Course: Calculus (3)

Lecture No: [20]

Chapter: [13]

PARTIAL DERIVATIVES

Section: [13.1]

FUNCTIONS OF TWO OR MORE VARIABLES

NOTATION AND TERMINOLOGY

- In this chapter we will extend many of the basic concepts of calculus to functions of two or more variables, commonly called *functions of several variables*.
- Many familiar quantities are functions of two or more variables. For example:

1. The work done by a force, $W = FD$, is a function of two variables.
2. The volume of a right circular cylinder, $V = \pi r^2 h$, is a function of two variables.
3. The volume of a rectangular solid, $V = lwh$, is a function of three variables.

NOTATION AND TERMINOLOGY

The notation for a function of two or more variables is similar to that for a function of a single variable.

$$z = f(\underbrace{x, y}_{\textbf{2 Variables}})$$

Function of two variables

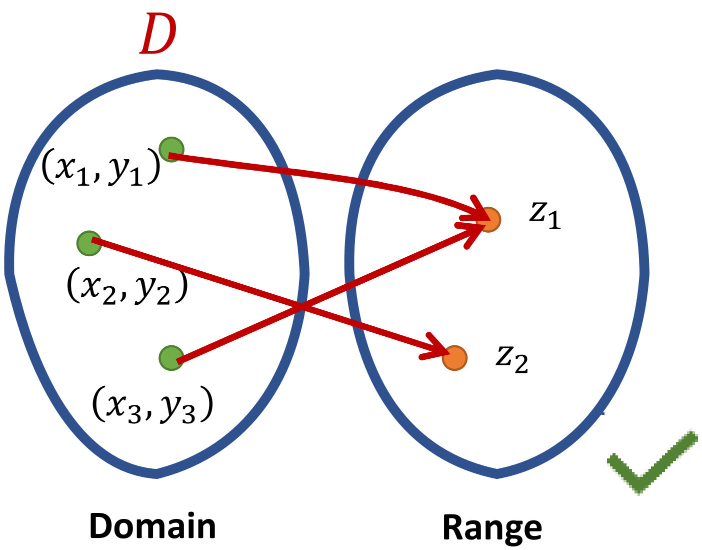
$$w = f(\underbrace{x, y, z}_{\textbf{3 Variables}})$$

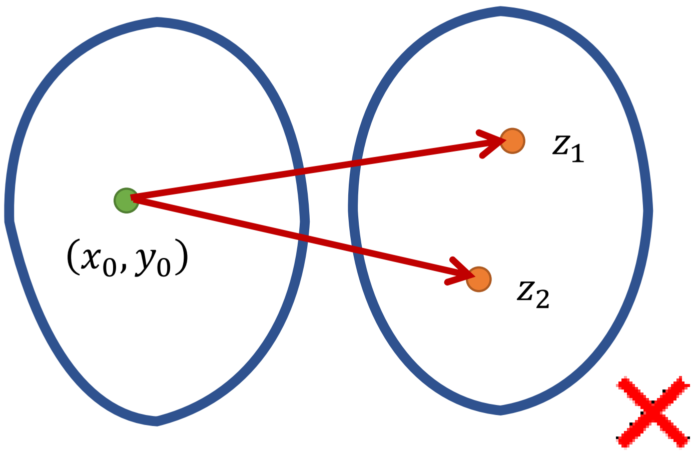
Function of three variables

NOTATION AND TERMINOLOGY

Definition of a Function of Two Variables

- ✓ Let D be a set of ordered pairs of real numbers.
- ✓ If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$ then f is a function of x and y.
- ✓ The set D is the domain of f and the corresponding set of values for $f(x, y)$ is the range of f.
- ✓ x and y are called the *independent variables* and z is called the *dependent variable*.





