Course: Calculus (3) Lecture No: [15]

Chapter: [12]

VECTOR-VALUED FUNCTIONS

Section: [12.1]

INTRODUCTION TO VECTOR-VALUED FUNCTIONS

IN THIS CHAPTER

- ✓ We will consider $\underbrace{\text{functions whose values are vectors}}$.

  Functions that associate vectors with real numbers.

- ✓ In this section we will discuss more general parametric curves, and we will show how vector notation can be used to express parametric equations in a more compact form.

PARAMETRIC CURVES IN 3 –SPACE

- A *space curve C* is the set of all ordered triples (x, y, z) together with their defining parametric equations

$$x = f(t), y = g(t) \text{ and } z = h(t)$$

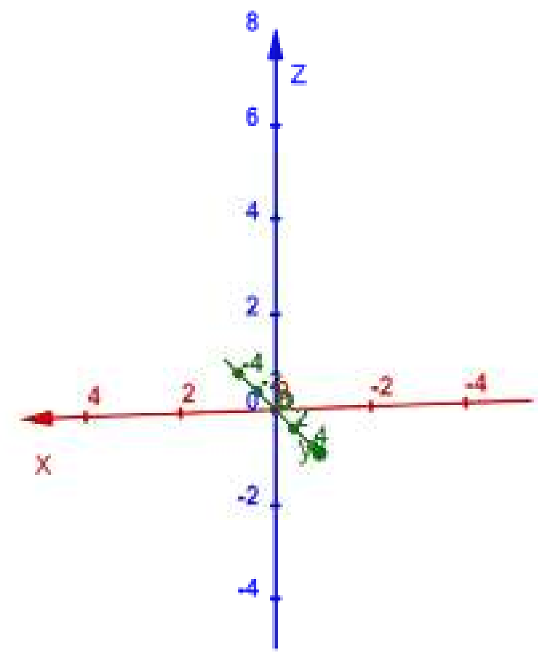
  where f, g and h are continuous functions of t on an interval I , that is traced in a *specific direction* (**orientation**) as the parameter t increases.
- The curve together with its orientation is called the *graph* of the parametric equations or the parametric curve represented by the equations.

PARAMETRIC CURVES IN 3 −SPACE

Example The parametric equations

$$x = 1 - t$$
$$y = 3t$$
$$z = 2t$$

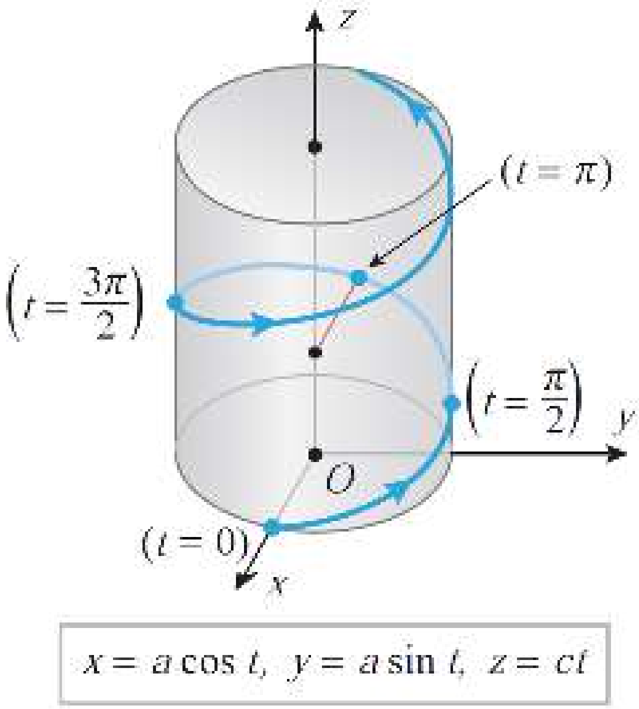
represent a line in 3 −space that passes through the point $(1,0,0)$ and is parallel to the vector $\langle -1, 3, 2 \rangle$.

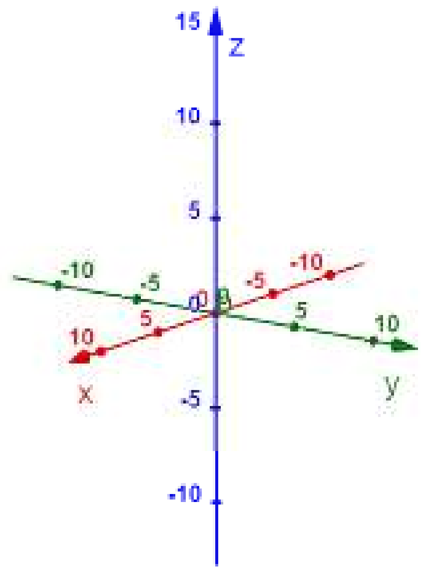


PARAMETRIC CURVES IN 3 −SPACE

Example Describe the parametric curve represented by the equations

$$x = 10\cos t$$
$$y = 10\sin t$$
$$z = t$$





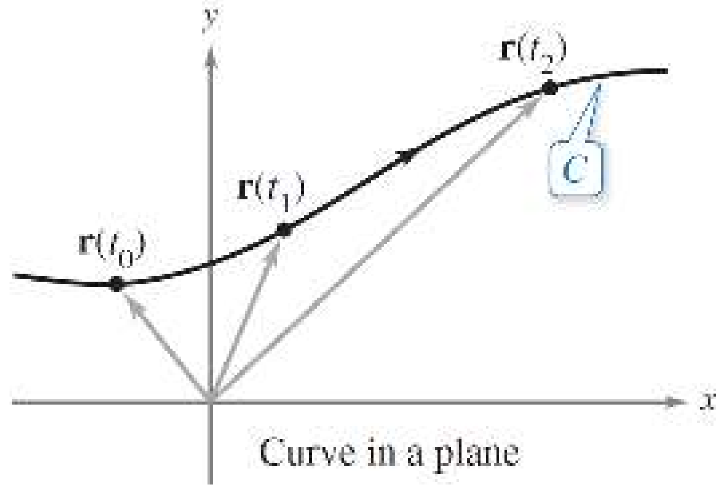
Circular HELIX

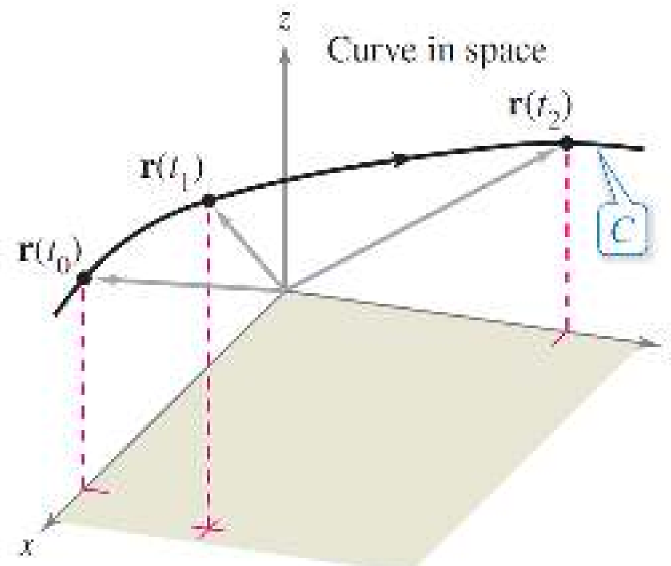
VECTOR-VALUED FUNCTIONS

A function of the form

$$\mathrm{r}(t) = f(t)\mathrm{i} + g(t)\mathrm{j} + h(t)\mathrm{k}$$
$$= \langle f(t), g(t), h(t) \rangle$$

is a vector-valued function, where the component functions f, g and h are real-valued functions of the parameter t.

