad x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t \qquad \mathrm{v}_1 = \langle 4, -4, 5 \rangle$$

$$\ell_2: \quad x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t \qquad \mathrm{v}_2 = \langle 8, -3, 1 \rangle$$

1. Are the lines parallel?

$$\ell_1 \parallel \ell_2 \quad \Leftrightarrow \quad \mathrm{v}_1 \parallel \mathrm{v}_2 \quad \Leftrightarrow \quad \mathrm{v}_2 = c\, \mathrm{v}_1$$

$$\begin{aligned} 4c &= 8 \\ -4c &= -3 \\ 5c &= 1 \end{aligned}$$

? **No such c**

$\therefore$ ℓ_1 and ℓ_2 are NOT parallel lines.

LINES DETERMINED BY A POINT AND A VECTOR

Example Let ℓ_1 and ℓ_2 be the lines

$$\ell_1: \quad x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$

$$\ell_2: \quad x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

2. Do the lines intersect?

   Suppose the point of intersection is

$$1 + 4t_1 = x^* = 2 + 8t_2$$

$$5 - 4t_1 = y^* = 4 - 3t_2$$

$$-1 + 5t_1 = z^* = 5 + t_2$$

LINES DETERMINED BY A POINT AND A VECTOR

Example Let ℓ_1 and ℓ_2 be the lines

$$\ell_1: \quad x = 1 + 4t, \quad y = 5 - 4t, \quad z = -1 + 5t$$

$$\ell_2: \quad x = 2 + 8t, \quad y = 4 - 3t, \quad z = 5 + t$$

2. Do the lines intersect?

Suppose the point of intersection is

$$\boxed{\begin{aligned} 1 + 4t_1 &= 2 + 8t_2 \\ 5 - 4t_1 &= 4 - 3t_2 \end{aligned}} \; +$$

$$-1 + 5t_1 = 5 + t_2$$

$$6 = 6 + 5t_2$$

$$t_2 = 0$$

$$t_1 = \frac{1}{4}$$

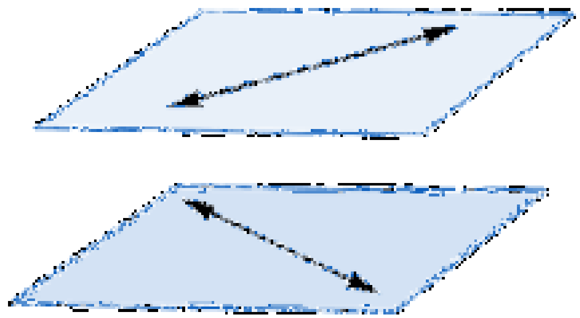
BUT !!

Do not satisfy the 3rd equation.

$\therefore$ ℓ_1 and ℓ_2 do NOT intersect.

LINES DETERMINED BY A POINT AND A VECTOR

- Two lines in $3-\text{space}$ that are *not parallel* and *do not intersect* are called skew lines.
- Any two skew lines lie in *parallel planes*.

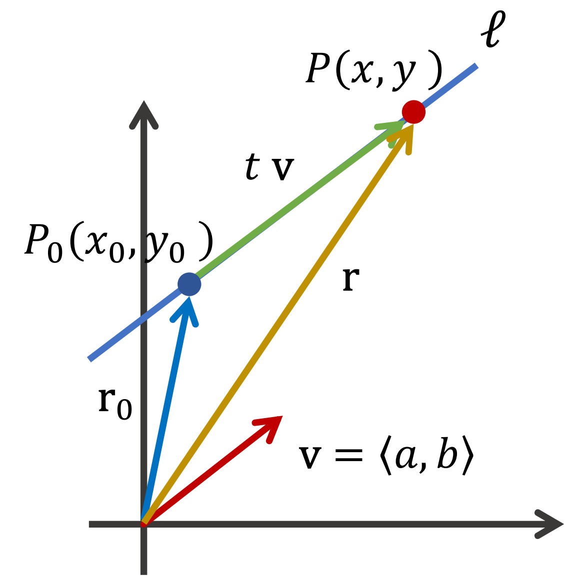


VECTOR EQUATIONS OF LINES

$$\mathrm{r} = \mathrm{r}_0 + t\,\mathrm{v}$$

$\langle x, y \rangle = \langle x_0, y_0 \rangle + t\,\langle a, b \rangle$ In 2-space

$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle$ In 3-space



NOTE Let ℓ be the line that passes through the point (x_0, y_0, z_0) and is parallel to the vector $\mathrm{v} = \langle a, b, c \rangle$, where a, b, and c are nonzero. Then the following equations are called the symmetric equations of ℓ.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Course: Calculus (3)

Lecture No: [11]

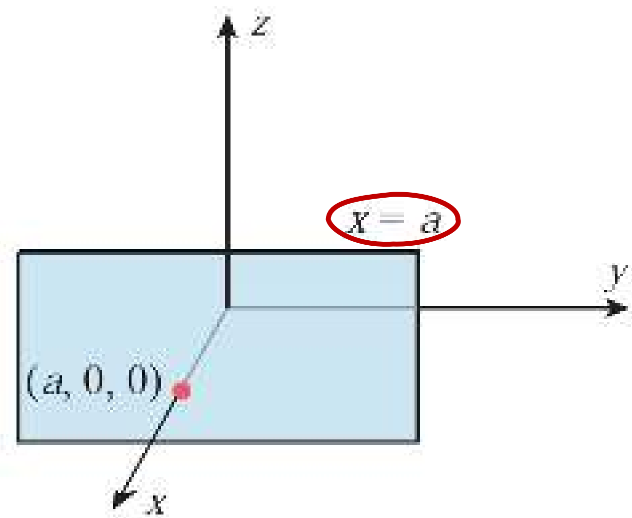
Chapter: [11]

THREE-DIMENSIONAL SPACE; VECTORS

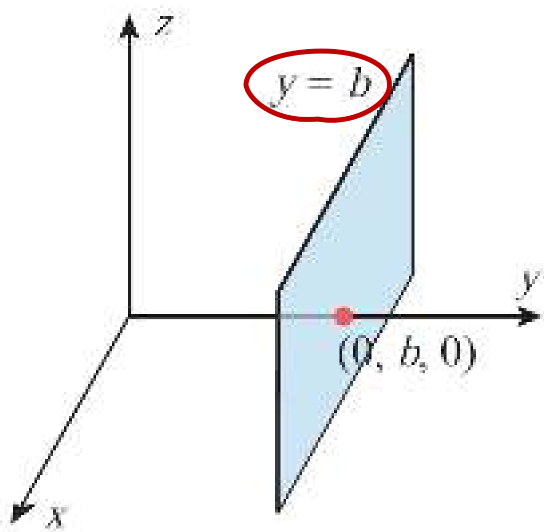
Section: [11.6]