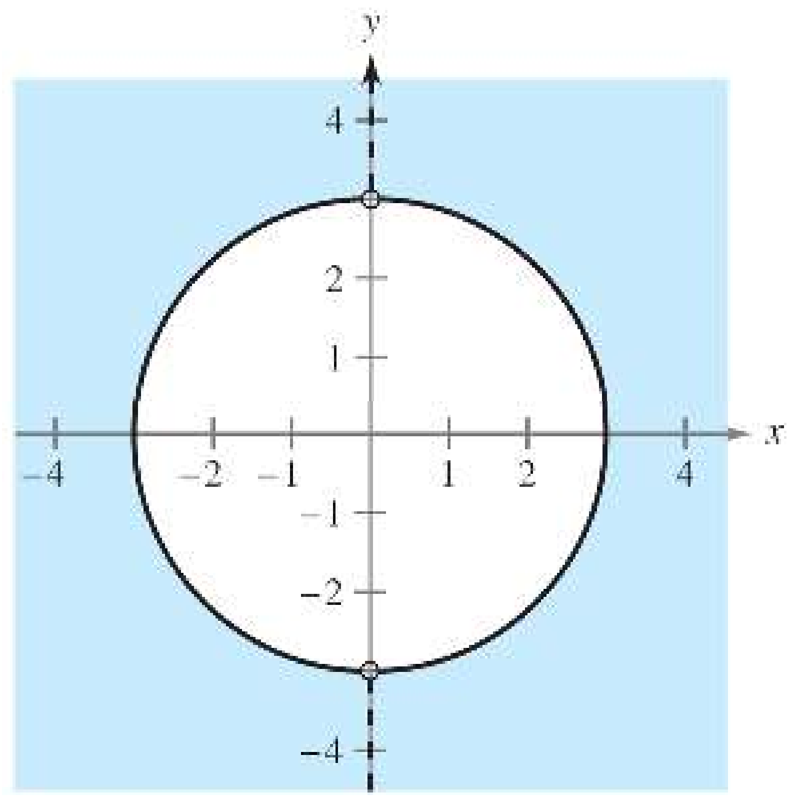
NOTATION AND TERMINOLOGY

Example Find the domain of the function $f(x,y) = \dfrac{\sqrt{x^2+y^2-9}}{x}$

The function f is defined for all points (x,y) such that $x \neq 0$ and

$$x^2 + y^2 - 9 \geq 0 \quad \Rightarrow \quad x^2 + y^2 \geq 9$$

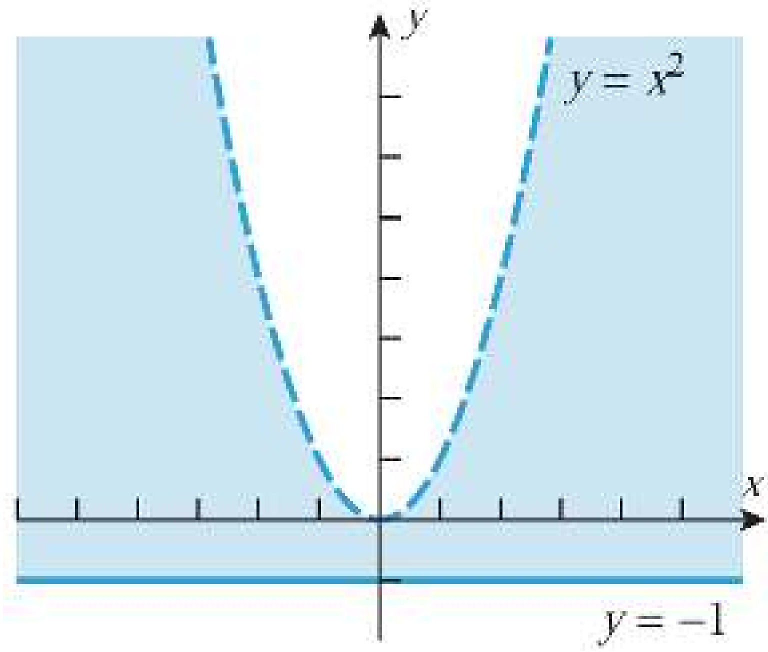
So, the domain is the set of all points lying on or outside the circle $x^2 + y^2 = 9$ *except* those points on the y –axis.



NOTATION AND TERMINOLOGY

Example Find the domain of the function $f(x,y) = \sqrt{y+1} + \ln(x^2 - y)$

- Note that $\sqrt{y+1}$ is defined only when $y \geq -1$.
- Also, $\ln(x^2 - y)$ is defined only when $x^2 - y > 0$ and hence $y < x^2$.
- Thus, the natural domain of f consists of all points in the xy-plane for which $-1 \leq y < x^2$.