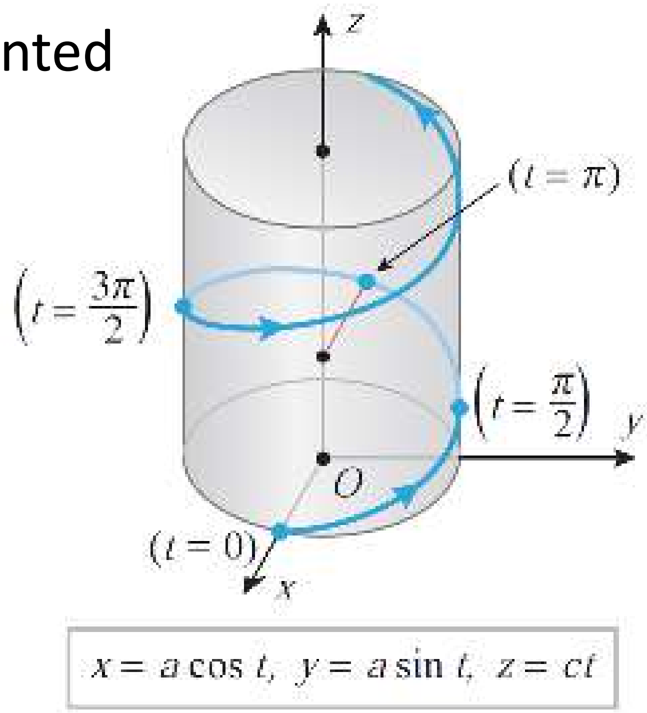




VECTOR-VALUED FUNCTIONS

Example Describe the parametric curve represented by the equations

$$x = 10\cos t$$
$$y = 10\sin t$$
$$z = t$$



Circular HELIX

$$\mathrm{r}(t) = 10\cos t\,\mathrm{i} + 10\sin t\,\mathrm{j} + t\mathrm{k}$$
$$= \langle 10\cos t\,, 10\sin t\,\ , t\rangle$$

VECTOR-VALUED FUNCTIONS

The domain of a vector-valued function $\mathrm{r}(t)$ is the set of allowable values for t.

NOTE Usual reasons to restrict a domain:

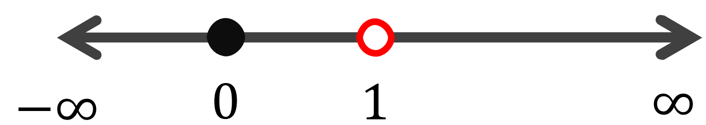
1. Avoid division by 0.
2. Avoid even roots of negative numbers.
3. Avoid logarithms of negative numbers or 0.

VECTOR-VALUED FUNCTIONS

Example Find the natural domain of $\mathrm{r}(t) = \ln|t-1|\,\mathrm{i} + e^{t}\,\mathrm{j} + \sqrt{t}\,\mathrm{k}$

$x(t) = \ln|t-1|$ ☑ Domain $= \mathbb{R} - \{1\}$

$y(t) = e^{t}$ ☑ Domain $= \mathbb{R}$



$z(t) = \sqrt{t}$ ☑ Domain $= [0, \infty)$

∴ The domain of $\mathrm{r}(t)$ is the *intersection of these sets*. $[0,1) \cup (1, \infty)$

Course: Calculus (3)

Lecture No: [16]

Chapter: [12]

VECTOR-VALUED FUNCTIONS

Section: [12.2]

CALCULUS OF VECTOR-VALUED FUNCTIONS

LIMITS AND CONTINUITY

- Many techniques and definitions used in the calculus of real-valued functions can be applied to vector-valued functions.
- For instance, you can add and subtract vector-valued functions, multiply a vector-valued function by a scalar, take the limit of a vector-valued function, differentiate a vector-valued function, and so on.

LIMITS AND CONTINUITY

$$\begin{aligned}\mathbf{r}_1(t) + \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j}.\end{aligned}$$

$$\begin{aligned}\mathbf{r}_1(t) - \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] - [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) - f_2(t)]\mathbf{i} + [g_1(t) - g_2(t)]\mathbf{j}.\end{aligned}$$

$$\begin{aligned}c\mathbf{r}(t) &= c[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] \\ &= cf_1(t)\mathbf{i} + cg_1(t)\mathbf{j}.\end{aligned}$$

$$\begin{aligned}\frac{\mathbf{r}(t)}{c} &= \frac{[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}]}{c}, \quad c \neq 0 \\ &= \frac{f_1(t)}{c}\mathbf{i} + \frac{g_1(t)}{c}\mathbf{j}.\end{aligned}$$

LIMITS AND CONTINUITY

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}$$

provided f, g, and h have limits as $t \to a$.

Example If $\mathrm{r}(t) = \frac{3}{t^2}\mathrm{i} + \frac{\ln t}{t^2-1}\mathrm{j} + \cos(\pi t)\,\mathrm{k}$, find $\lim_{t\to 1} \mathrm{r}(t)$.

$$\lim_{t\to 1} \mathrm{r}(t) = \langle\ ,\ ,\ \ \rangle \qquad \lim_{t\to 1} \frac{3}{t^2} = 3$$

LIMITS AND CONTINUITY

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}$$

provided f, g, and h have limits as $t \to a$.

Example If $\mathrm{r}(t) = \frac{3}{t^2}\mathrm{i} + \frac{\ln t}{t^2-1}\mathrm{j} + \cos(\pi t)\,\mathrm{k}$, find $\lim_{t\to 1} \mathrm{r}(t)$.

$$\lim_{t\to 1} \mathrm{r}(t) = \langle 3, \quad , \quad \rangle$$

$$\lim_{t\to 1} \frac{\ln t}{t^2-1} = \lim_{t\to 1} \frac{1/t}{2t} = \frac{1}{2}$$

LIMITS AND CONTINUITY

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}$$

provided f, g, and h have limits as $t \to a$.

Example If $\mathrm{r}(t) = \frac{3}{t^2}\mathrm{i} + \frac{\ln t}{t^2-1}\mathrm{j} + \cos(\pi t)\mathrm{k}$, find $\lim_{t\to 1} \mathrm{r}(t)$.

$\lim_{t\to 1} \mathrm{r}(t) = \langle 3, \frac{1}{2}, \quad \rangle$ $\qquad \lim_{t\to 1} \cos(\pi t) = -1$

LIMITS AND CONTINUITY

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j} + \left[\lim_{t \to a} h(t)\right]\mathbf{k}$$

provided f, g, and h have limits as $t \to a$.