PLANES IN 3-SPACE

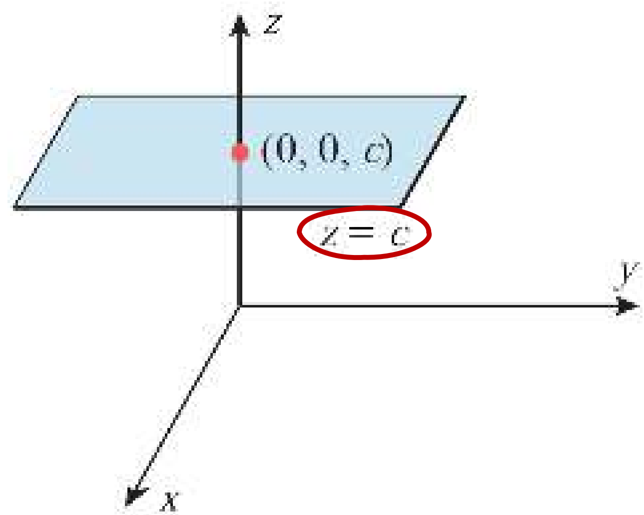
PLANES PARALLEL TO THE COORDINATE PLANES



Parallel to
yz −plane



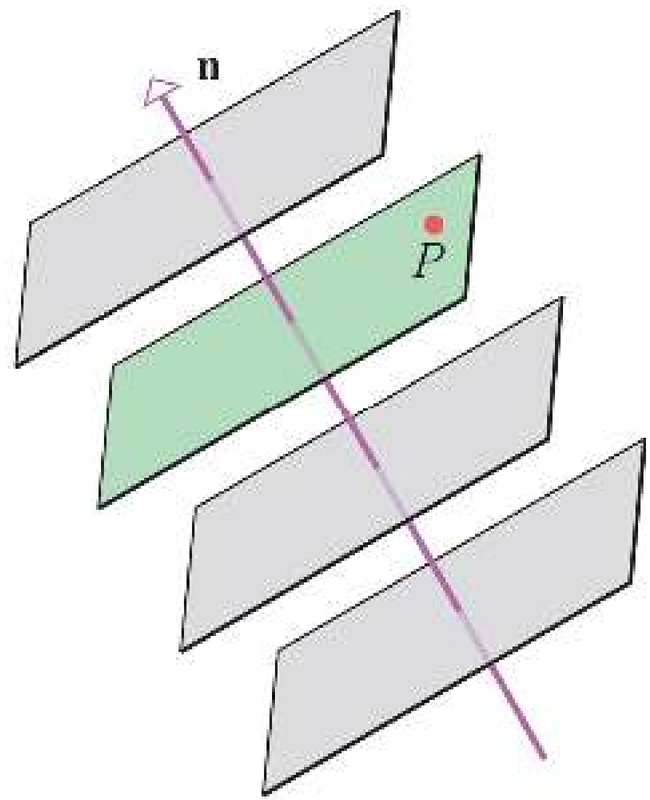
Parallel to
xz −plane



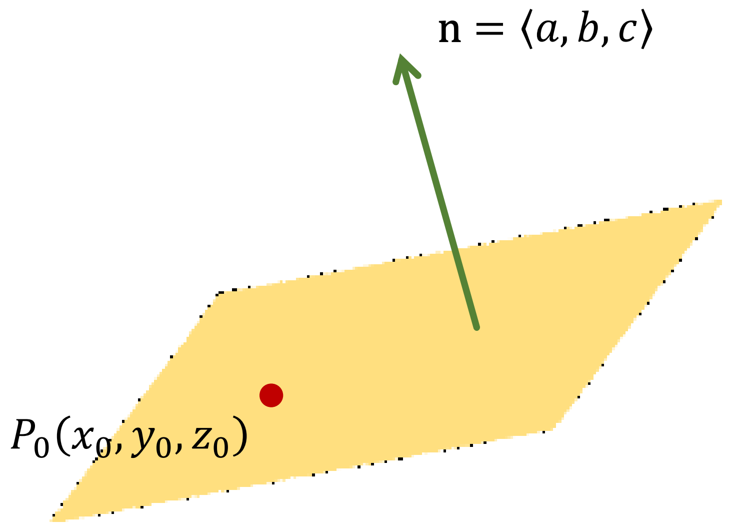
Parallel to
xy −plane

PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

- A plane in 3 $-$space can be determined uniquely by specifying a *point* in the plane and a *vector perpendicular* to the plane.
- A vector perpendicular to a plane is called a normal to the plane.



PLANES DETERMINED BY A POINT AND A NORMAL VECTOR



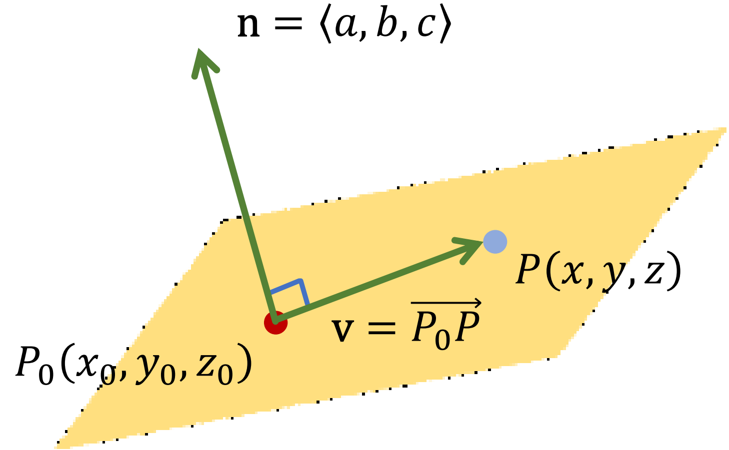
PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

$$\mathrm{n} \cdot \mathrm{v} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

This is called the point-normal form of the equation of a plane.



$$ax + by + cz + d = 0$$

PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

Example Find an equation of the plane passing through the point $(3, -1, 7)$ and perpendicular to the vector $\mathbf{n} = \langle 4, 2, -5 \rangle$.

$$4(x-3) + 2(y+1) - 5(z-7) = 0$$

$$4x - 12 + 2y + 2 - 5z + 35 = 0$$

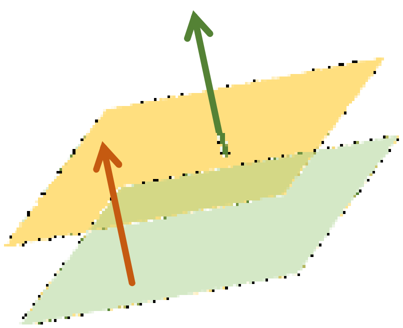
$$4x + 2y - 5z + 25 = 0$$

PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

Example Determine whether the two planes are parallel.

$$P_1: \quad 3x - 4y + 5z = 0 \qquad \mathbf{n}_1 = \langle 3, -4, 5 \rangle$$
$$P_2: \quad -6x + 8y - 10z - 4 = 0 \qquad \mathbf{n}_2 = \langle -6, 8, -10 \rangle$$

$$P_1 \parallel P_2 \quad \Leftrightarrow \quad \mathbf{n}_1 \parallel \mathbf{n}_2 \quad \Leftrightarrow \quad \mathbf{n}_2 = k\,\mathbf{n}_1$$



$$\Leftrightarrow \quad \langle -6, 8, -10 \rangle = k\langle 3, -4, 5 \rangle$$

$$\Leftrightarrow \quad \boxed{\begin{aligned} -6 &= 3k \\ 8 &= -4k \\ -10 &= 5k \end{aligned}}$$

$$\Leftrightarrow \quad k = -2$$

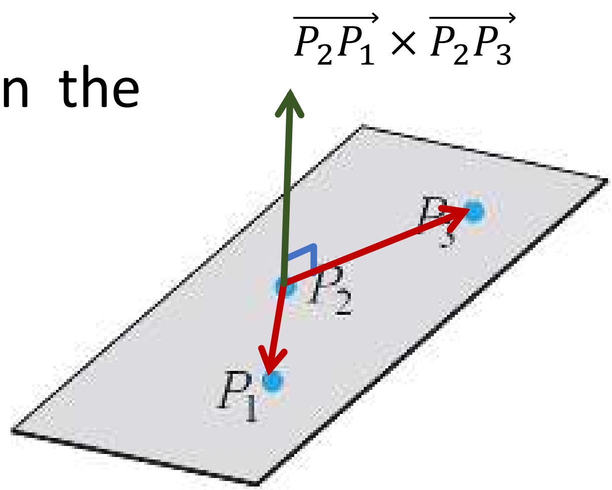
$\therefore$ P_1 and P_2 are parallel planes

PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

Example Find an equation of the plane through the points $P_1(1, 2, -1)$, $P_2(2, 3, 1)$, and $P_3(3, -1, 2)$.

$$\mathrm{n} = \overrightarrow{P_2P_1} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -1 & -1 & -2 \\ 1 & -4 & 1 \end{vmatrix} = \langle -9, -1, 5 \rangle$$

By using this normal and the point $P_3(3, -1, 2)$ in the plane, we obtain the point-normal form



$$-9(x-3) - (y+1) + 5(z-2) = 0$$

$$-9x - y + 5z + 16 = 0$$

$$\boxed{9x + y - 5z - 16 = 0}$$

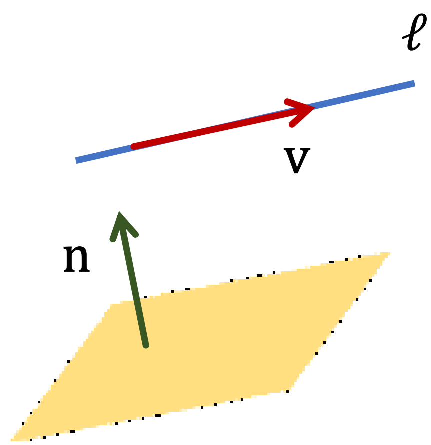
PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

Example Determine whether the line

$$\ell: \quad x = 3 + 8t \quad , \quad y = 4 + 5t \quad , \quad z = -3 - t$$

is parallel to the plane $x - 3y + 5z = 12$.

$\mathrm{v} = \langle 8,5,-1 \rangle \qquad \mathrm{n} = \langle 1,-3,5 \rangle$



PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

Example Determine whether the line

$$\ell: \quad x = 3 + 8t \quad , \quad y = 4 + 5t \quad , \quad z = -3 - t$$

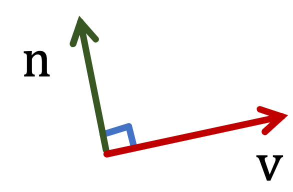
is parallel to the plane $x - 3y + 5z = 12$.

$\mathrm{v} = \langle 8,5,-1 \rangle \qquad \mathrm{n} = \langle 1,-3,5 \rangle$

$\mathrm{n} \cdot \mathrm{v} = (1)(8) + (-3)(5) + (5)(-1) = 12 \neq 0$

$\therefore$ The line and the plane are not parallel.

$\therefore$ The line and the plane **intersects**.



PLANES DETERMINED BY A POINT AND A NORMAL VECTOR

Example Find the intersection of the line

$$\ell: \quad x = 3 + 8t \quad , \quad y = 4 + 5t \quad , \quad z = -3 - t$$

and the plane $x - 3y + 5z = 12$.

Suppose the point of intersection is (x_0, y_0, z_0)

LINE

$$x_0 = 3 + 8t_0$$
$$y_0 = 4 + 5t_0$$
$$z_0 = -3 - t_0$$

PLANE

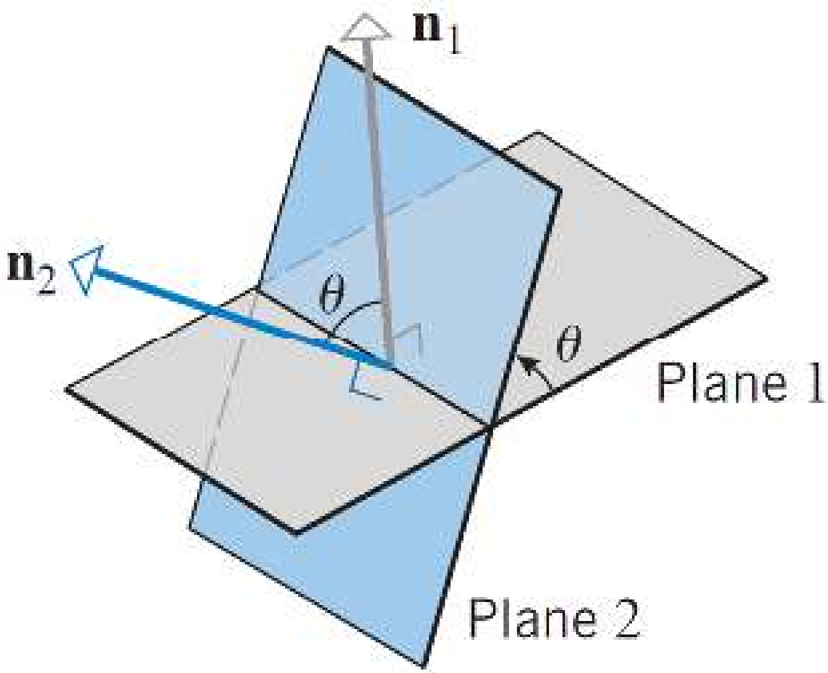
$$x_0 - 3y_0 + 5z_0 = 12$$
$$(3 + 8t_0) - 3(4 + 5t_0) + 5(-3 - t_0) = 12$$
$$t_0 = -3$$

POINT

$$(-21, -11, 0)$$

INTERSECTING PLANES

Two distinct intersecting planes determine two positive angles of intersection



If $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes, then the acute angle θ between the planes satisfies:

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}$$

INTERSECTING PLANES

Example Find the acute angle of intersection between the two planes

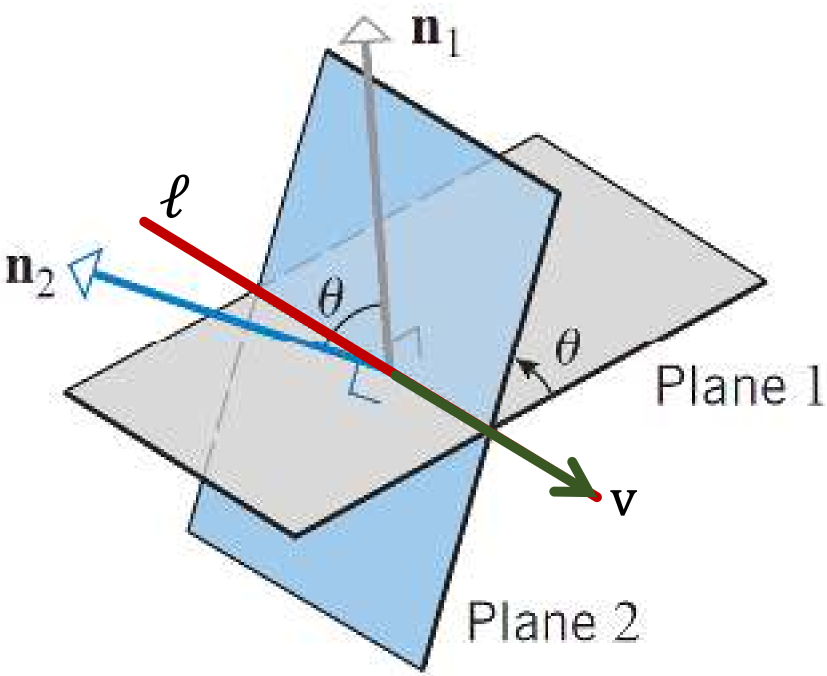
$$\underbrace{2x - 4y + 4z = 6}_{\mathbf{n}_1 = \langle 2, -4, 4 \rangle} \quad \text{and} \quad \underbrace{6x + 2y - 3z = 4}_{\mathbf{n}_2 = \langle 6, 2, -3 \rangle}$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|} = \frac{|-8|}{\sqrt{36}\sqrt{49}} = \frac{4}{21} \qquad \theta = \cos^{-1}\left(\frac{4}{21}\right) \approx 79^\circ$$

INTERSECTING PLANES

Example Find an equation for the line ℓ of intersection of the planes

$$2x - 4y + 4z = 6 \quad \text{and} \quad 6x + 2y - 3z = 4$$



$$\mathrm{v} \parallel \text{Plane } 1 \quad \Rightarrow \quad \mathrm{v} \perp \mathrm{n1}$$

$$\mathrm{v} \parallel \text{Plane } 2 \quad \Rightarrow \quad \mathrm{v} \perp \mathrm{n2}$$

$$\therefore \mathrm{v} = \mathrm{n1} \times \mathrm{n2} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ 2 & -4 & 4 \\ 6 & 2 & -3 \end{vmatrix}$$

$$\mathrm{v} = \langle 4,30,28 \rangle$$

INTERSECTING PLANES

Example Find an equation for the line ℓ of intersection of the planes

$$2x - 4y + 4z = 6 \quad \text{and} \quad 6x + 2y - 3z = 4$$

$\mathrm{v} = \langle 4,30,28 \rangle$

To find a point on ℓ

ℓ is not perpendicular to $\mathrm{k} = \langle 0,0,1 \rangle$

$\mathrm{v} \cdot \mathrm{k} = 0 + 0 + 28 \neq 0$

$\therefore$ ℓ intersects the xy −plane $(z = 0)$

Solve the equations:

$$\underbrace{\begin{matrix} 2x - 4y = 6 \\ 6x + 2y = 4 \end{matrix}}$$

$x = 1, y = -1$

$\therefore \text{point} = (1, -1, 0)$

INTERSECTING PLANES

Example Find an equation for the line ℓ of intersection of the planes

$$2x - 4y + 4z = 6 \quad \text{and} \quad 6x + 2y - 3z = 4$$

$$\mathrm{v} = \langle 4,30,28 \rangle$$

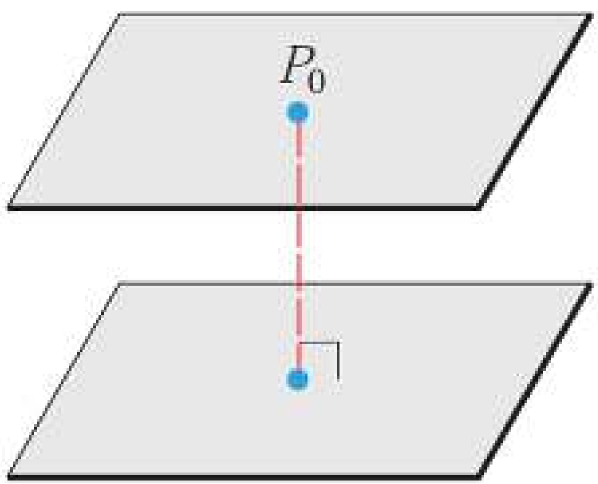
$$\therefore \text{point} = (1,-1,0)$$

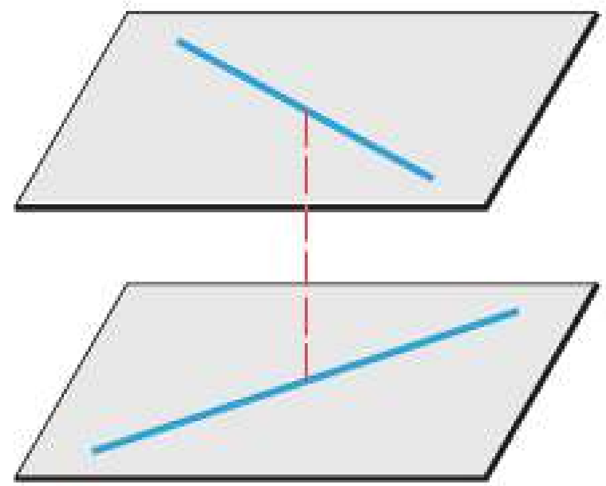
The parametric equations of ℓ are

$$x = 1 + 4t$$
$$y = -1 + 30t$$
$$z = 28t$$

DISTANCE PROBLEMS INVOLVING PLANES

- The distance between a point and a plane.
- The distance between two parallel planes.
- The distance between two skew lines.





DISTANCE PROBLEMS INVOLVING PLANES

The distance D between a point $P_0(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example Find the distance D between the point $(1, -4, -3)$ and the plane $2x - 3y + 6z = -1$.

$$D = \frac{|(2)(1) + (-3)(-4) + 6(-3) + 1|}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{|-3|}{7} = \frac{3}{7}$$

Course: Calculus (3) Lecture No: [13]

Chapter: [11]

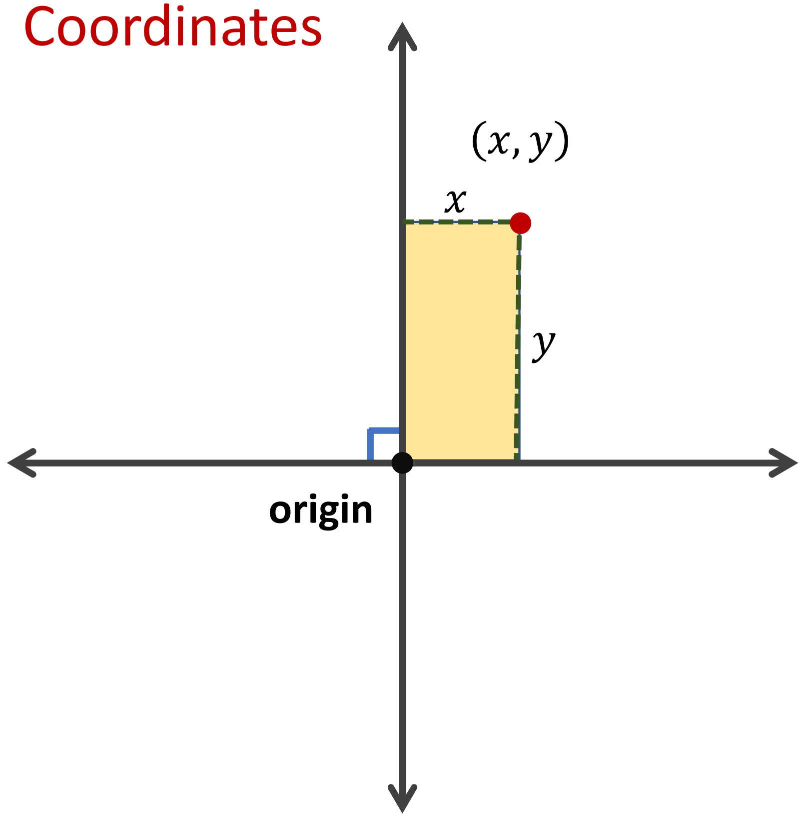
THREE-DIMENSIONAL SPACE; VECTORS

Section: [11.8]