NOTATION AND TERMINOLOGY

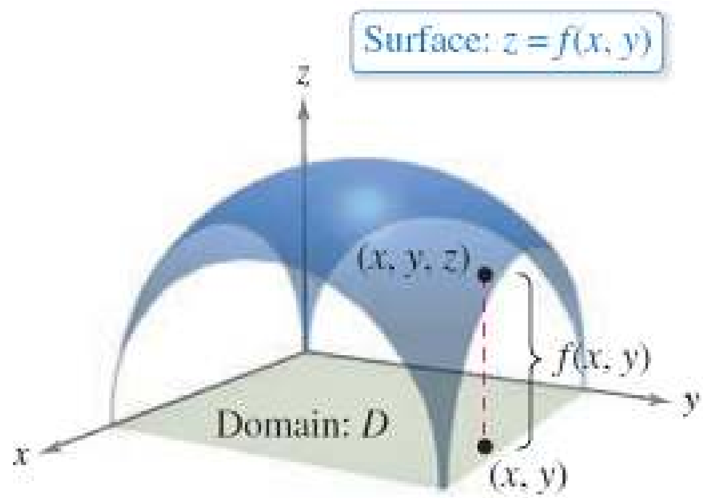
Example Find the domain of the function $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

$$1 - x^2 - y^2 - z^2 \geq 0 \quad \Rightarrow \quad x^2 + y^2 + z^2 \leq 1$$

The natural domain of f consists of all points on or within the sphere whose center is $(0,0,0)$ and radius 1.

LEVEL CURVES

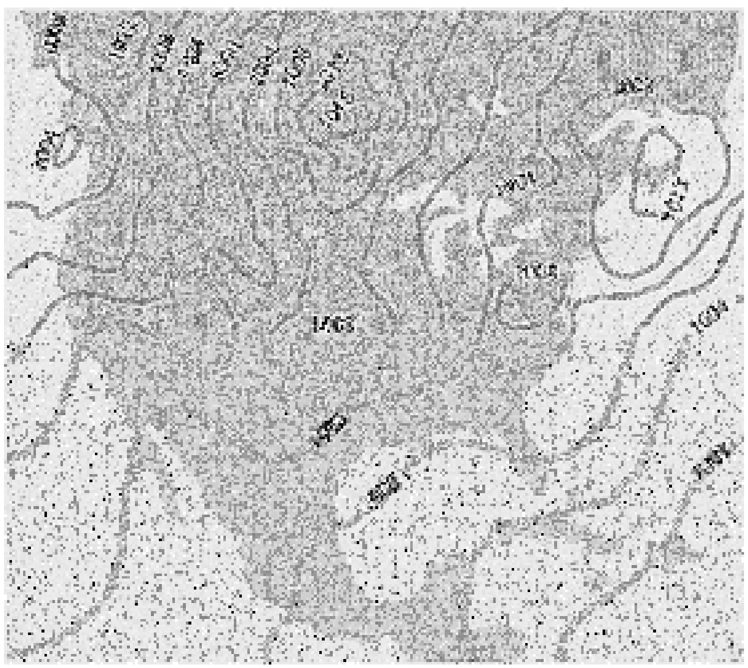
The set of all points $\left(x, y, f(x, y)\right)$ in space, for (x, y) in the domain of f, is called the **graph** of f.