Example If $\mathrm{r}(t) = \frac{3}{t^2}\mathrm{i} + \frac{\ln t}{t^2-1}\mathrm{j} + \cos(\pi t)\,\mathrm{k}$, find $\lim_{t \to 1} \mathrm{r}(t)$.

$$\lim_{t \to 1} \mathrm{r}(t) = \left\langle 3, \tfrac{1}{2}, -1 \right\rangle$$

$$= 3\mathrm{i} + \frac{1}{2}\mathrm{j} - \mathrm{k}$$

LIMITS AND CONTINUITY

Example If $\mathrm{r}(t) = \frac{2t^2-1}{t^2+t}\mathrm{i} + \sin\left(\frac{1}{t}\right)\mathrm{j} + te^{-t}\,\mathrm{k}$, find $\lim\limits_{t\to\infty} \mathrm{r}(t)$.

$$\lim_{t\to\infty} \mathrm{r}(t) = \langle\ \ ,\ \ ,\ \ \rangle$$

$$\lim_{t\to\infty} \frac{2t^2-1}{t^2+t} = 2$$

LIMITS AND CONTINUITY

Example If $\mathrm{r}(t) = \frac{2t^2-1}{t^2+t}\mathrm{i} + \sin\left(\frac{1}{t}\right)\mathrm{j} + te^{-t}\,\mathrm{k}$, find $\lim_{t\to\infty} \mathrm{r}(t)$.

$$\lim_{t\to\infty} \mathrm{r}(t) = \langle 2,\ \ ,\ \ \rangle$$

$$\lim_{t\to\infty} \sin\left(\frac{1}{t}\right) = \ 0$$

LIMITS AND CONTINUITY

Example If $\mathrm{r}(t) = \frac{2t^2-1}{t^2+t}\mathrm{i} + \sin\left(\frac{1}{t}\right)\mathrm{j} + te^{-t}\mathrm{k}$, find $\lim_{t\to\infty} \mathrm{r}(t)$.

$$\lim_{t\to\infty} \mathrm{r}(t) = \langle 2,0, \quad \rangle$$

$$\lim_{t\to\infty} te^{-t} = 0 \cdot \infty$$

$$= \lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0$$

LIMITS AND CONTINUITY

Example If $\mathrm{r}(t) = \frac{2t^2-1}{t^2+t}\mathrm{i} + \sin\left(\frac{1}{t}\right)\mathrm{j} + te^{-t}\,\mathrm{k}$, find $\lim_{t\to\infty} \mathrm{r}(t)$.

$$\lim_{t\to\infty} \mathrm{r}(t) = \langle 2,0,0 \rangle = 2\mathrm{i}$$

LIMITS AND CONTINUITY

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ when the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** I when it is continuous at every point in the interval.

Example The vector-valued function $\mathrm{r}(t) = t^2\mathrm{i} + \frac{1}{t^2-1}\mathrm{j} + t\mathrm{k}$, is discontinuous at $t = \pm 1$.

It is continuous for all $t \in \mathbb{R} - \{-1,1\}$

DERIVATIVES

- The derivative of a vector-valued function is *defined by a limit* like that for the derivative of a real-valued function.

$$\mathrm{r}'(t) = \lim_{h \to 0} \frac{\mathrm{r}(t+h) - \mathrm{r}(t)}{h}$$

- The derivative of $\mathrm{r}(t)$ can be *expressed as*

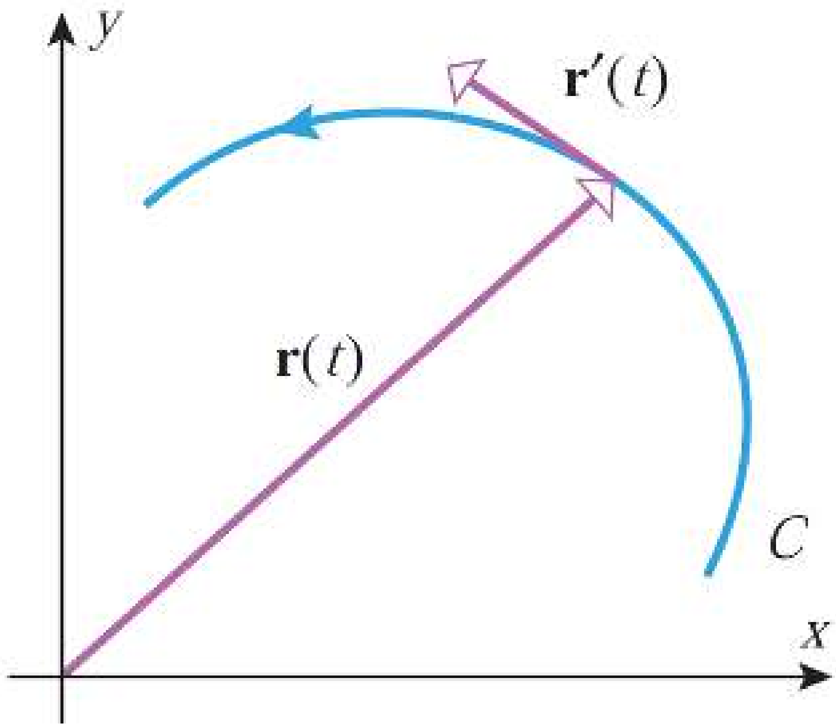
$$\frac{d}{dt}[\mathrm{r}(t)], \qquad \frac{d\mathrm{r}}{dt}, \qquad \mathrm{r}'(t), \qquad \mathrm{r}'$$

- Keep in mind that $\mathrm{r}(t)$ is a *vector*, not a number, and hence *has a magnitude and a direction* for each value of t, except if $\mathrm{r}(t) = 0$.

DERIVATIVES

Suppose that C is the graph of a vector-valued function $\mathrm{r}(t)$ and that $\mathrm{r}'(t)$ exists and is nonzero for a given value of t.

If the vector $\mathrm{r}'(t)$ is positioned with its initial point at the terminal point of the radius vector $\mathrm{r}(t)$, then $\mathrm{r}'(t)$ is tangent to C and points in the direction of increasing parameter.