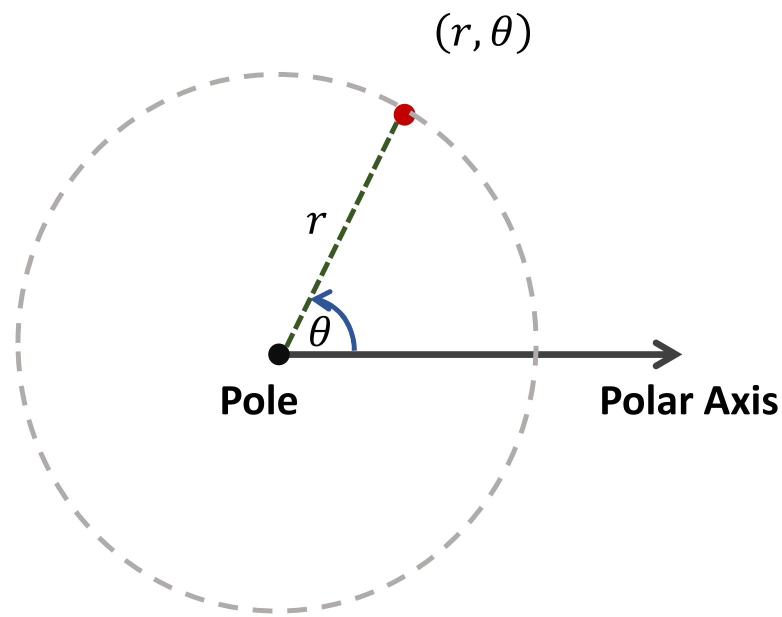
CYLINDRICAL AND SPHERICAL COORDINATES

REVIEW OF POLAR COORDINATES

Rectangular Coordinates



Polar Coordinates

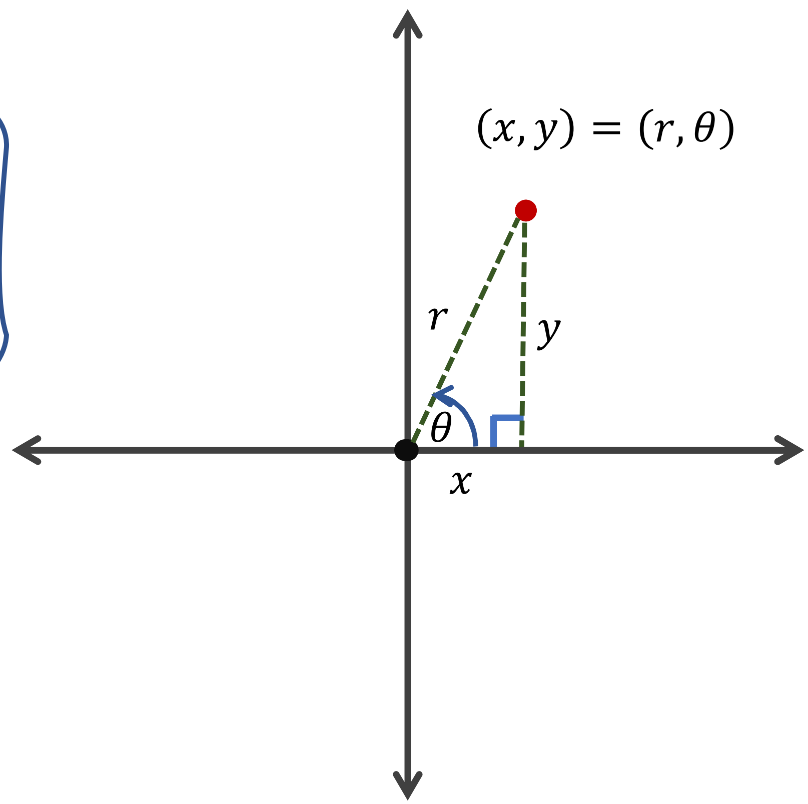


REVIEW OF POLAR COORDINATES

From Rectangular
To Polar

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$



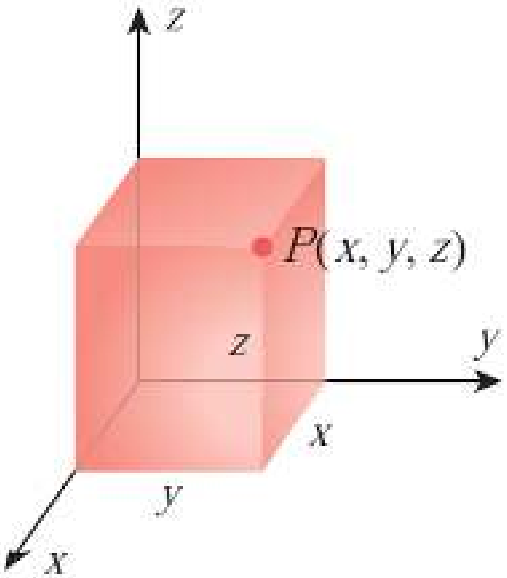
From Polar
To Rectangular

$$\cos\theta = \frac{x}{r} \quad , \quad \sin\theta = \frac{y}{r}$$

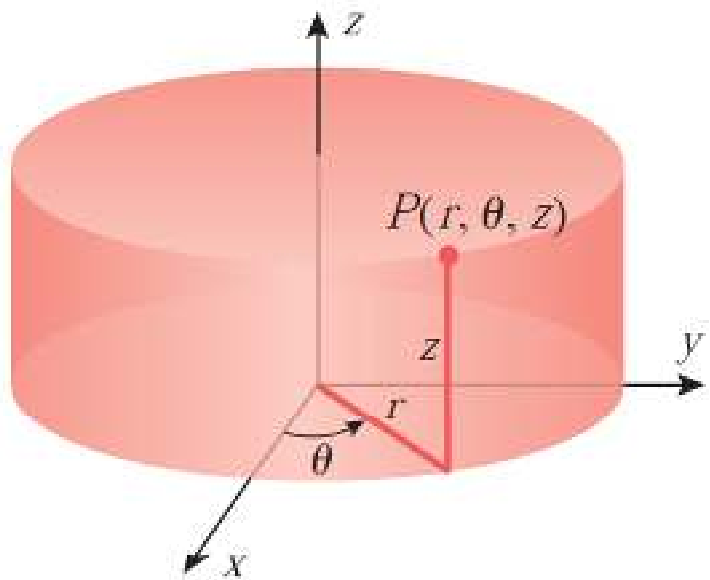
$$x = r\cos\theta$$

$$y = r\sin\theta$$

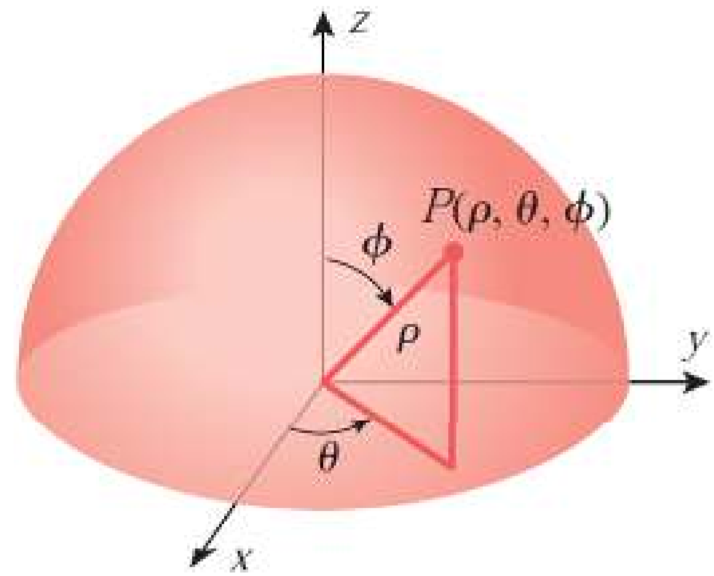
CYLINDRICAL AND SPHERICAL COORDINATE SYSTEMS



Rectangular coordinates
(x, y, z)



Cylindrical coordinates
(r, θ, z)
$(r \geq 0,\ 0 \leq \theta < 2\pi)$

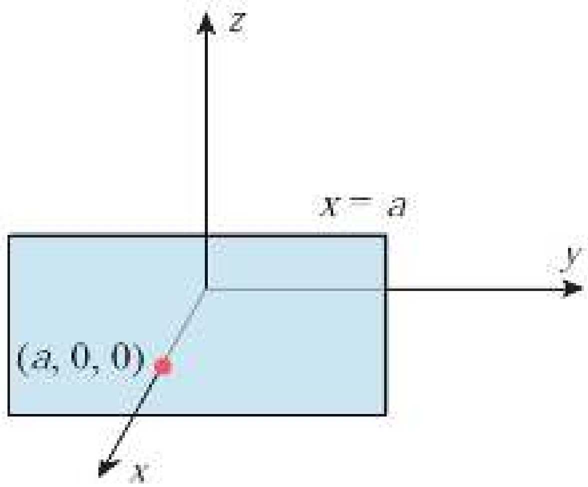


Spherical coordinates
(ρ, θ, ϕ)
$(\rho \geq 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \phi \leq \pi)$