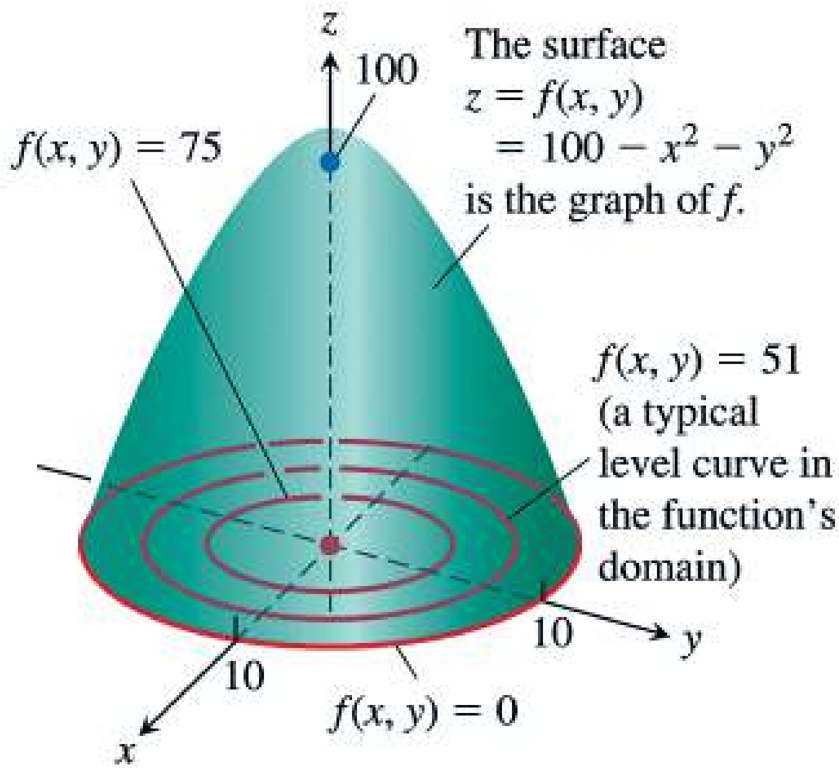


The graph of f is also called the **surface** $z = f(x, y)$.

LEVEL CURVES

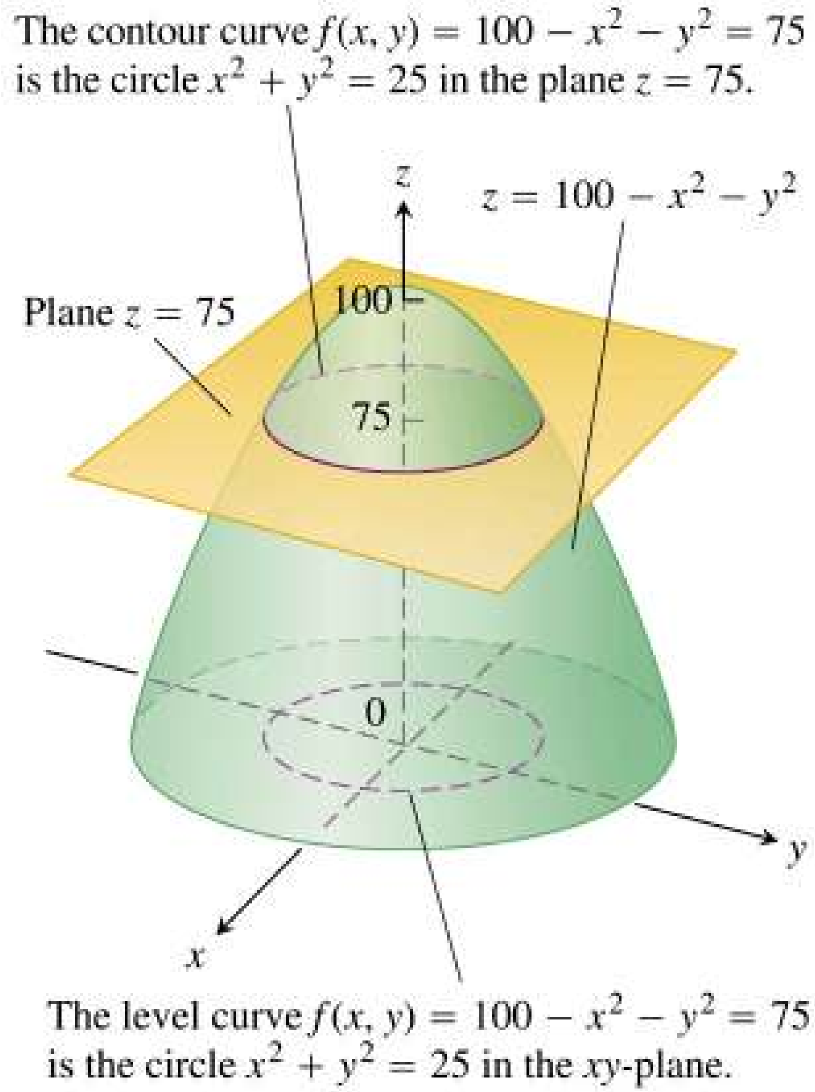
The set of points in the plane where a function $f(x,y)$ has a constant value $f(x,y) = c$ is called a **level curve** of f.