DERIVATIVES

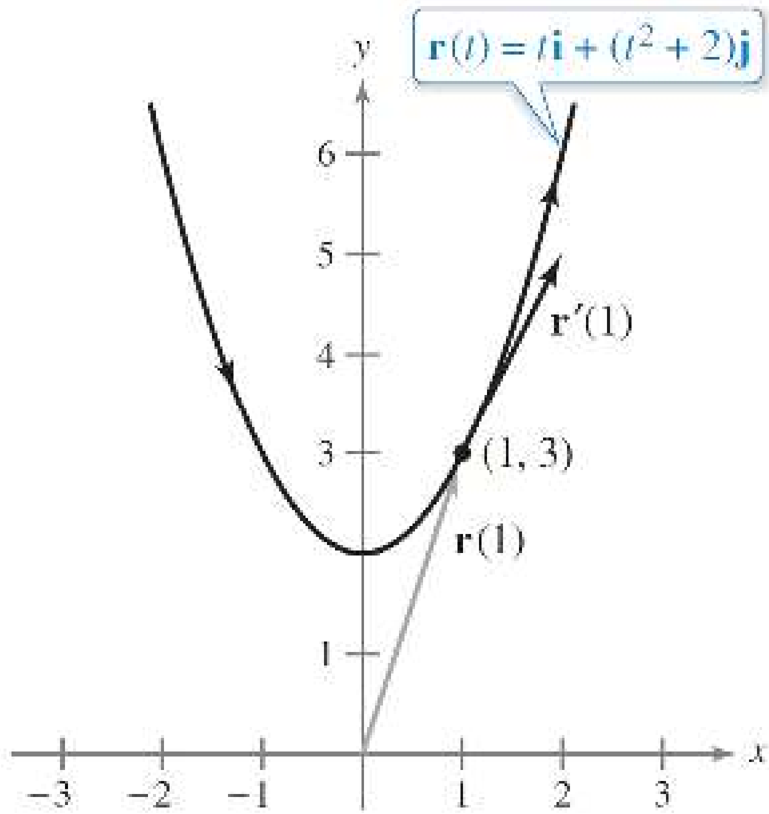
If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Example For the vector-valued function $\mathrm{r}(t) = t\mathrm{i} + (t^2 + 2)\mathrm{j}$, find $\mathrm{r}'(1)$.

$$\mathrm{r}'(t) = \mathrm{i} + 2t\mathrm{j}$$

$$\mathrm{r}'(1) = \mathrm{i} + 2\mathrm{j}$$



DERIVATIVES

Example For the vector-valued function $\mathrm{r}(t) = \cos t\,\mathrm{i} + \sin t\,\mathrm{j} + 2t\mathrm{k}$, find:

1. $\mathrm{r}'(t)$ $\qquad \mathrm{r}'(t) = -\sin t\,\mathrm{i} + \cos t\,\mathrm{j} + 2\mathrm{k}$

2. $\mathrm{r}''(t)$ $\qquad \mathrm{r}''(t) = -\cos t\,\mathrm{i} - \sin t\,\mathrm{j}$

3. $\mathrm{r}'(t) \cdot \mathrm{r}''(t)$ $\qquad \mathrm{r}'(t) \cdot \mathrm{r}''(t) = \sin t \cos t - \cos t \sin t = 0$

4. $\mathrm{r}'(t) \times \mathrm{r}''(t)$

$$\mathrm{r}'(t) \times \mathrm{r}''(t) = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = 2\sin t\,\mathrm{i} - 2\cos t\,\mathrm{j} + \mathrm{k}$$

DERIVATIVE RULES

1. $\dfrac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

2. $\dfrac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

3. $\dfrac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$

4. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

5. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

6. $\dfrac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$

7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

DERIVATIVE RULES

Example For $\mathrm{u}(t) = \frac{1}{t}\mathrm{i} - \mathrm{j} + \ln t\,\mathrm{k}$ and $\mathrm{v}(t) = t^2\mathrm{i} - 2t\mathrm{j} + \mathrm{k}$ then:

1

$$
\begin{aligned}
\frac{d}{dt}[\mathrm{u}(t) \cdot \mathrm{v}(t)] &= \mathrm{u}(t) \cdot \mathrm{v}'(t) + \mathrm{u}'(t) \cdot \mathrm{v}(t) \\
&= \left\langle \frac{1}{t}, -1, \ln t \right\rangle \cdot \langle 2t, -2, 0 \rangle + \left\langle \frac{-1}{t^2}, 0, \frac{1}{t} \right\rangle \cdot \langle t^2, -2t, 1 \rangle \\
&= (2 + 2 + 0) + \left(-1 + 0 + \frac{1}{t}\right) \\
&= 3 + \frac{1}{t}
\end{aligned}
$$

DERIVATIVE RULES

Example For $\mathrm{u}(t) = \frac{1}{t}\mathrm{i} - \mathrm{j} + \ln t\,\mathrm{k}$ and $\mathrm{v}(t) = t^2\mathrm{i} - 2t\mathrm{j} + \mathrm{k}$ then:

2 $$\frac{d}{dt}[\mathrm{v}(t) \times \mathrm{v}'(t)] = \mathrm{v}(t) \times \mathrm{v}''(t) + \mathrm{v}'(t) \times \mathrm{v}'(t)$$

$$= \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + 0$$

$$= 2\mathrm{j} + 4t\mathrm{k}$$

TANGENT LINES TO GRAPHS OF VECTOR-VALUED FUNCTIONS

Example Find parametric equations of the tangent line to the circular helix $\mathrm{r}(t) = \cos t\,\mathrm{i} + \sin t\,\mathrm{j} + t\mathrm{k}$ at the point where $t = \pi$.

$t = \pi$

POINT

$$(\cos\pi, \sin\pi, \pi) = (-1, 0, \pi)$$

TANGENT VECTOR

$$\mathrm{r}'(t) = -\sin t\,\mathrm{i} + \cos t\,\mathrm{j} + \mathrm{k}$$

$$\mathrm{r}'(\pi) = -\mathrm{j} + \mathrm{k}$$

∴ The parametric equations of the tangent line are

$$x = -1$$

$$y = -t$$

$$z = \pi + t$$

DEFINITE AND INDEFINITE INTEGRALS OF VECTOR-VALUED FUNCTIONS

In general, we have

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\mathbf{i} + \left(\int_a^b y(t)\,dt\right)\mathbf{j} + \left(\int_a^b z(t)\,dt\right)\mathbf{k}$$

Example Let $\mathrm{r}(t) = t^2\mathrm{i} + e^t\mathrm{j} - 2\cos(\pi t)\,\mathrm{k}$. Then

$$\int_0^1 \mathbf{r}(t)\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 e^t\,dt\right)\mathbf{j} - \left(\int_0^1 2\cos\pi t\,dt\right)\mathbf{k}$$

$$= \frac{t^3}{3}\bigg]_0^1 \mathbf{i} + e^t\bigg]_0^1 \mathbf{j} - \frac{2}{\pi}\sin\pi t\bigg]_0^1 \mathbf{k} = \frac{1}{3}\mathbf{i} + (e-1)\mathbf{j}$$

DEFINITE AND INDEFINITE INTEGRALS OF VECTOR-VALUED FUNCTIONS

Example $\int (2t\mathbf{i} + 3t^2\mathbf{j})\, dt = \left(\int 2t\, dt\right)\mathbf{i} + \left(\int 3t^2\, dt\right)\mathbf{j}$

$$= (t^2 + C_1)\mathbf{i} + (t^3 + C_2)\mathbf{j}$$

$$= (t^2\mathbf{i} + t^3\mathbf{j}) + (C_1\mathbf{i} + C_2\mathbf{j}) = (t^2\mathbf{i} + t^3\mathbf{j}) + \mathbf{C}$$