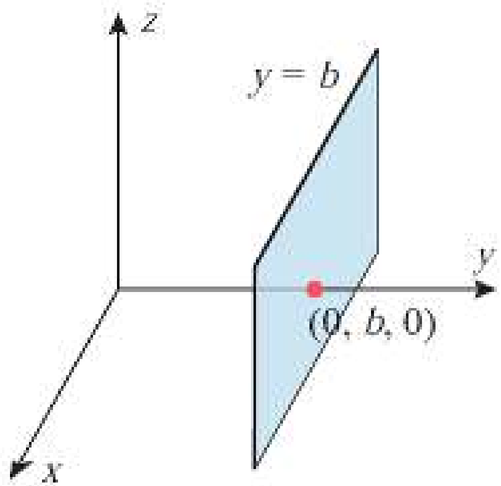
CONSTANT SURFACES

In rectangular coordinates

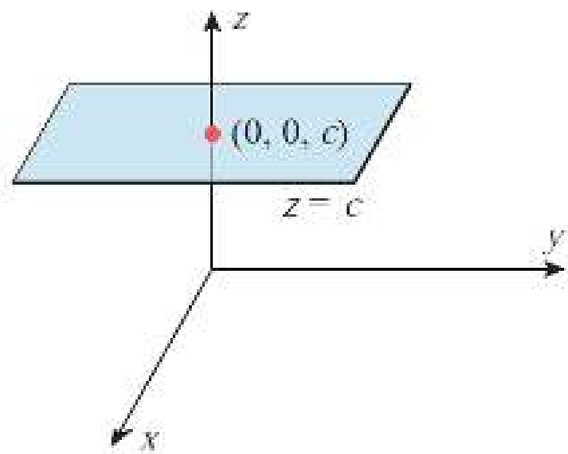
$x = a$



$y = b$



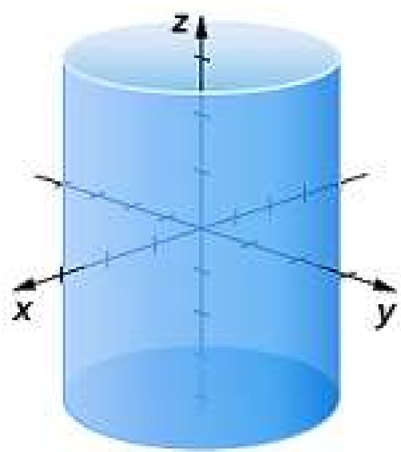
$z = c$



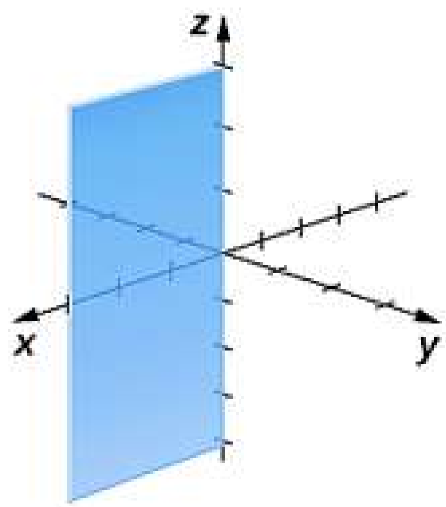
CONSTANT SURFACES

In cylindrical coordinates

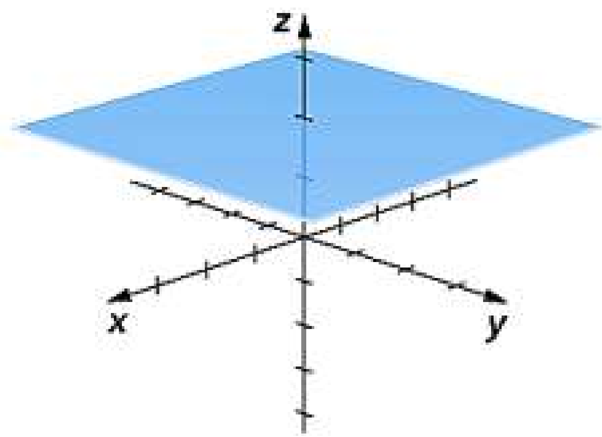
$r = r_0$



$\theta = \theta_0$



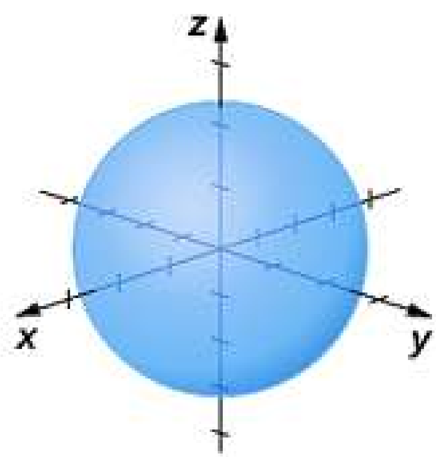
$z = c$



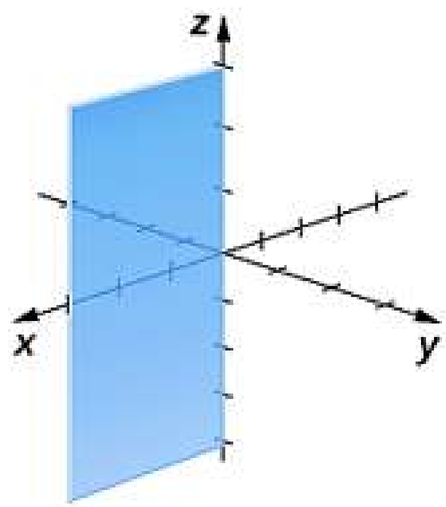
CONSTANT SURFACES

In spherical coordinates

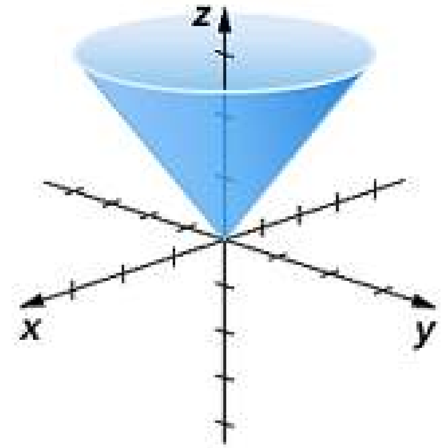
$\rho = \rho_0$



$\theta = \theta_0$



$\phi = \phi_0$



CONVERTING COORDINATES

From	Cylindrical	**From**	Spherical	**From**	Spherical
To	Rectangular	**To**	Cylindrical	**To**	Rectangular
	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$		$r = \rho\sin\phi$ $\theta = \theta$ $z = \rho\cos\phi$		$x = \rho\sin\phi\cos\theta$ $y = \rho\sin\phi\sin\theta$ $z = \rho\cos\phi$
From	Rectangular	**From**	Cylindrical	**From**	Rectangular
To	Cylindrical	**To**	Spherical	**To**	Spherical
	$r = \sqrt{x^2 + y^2}$ $\tan\theta = y/x$ $z = z$		$\rho = \sqrt{r^2 + z^2}$ $\theta = \theta$ $\tan\phi = r/z$		$\rho = \sqrt{x^2 + y^2 + z^2}$ $\tan\theta = y/x$ $\cos\phi = z/\rho$

CONVERTING COORDINATES

Example Find the rectangular coordinates of the point with cylindrical coordinates

$$(r, \theta, z) = \left(4, \frac{\pi}{3}, -3\right)$$

$$x = 4\cos\frac{\pi}{3} = 2$$

$$y = 4\sin\frac{\pi}{3} = 2\sqrt{3}$$

$$z = -3$$

$$\therefore (x, y, z) = \left(2, 2\sqrt{3}, -3\right)$$

From	Cylindrical
To	Rectangular

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

CONVERTING COORDINATES