LEVEL CURVES

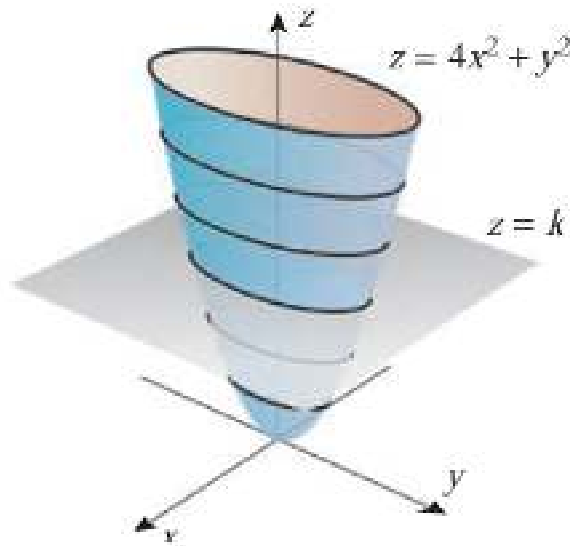
The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour curve** $f(x, y) = c$.



LEVEL CURVES

Example Sketch the contour plot of $f(x,y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

$f(x,y) = k \qquad 4x^2 + y^2 = k$



$k = 0 \qquad 4x^2 + y^2 = 0 \qquad (0,0)$

$k > 0 \qquad \dfrac{x^2}{k/4} + \dfrac{y^2}{k} = 1$

Which represents a family of ellipses with x −intercepts $\pm\frac{\sqrt{k}}{2}$ and y −intercepts $\pm\sqrt{k}$.

LEVEL CURVES

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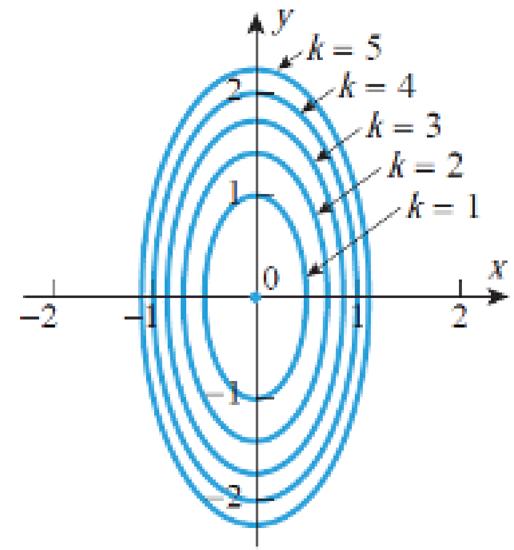

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Course: Calculus (3)

Lecture No: [21]

Chapter: [13]

PARTIAL DERIVATIVES

Section: [13.2]

LIMITS AND CONTINUITY

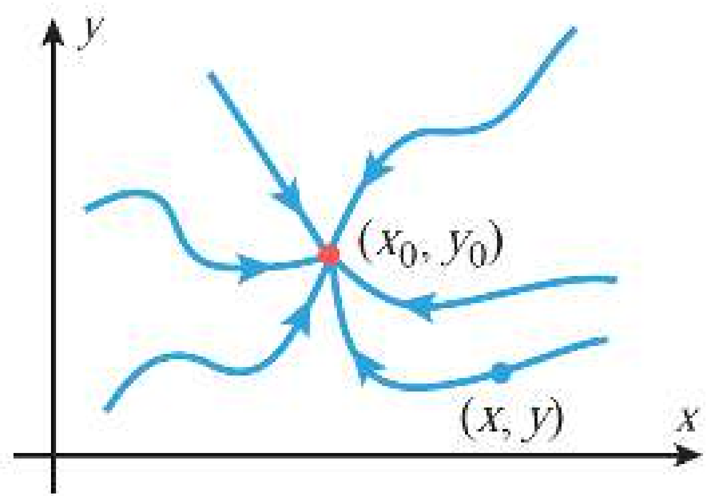
LIMITS ALONG CURVES

- For a function of one variable there are two one-sided limits at a point x_0, namely,

$$\lim_{x \to x_0^+} f(x) \text{ and } \lim_{x \to x_0^-} f(x)$$

reflecting the fact that *there are only two directions* from which x can approach x_0, *the right or the left*.