DEFINITE AND INDEFINITE INTEGRALS OF VECTOR-VALUED FUNCTIONS

Example Find $\mathrm{r}(t)$ given that $\mathrm{r}'(t) = \langle 3, 2t\rangle$ and $\mathrm{r}(1) = \langle 2, 5\rangle$.

$$\mathrm{r}(t) = \int \mathrm{r}'(t)dt = \int \langle 3,2t\rangle dt = \langle 3t, t^2\rangle + \mathrm{C}$$

But

$$\mathrm{r}(1) = \langle 2, 5\rangle$$

$$\langle 3,1\rangle + \mathrm{C} = \langle 2, 5\rangle$$

$$\mathrm{C} = \langle -1, 4\rangle$$

So

$$\mathrm{r}(t) = \langle 3t, t^2\rangle + \langle -1,4\rangle$$

$$\mathrm{r}(t) = \langle 3t - 1, t^2 + 4\rangle$$

Course: Calculus (3)

Lecture No: [18]

Chapter: [12]

VECTOR-VALUED FUNCTIONS

Section: [12.3]

CHANGE OF PARAMETER; ARC LENGTH

SMOOTH PARAMETRIZATIONS

- We will say that a curve represented by $\mathrm{r}(t)$ is *smoothly parametrized* by $\mathrm{r}(t)$, or that $\mathrm{r}(t)$ is a smooth function of t if:
  - ✓ $\mathrm{r}'(t)$ is continuous, and
  - ✓ $\mathrm{r}'(t) \neq \mathbf{0}$ for any allowable value of t.
- Geometrically, this means that a smoothly parametrized curve can have no abrupt (مفاجئ) changes in direction as the parameter increases.

SMOOTH PARAMETRIZATIONS

Example Determine whether the following vector-valued functions are smooth.

1. $\mathrm{r}(t) = a\cos t\,\mathrm{i} + a\sin t\,\mathrm{j} + ct\mathrm{k}$ $\qquad a > 0, c > 0$

$$\mathrm{r}'(t) = -a\sin t\,\mathrm{i} + a\cos t\,\mathrm{j} + c\mathrm{k}$$

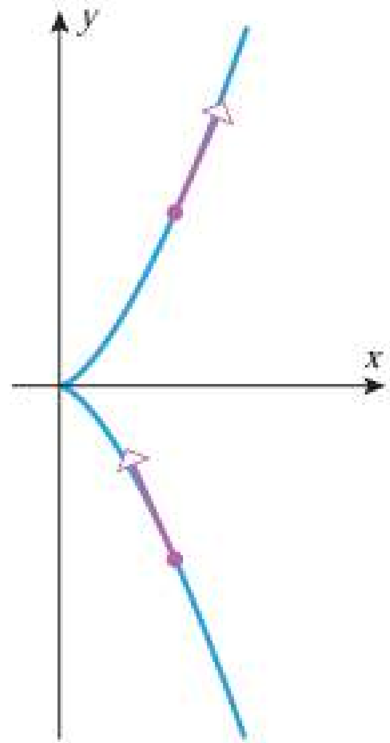
- ✓ The components are continuous functions, and
- ✓ there is no value of t for which all three of them are zero.
- ✓ So $\mathrm{r}(t)$ is a smooth function.

SMOOTH PARAMETRIZATIONS

Example Determine whether the following vector-valued functions are smooth.

2. $\mathrm{r}(t) = t^2\mathrm{i} + t^3\mathrm{j}$

$$\mathrm{r}'(t) = 2t\mathrm{i} + 3t^2\mathrm{j}$$



- ✓ The components are continuous functions, and
- ✓ they are both equal to zero if $t = 0$.
- ✓ So, $\mathrm{r}(t)$ is NOT a smooth function.

ARC LENGTH FROM THE VECTOR VIEWPOINT

If C is the graph of a smooth vector-valued function $\mathrm{r}(t)$, then its arc length ℓ from $t = a$ to $t = b$ is

$$\ell = \int_a^b \left\| \frac{d\mathrm{r}}{dt} \right\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example Find the arc length of that portion of the circular helix $\mathbf{r}(t) = \langle \cos t \, , \sin t \, , t \rangle$ from $t = 0$ to $t = \pi$.

ARC LENGTH FROM THE VECTOR VIEWPOINT

$$\ell = \int_a^b \left\|\frac{d\mathrm{r}}{dt}\right\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example Find the arc length of that portion of the circular helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ from $t = 0$ to $t = \pi$.

$$\mathrm{r}'(t) = \langle -\sin t, \cos t, 1\rangle$$

$$\|\mathrm{r}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1}$$

$$= \sqrt{2}$$

$$\ell = \int_0^\pi \|\mathrm{r}'(t)\| dt$$

$$= \int_0^\pi \sqrt{2}\, dt$$

$$= \sqrt{2}\,\pi$$

Course: Calculus (3)

Lecture No: [18]

Chapter: [12]

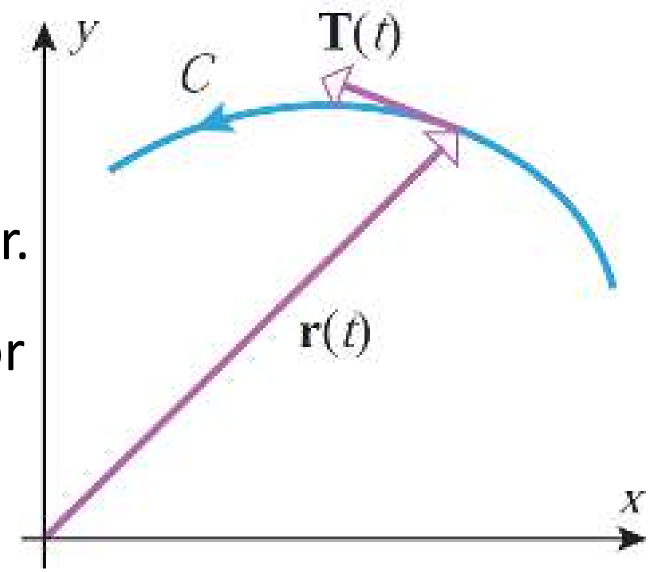
VECTOR-VALUED FUNCTIONS

Section: [12.4]

UNIT TANGENT, NORMAL, AND BINORMAL VECTORS

UNIT TANGENT VECTORS

- Recall that if C is the graph of a *smooth* vector-valued function $\mathbf{r}(t)$, then the vector $\mathbf{r}'(t)$ is:
  - ✓ nonzero, tangent to C, and
  - ✓ points in the direction of increasing parameter.



- Thus, by normalizing $\mathbf{r}'(t)$ we obtain a unit vector

$$\mathrm{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

that is tangent to C and points in the direction of increasing parameter.

- We call $\mathrm{T}(t)$ the unit tangent vector to C at t.