Example Find the rectangular coordinates of the point with spherical coordinates

$$(\rho, \theta, \phi) = \left(4, \frac{\pi}{3}, \frac{\pi}{4}\right)$$

$$x = 4\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$y = 4\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{2\sqrt{3}}{\sqrt{2}} = \sqrt{6}$$

$$z = 4\cos\frac{\pi}{4} = 2\sqrt{2}$$

$$\therefore (x, y, z) = \left(\sqrt{2}, \sqrt{6}, 2\sqrt{2}\right)$$

From	Spherical
To	Rectangular

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

CONVERTING COORDINATES

Example Find the spherical coordinates of the point that has rectangular coordinates

$$(x, y, z) = \left(4, -4, 4\sqrt{6}\right)$$

$$\rho = \sqrt{4^2 + (-4)^2 + \left(4\sqrt{6}\right)^2} = \sqrt{128} = 8\sqrt{2}$$

$$\tan\theta = \frac{-4}{4} = -1$$

$$\cos\phi = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2}$$

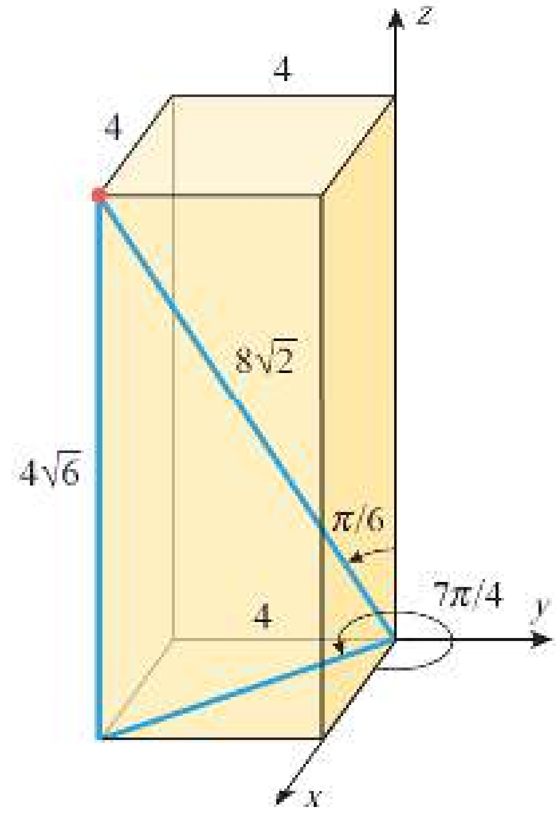
From	Rectangular
To	Spherical

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = y/x$$
$$\cos\phi = z/\rho$$

CONVERTING COORDINATES

Example Find the spherical coordinates of the point that has rectangular coordinates

$$(x, y, z) = \left(4, -4, 4\sqrt{6}\right)$$



$$\rho = \sqrt{4^2 + (-4)^2 + \left(4\sqrt{6}\right)^2} = \sqrt{128} = 8\sqrt{2}$$

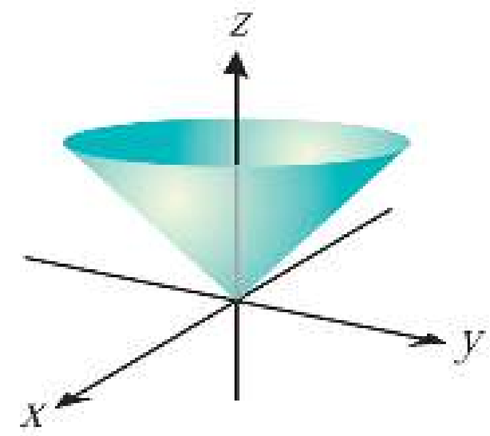
$$\tan\theta = \frac{-4}{4} = -1 \qquad \theta = \frac{7\pi}{4}$$

$$\cos\phi = \frac{4\sqrt{6}}{8\sqrt{2}} = \frac{\sqrt{3}}{2} \qquad \phi = \frac{\pi}{6}$$

$$\therefore (\rho, \theta, \phi) = \left(8\sqrt{2}, \frac{7\pi}{4}, \frac{\pi}{6}\right)$$

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

Example Find equations of the cone $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates.