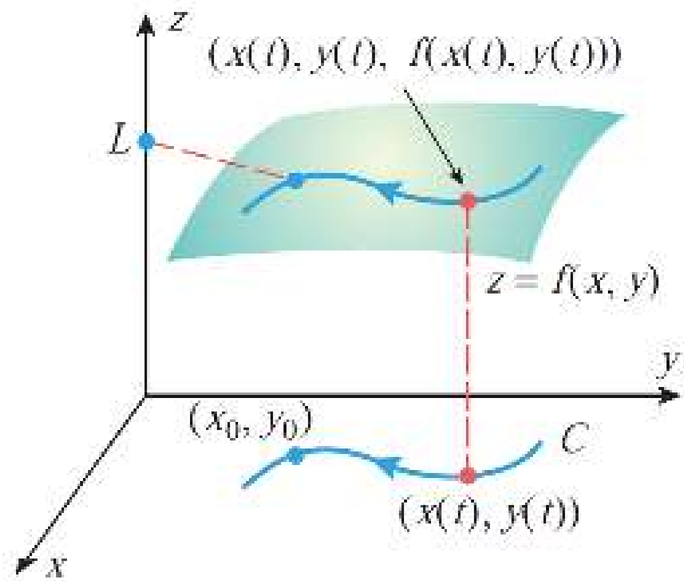
- For functions of several variables the situation is more complicated because *there are infinitely many different curves* along which one point can approach another.



LIMITS ALONG CURVES

If C is a smooth parametric curve in $2-$space that is represented by the equations $x = x(t)$ and $y = y(t)$, and if $x_0 = x(t_0)$ and $y_0 = y(t_0)$, then

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{(along } c\text{)}}} f(x,y) = \lim_{t\to t_0} f\big(x(t), y(t)\big)$$



RELATIONSHIPS BETWEEN GENERAL LIMITS AND LIMITS ALONG SMOOTH CURVES

- If $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$, then $f(x, y) \to L$ as $(x, y) \to (x_0, y_0)$ along *any* smooth curve.
- If the limit of $f(x, y)$ *fails to exist* as $(x, y) \to (x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ **does not exist** as $(x, y) \to (x_0, y_0)$.

LIMITS ALONG CURVES

Example Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ along: $= \dfrac{0}{0}$

1. the x –axis $(y = 0)$

$$\lim_{(x,0)\to(0,0)} \frac{x \times 0}{x^2 + 0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

2. the y –axis $(x = 0)$

$$\lim_{(0,y)\to(0,0)} \frac{0 \times y}{0^2 + y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

LIMITS ALONG CURVES

Example Evaluate $\lim_{(x,y)\to(0,0)} -\dfrac{xy}{x^2+y^2}$ along: $= \dfrac{0}{0}$

3. the line $y = x$

$$\lim_{(x,x)\to(0,0)} -\frac{x \times x}{x^2 + x^2} = \lim_{x\to 0} \frac{-x^2}{2x^2} = \frac{-1}{2}$$

4. The parabola $y = x^2$

$$\lim_{(x,x^2)\to(0,0)} -\frac{x \times x^2}{x^2 + x^4} = \lim_{x\to 0} \frac{-x^3}{x^2(1+x^2)} = 0$$

Since we found two different smooth curves along which this limit had different values then the limits does not exist

LIMITS ALONG CURVES

Example Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 3y^2}{x^2 + 2y^2} \quad = \frac{0}{0}$$

1. the x −axis

$$\lim_{(x,0)\to(0,0)} \frac{x^2 - 0}{x^2 + 0} = 1$$

2. the y −axis

$$\lim_{(0,y)\to(0,0)} \frac{0 - 3y^2}{0 + 2y^2} = -\frac{3}{2}$$

The limit does not exist

LIMITS ALONG CURVES

Example Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2} \qquad = \frac{0}{0}$$