UNIT TANGENT VECTORS

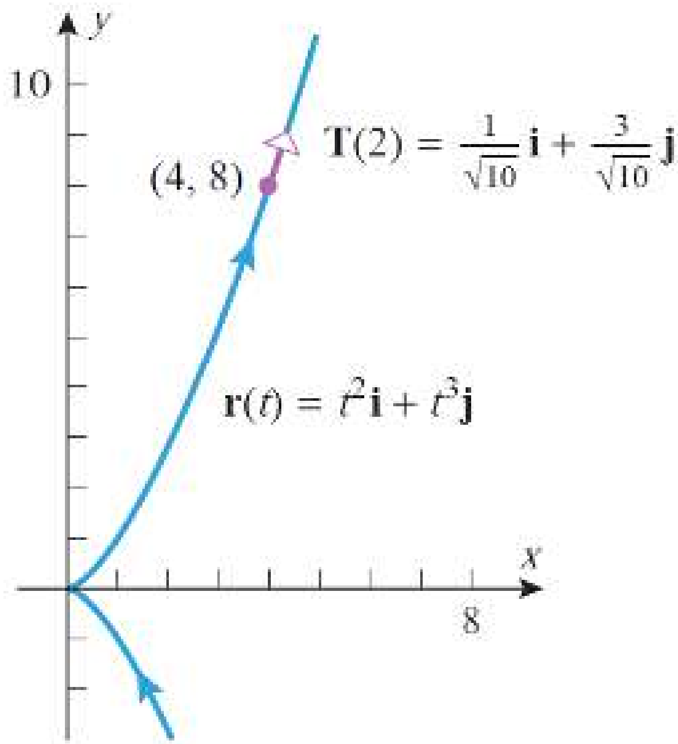
Example Find the unit tangent vector to the graph of $\mathrm{r}(t) = t^2\mathrm{i} + t^3\mathrm{j}$ at the point where $t = 2$.

$$\mathrm{r}'(t) = 2t\mathrm{i} + 3t^2\mathrm{j}$$

$$\mathrm{r}'(2) = 4\mathrm{i} + 12\mathrm{j}$$

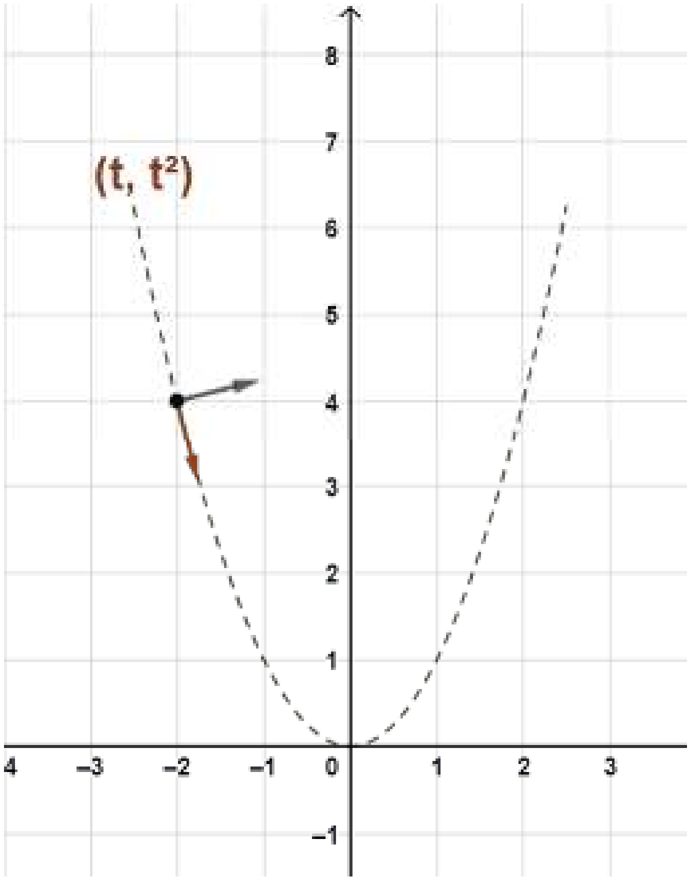
$$\mathrm{T}(2) = \frac{\mathrm{r}'(2)}{\|\mathrm{r}'(2)\|}$$

$$= \frac{4\mathrm{i} + 12\mathrm{j}}{\sqrt{160}} = \frac{1}{\sqrt{10}}\mathrm{i} + \frac{3}{\sqrt{10}}\mathrm{j}$$



UNIT NORMAL VECTORS

- Recall if $\|\mathrm{r}(t)\| = c$, then $\mathrm{r}(t)$ and $\mathrm{r}'(t)$ are orthogonal vectors.
- $\mathrm{T}(t)$ has constant norm 1, so $\mathrm{T}(t)$ and $\mathrm{T}'(t)$ are orthogonal vectors.
- This implies that $\mathrm{T}'(t)$ is perpendicular to the tangent line to C at t, so we say that $\mathrm{T}'(t)$ is *normal* to C at t.

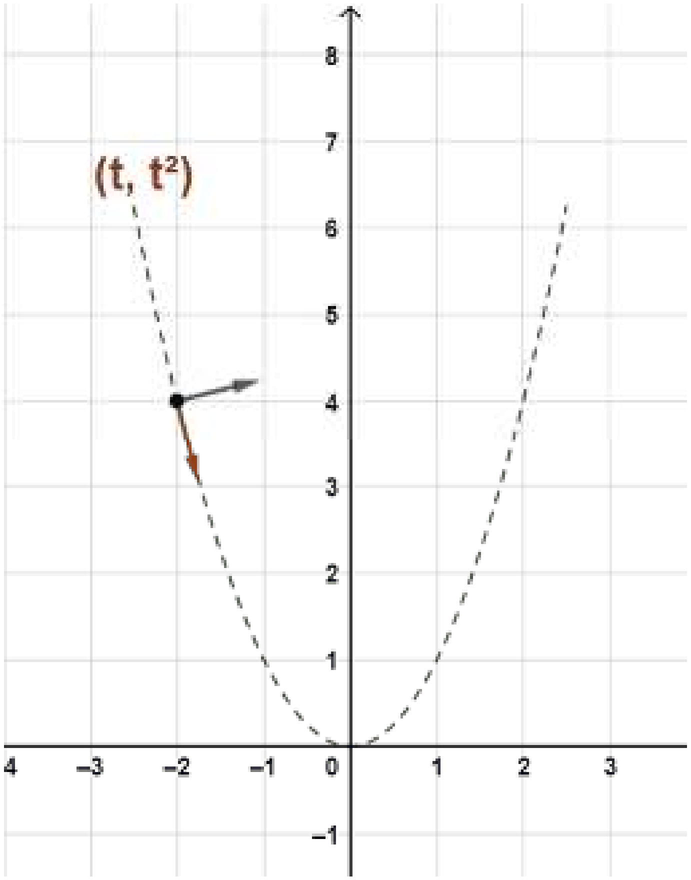


UNIT NORMAL VECTORS

- It follows that if $\mathrm{T}'(t) \neq 0$, and if we normalize $\mathrm{T}'(t)$, then we obtain a unit vector

$$\mathrm{N}(t) = \frac{\mathrm{T}'(t)}{\|\mathrm{T}'(t)\|}$$

that is normal to C and points in the same direction as $\mathrm{T}'(t)$.