From	Rectangular
To	Cylindrical

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = y/x$$
$$z = z$$

$$\boxed{z = r}$$

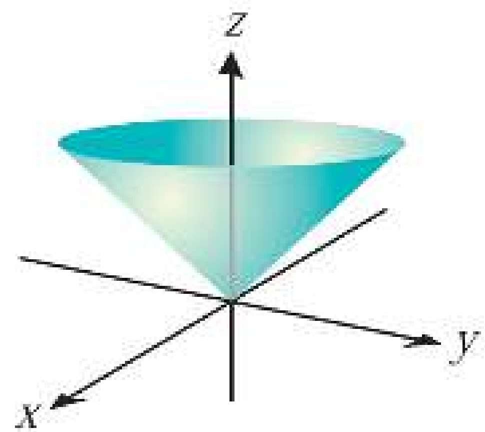
From	Spherical
To	Rectangular

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$$

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

Example Find equations of the cone $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates.



From	Rectangular
To	Cylindrical

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = y/x$$
$$z = z$$

$$\boxed{z = r}$$

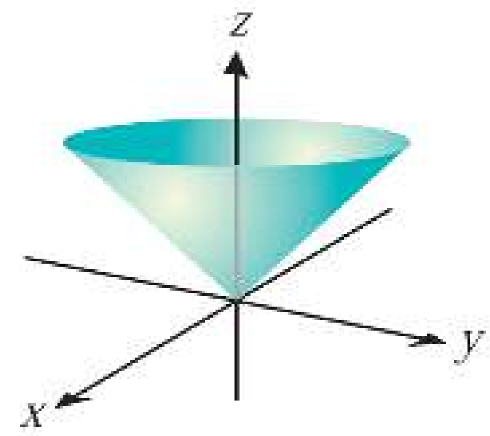
From	Spherical
To	Rectangular

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi\,(\cos^2\theta + \sin^2\theta)}$$

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

Example Find equations of the cone $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates.



From	Rectangular
To	Cylindrical

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = y/x$$
$$z = z$$

$$\boxed{z = r}$$

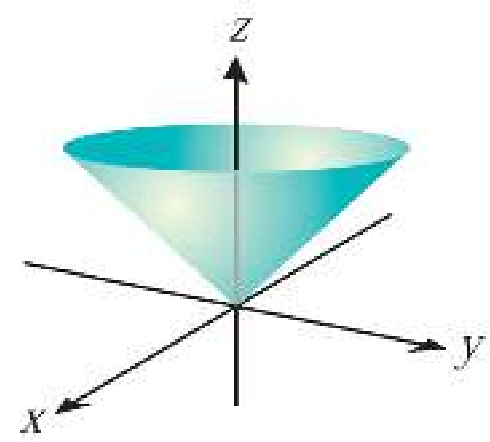
From	Spherical
To	Rectangular

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$\rho \cos\phi = \sqrt{\rho^2 \sin^2 \phi}$$

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

Example Find equations of the cone $z = \sqrt{x^2 + y^2}$ in cylindrical and spherical coordinates.



From	Rectangular
To	Cylindrical

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = y/x$$

$$z = z$$

$$\boxed{z = r}$$

From	Spherical
To	Rectangular

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\rho \cos\phi = \rho \sin\phi$$

$$1 = \tan\phi$$

$$\boxed{\phi = \frac{\pi}{4}}$$

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

Example Find equations of the paraboloid $\rho = \cos\phi \csc^2\phi$ in cylindrical coordinates.

$$\rho = \cos\phi \csc^2\phi$$

$$\sin^2\phi\, \rho = \cos\phi$$

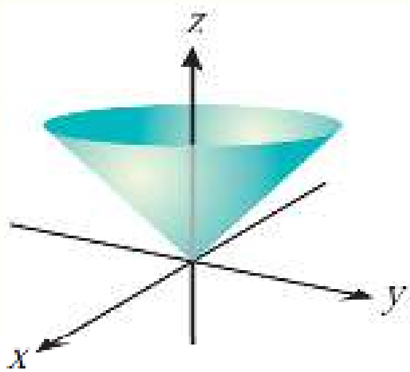
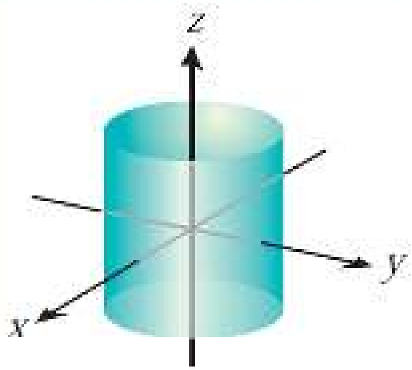
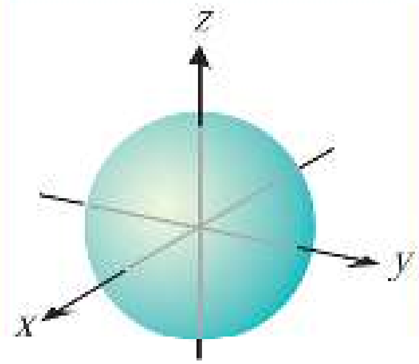
$$\frac{r^2}{\rho^2}\rho = \frac{z}{\rho}$$

$$z = r^2$$

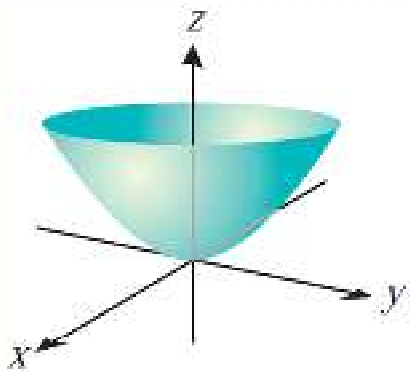
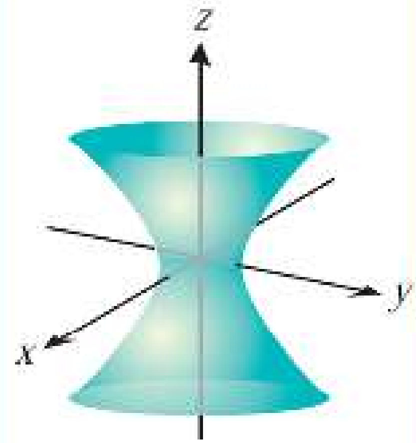
From	Spherical
To	Cylindrical

$$r = \rho \sin\phi$$
$$\theta = \theta$$
$$z = \rho\cos\phi$$

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

	CONE	CYLINDER	SPHERE
			
RECTANGULAR	$z = \sqrt{x^2 + y^2}$	$x^2 + y^2 = 1$	$x^2 + y^2 + z^2 = 1$
CYLINDRICAL	$z = r$	$r = 1$	$z^2 = 1 - r^2$
SPHERICAL	$\phi = \pi/4$	$\rho = \csc \phi$	$\rho = 1$

EQUATIONS OF SURFACES IN CYLINDRICAL AND SPHERICAL COORDINATES

	PARABOLOID	HYPERBOLOID
		
RECTANGULAR	$z = x^2 + y^2$	$x^2 + y^2 - z^2 = 1$
CYLINDRICAL	$z = r^2$	$z^2 = r^2 - 1$
SPHERICAL	$\rho = \cos\phi \csc^2\phi$	$\rho^2 = -\sec 2\phi$