1. the x –axis

$$\lim_{(x,0)\to(0,0)} \frac{0}{x^6 + 0} = 0$$

2. The curve $y = x^3$

$$\lim_{(x,x^3)\to(0,0)} \frac{(x^3)(x^3)}{x^6 + x^6} = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$$

The limit does not exist

LIMITS ALONG CURVES

Example Evaluate $\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+2^2} = -\frac{2}{5}$

Example Evaluate $\lim_{(x,y)\to(1,4)} (5x^3y^2+9) = 5(1^3)(4^2)+9 = 89$

Example Evaluate $\lim_{(x,y)\to(0,0)} \frac{1}{x^2+y^2} = \frac{1}{0+0} = +\infty$ does not exist

LIMITS ALONG CURVES

Example Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \frac{0}{0}$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2 - y^2)(x^2 + y^2)}{x^2 + y^2}$$

$$= \lim_{(x,y)\to(0,0)} (x^2 - y^2)$$

$$= 0$$

LIMITS ALONG CURVES

Example Evaluate $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) \quad = 0 \cdot \infty$

- It is not evident whether this limit exists because it is an indeterminate form of type $0 \cdot \infty$.
- Although L'Hospital's rule cannot be applied directly, we can find this limit by converting to polar coordinates.

$x = r\cos\theta \qquad y = r\sin\theta$

$r^2 = x^2 + y^2 \qquad \tan\theta = y/x$

Note

Since $r \geq 0$ then $r = \sqrt{x^2+y^2}$, so that $r \to 0^+$ if and only if $(x,y) \to (0,0)$

LIMITS ALONG CURVES

Example Evaluate $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$ $= 0 \cdot \infty$

$$= \lim_{r\to 0^+} r^2 \ln(r^2)$$

$$= \lim_{r\to 0^+} \frac{2\ln r}{1/r^2}$$

$$= \lim_{r\to 0^+} \frac{2/r}{-1/r^3}$$

$$= \lim_{r\to 0^+} (-r^2) = 0$$

$x = r\cos\theta$ $\quad y = r\sin\theta$

$r^2 = x^2 + y^2$ $\quad \tan\theta = y/x$

Note

Since $r \geq 0$ then $r = \sqrt{x^2+y^2}$, so that $r \to 0^+$ if and only if $(x,y) \to (0,0)$

LIMITS ALONG CURVES

Example Show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)}{x^2+y^2} = \frac{1}{0}$$

Along the x −axis $\displaystyle \lim_{(x,0)\to(0,0)} \frac{\cos 0}{x^2+0} = \lim_{x\to 0} \frac{1}{x^2} = \infty$

The limit does not exist

LIMITS ALONG CURVES

Example Determine whether the following limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{3x^2 + 2y^2} = \frac{0}{0}$$

Along the line $y = mx$ where $m \neq 0$

$$\lim_{(x,mx)\to(0,0)} \frac{(x)(mx)}{3x^2 + 2(mx)^2} = \lim_{x\to 0} \frac{mx^2}{(3 + 2m^2)x^2} = \frac{m}{3 + 2m^2}$$

Since the limit depends on the slope m of the line in which approach the origin, we conclude that the limit does not exist.

LIMITS ALONG CURVES

Example Evaluate the following limit by converting to polar coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}} = \frac{0}{0}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

Remember that $r \to 0^+$ if and only if $(x, y) \to (0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{(r\cos\theta)^2(r\sin\theta)^2}{r}$$

$$= \lim_{r\to 0^+} r^3\cos^2\theta\sin^2\theta = 0$$

LIMITS ALONG CURVES

Example Evaluate the following limit.