UNIT NORMAL VECTORS

- We call $\mathrm{N}(t)$ the *principal unit normal vector to C at t* , or more simply, the *unit normal vector*.
- Observe that the unit normal vector is defined only at points where $\mathrm{T}'(t) \neq 0$. Unless stated otherwise, we will assume that this condition is satisfied.
- In particular, this excludes straight lines.

UNIT NORMAL VECTORS

Example Find $\mathrm{T}(t)$ and $\mathrm{N}(t)$ for the circular helix $\mathrm{r}(t) = \langle 3\cos t\,, 3\sin t\,, 4t\rangle$.

$$\mathrm{r}'(t) = \langle -3\sin t\,, 3\cos t\,, 4\rangle$$

$$\|\mathrm{r}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$$

$$\mathrm{T}(t) = \frac{\langle -3\sin t\,, 3\cos t\,, 4\rangle}{5} = \left\langle \frac{-3}{5}\sin t\,, \frac{3}{5}\cos t\,, \frac{4}{5}\right\rangle$$

UNIT NORMAL VECTORS

Example Find $\mathrm{T}(t)$ and $\mathrm{N}(t)$ for the circular helix $\mathrm{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

$$\mathrm{T}(t) = \frac{\langle -3\sin t, 3\cos t, 4\rangle}{5} = \left\langle \frac{-3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle$$

$$\mathrm{T}'(t) = \left\langle \frac{-3}{5}\cos t, \frac{-3}{5}\sin t, 0\right\rangle$$

$$\|\mathrm{T}'(t)\| = \sqrt{\frac{9}{25}\cos^2 t + \frac{9}{25}\sin^2 t + 0} = \frac{3}{5}$$

UNIT NORMAL VECTORS

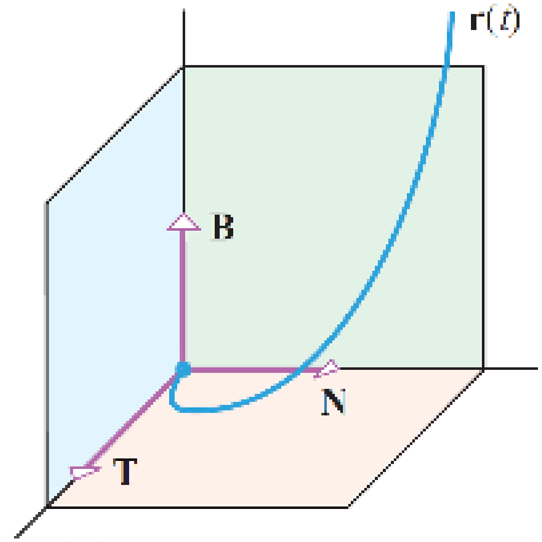
Example Find $\mathrm{T}(t)$ and $\mathrm{N}(t)$ for the circular helix $\mathrm{r}(t) = \langle 3\cos t\,, 3\sin t\,, 4t\rangle$.

$$\mathrm{T}'(t) = \left\langle \frac{-3}{5}\cos t\,, \frac{-3}{5}\sin t\,, 0 \right\rangle$$

$$\|\mathrm{T}'(t)\| = \sqrt{\frac{9}{25}\cos^2 t + \frac{9}{25}\sin^2 t + 0} = \frac{3}{5}$$

$$\mathrm{N}(t) = \frac{\left\langle \frac{-3}{5}\cos t\,, \frac{-3}{5}\sin t\,, 0 \right\rangle}{\frac{3}{5}} = \langle -\cos t\,, -\sin t\,, 0\rangle$$

BINORMAL VECTORS IN 3 –SPACE



If C is the graph of a vector-valued function $\mathrm{r}(t)$ in $3-$space, then we define the *binormal vector* to C at t to be

$$\mathrm{B}(t) = \mathrm{T}(t) \times \mathrm{N}(t)$$

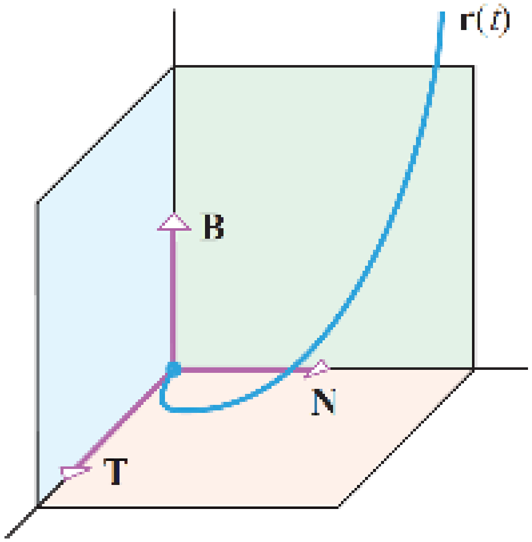
- It follows from properties of the cross product that $\mathrm{B}(t)$ is orthogonal to both $\mathrm{T}(t)$ and $\mathrm{N}(t)$ and is oriented relative to $\mathrm{T}(t)$ and $\mathrm{N}(t)$ by the right-hand rule.
- $\mathrm{B}(t)$ is unit vector !!.

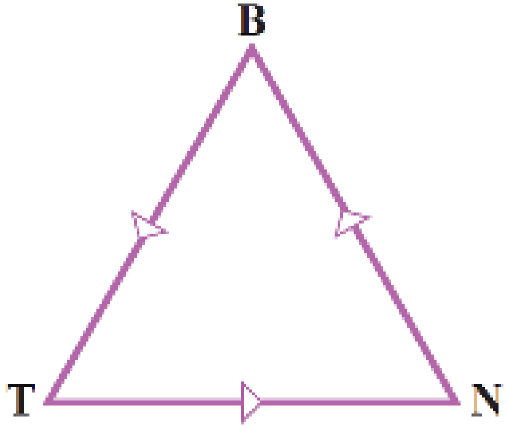
$$\|\mathrm{B}(t)\| = \|\mathrm{T}(t) \times \mathrm{N}(t)\| = \|\mathrm{T}(t)\|\|\mathrm{N}(t)\| \sin\frac{\pi}{2} = 1$$

BINORMAL VECTORS IN $3-$SPACE

Note that $\mathrm{T}(t)$, $\mathrm{N}(t)$, $\mathrm{B}(t)$ are three *mutually orthogonal unit vectors*.

$$\mathrm{B}(t) = \mathrm{T}(t) \times \mathrm{N}(t)$$
$$\mathrm{N}(t) = \mathrm{B}(t) \times \mathrm{T}(t)$$
$$\mathrm{T}(t) = \mathrm{N}(t) \times \mathrm{B}(t)$$





The binormal $\mathrm{B}(t)$ can be expressed directly in terms of $\mathrm{r}(t)$ as:

$$\mathrm{B}(t) = \frac{\mathrm{r}'(t) \times \mathrm{r}''(t)}{\|\mathrm{r}'(t) \times \mathrm{r}''(t)\|}$$

BINORMAL VECTORS IN 3 –SPACE

Example Find $\mathrm{B}(t)$ for the circular helix $\mathrm{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

$$\mathrm{T}(t) = \left\langle \frac{-3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5} \right\rangle \qquad \mathrm{N}(t) = \langle -\cos t, -\sin t, 0\rangle$$

$$\mathrm{B}(t) = \mathrm{T}(t) \times \mathrm{N}(t) = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \left\langle \frac{4}{5}\sin t, \frac{-4}{5}\cos t, \frac{3}{5} \right\rangle$$

Course: Calculus (3)

Lecture No: [19]

Chapter: [12]

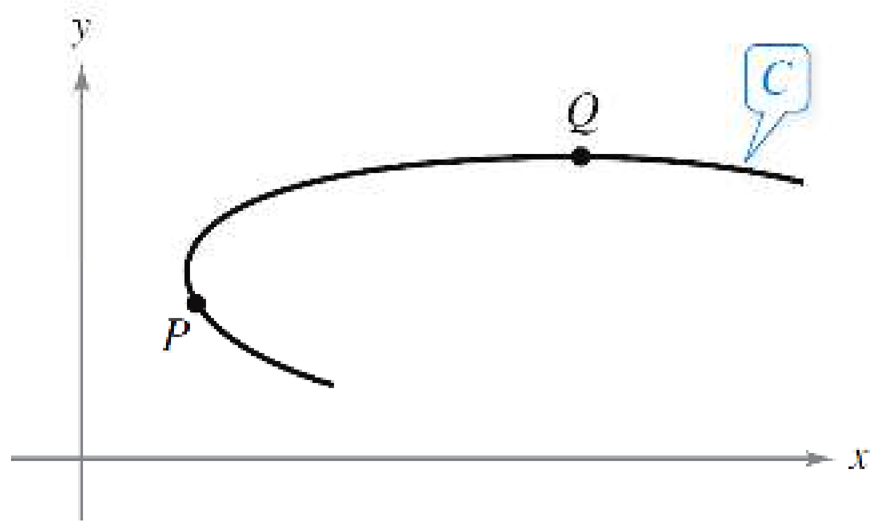
VECTOR-VALUED FUNCTIONS

Section: [12.5]

CURVATURE

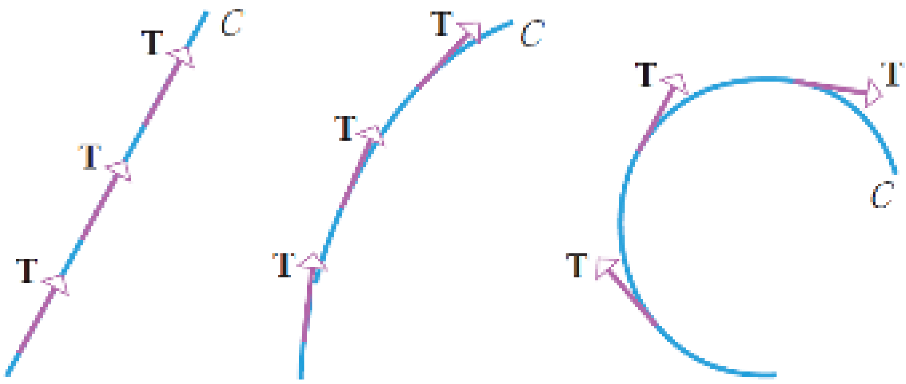
DEFINITION OF CURVATURE

- We will consider the problem of obtaining a *numerical measure of how sharply a curve bends*.
- For instance, in the figure, the curve bends more sharply at P than at Q and you can say that the curvature is greater at P than at Q.



DEFINITION OF CURVATURE

You can calculate curvature by calculating the magnitude of the rate of change of the unit tangent vector T with respect to the arc length s.



- If C is a *straight line (no bend)*, then the direction of T remains constant.
- If C *bends slightly*, then T undergoes a gradual change of direction.
- If C *bends sharply*, then T undergoes a rapid change of direction.

DEFINITION OF CURVATURE

If $\mathrm{r}(t)$ is a smooth vector-valued function, then for each value of t at which $\mathrm{T}'(t)$ and $\mathrm{r}''(t)$ exist, the curvature κ can be expressed as

$$\kappa(t) = \frac{\|\mathrm{T}'(t)\|}{\|\mathrm{r}'(t)\|} = \frac{\|\mathrm{r}'(t) \times \mathrm{r}''(t)\|}{\|\mathrm{r}'(t)\|^3}$$

Example Show that the curvature of a circle of radius R is $\kappa = \frac{1}{R}$.

1 $$\kappa(t) = \frac{\|\mathrm{T}'(t)\|}{\|\mathrm{r}'(t)\|}$$

DEFINITION OF CURVATURE

Example Show that the curvature of a circle of radius R is $\kappa = \frac{1}{R}$.

1

$$\kappa(t) = \frac{\|\mathrm{T}'(t)\|}{\|\mathrm{r}'(t)\|} \qquad \mathrm{r}(t) = R\cos t\,\mathrm{i} + R\sin t\,\mathrm{j} \qquad t \in [0,2\pi]$$

$$\mathrm{r}'(t) = -R\sin t\,\mathrm{i} + R\cos t\,\mathrm{j}$$

$$\mathrm{T}(t) = \frac{\mathrm{r}'(t)}{\|\mathrm{r}'(t)\|} = \frac{\langle -R\sin t, R\cos t\rangle}{\sqrt{(-R\sin t)^2 + (R\cos t)^2}} = \langle -\sin t, \cos t\rangle$$

$$\mathrm{T}'(t) = \langle -\cos t, -\sin t\rangle$$

$$\kappa(t) = \frac{\sqrt{(-\cos t)^2 + (-\sin t)^2}}{\sqrt{(-R\sin t)^2 + (R\cos t)^2}} = \frac{1}{R}$$

DEFINITION OF CURVATURE

Example Show that the curvature of a circle of radius R is $\kappa = \frac{1}{R}$.

2 $$\kappa(t) = \frac{\|\mathrm{r}'(t) \times \mathrm{r}''(t)\|}{\|\mathrm{r}'(t)\|^3}$$

$$\mathrm{r}(t) = R\cos t\,\mathrm{i} + R\sin t\,\mathrm{j} + 0\mathrm{k} \quad t \in [0,2\pi]$$

$$\mathrm{r}'(t) = -R\sin t\,\mathrm{i} + R\cos t\,\mathrm{j} + 0\mathrm{k}$$

$$\mathrm{r}''(t) = -R\cos t\,\mathrm{i} - R\sin t\,\mathrm{j} + 0\mathrm{k}$$

$$\mathrm{r}'(t) \times \mathrm{r}''(t) = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ -R\sin t & R\cos t & 0 \\ -R\cos t & -R\sin t & 0 \end{vmatrix} = R^2\mathrm{k}$$

$$\|\mathrm{r}'(t) \times \mathrm{r}''(t)\| = R^2$$

$$\|\mathrm{r}'(t)\| = R$$

$$\kappa(t) = \frac{R^2}{R^3} = \frac{1}{R}$$