$$\lim_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\frac{1}{x^2+y^2+z^2}\right] = \tan^{-1}\left(\frac{1}{0}\right)$$

$$= \tan^{-1}\infty$$

$$= \frac{\pi}{2}$$

CONTINUITY

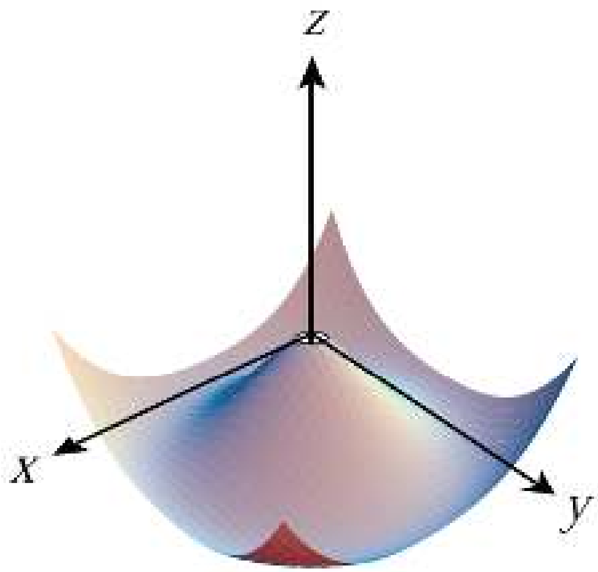
A function $f(x,y)$ is said to be continuous at (x_0, y_0) if $f(x_0, y_0)$ is defined and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

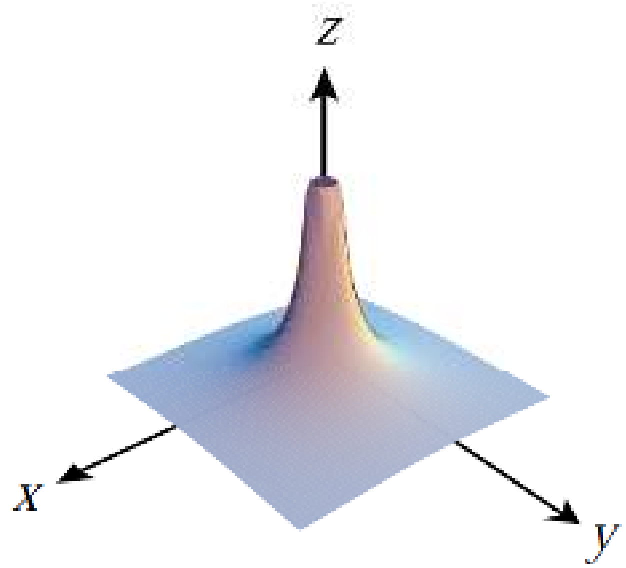
In addition, if f is continuous at every point in an open set D, then we say that f is continuous on D, and if f is continuous at every point in the xy −plane, then we say that f is continuous everywhere.

CONTINUITY

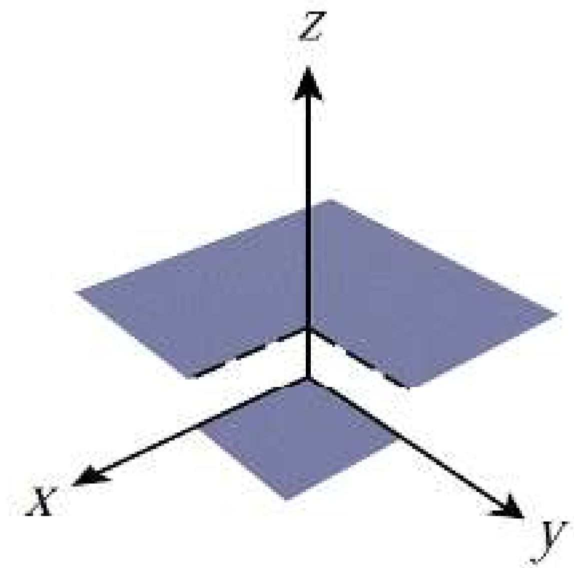
NOTE We will regard f as being continuous if the surface has no tears or holes.



Hole at the origin



Infinite at the origin



Vertical jump at the origin

CONTINUITY

Example $f(x,y) = \dfrac{x^3y^2}{1-xy}$ is continuous except where $1 - xy = 0$

$$y = \frac{1}{x}$$

Example Let $f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & : \quad (x,y) \neq (0,0) \\ 1 & : \quad (x,y) = (0,0) \end{cases}$

Show that f is continuous at $(0,0)$.

CONTINUITY

Example Let $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & : \quad (x,y) \neq (0,0) \\ 1 & : \quad (x,y) = (0,0) \end{cases}$

Show that f is continuous at $(0,0)$. ✓

1. $f(0,0) = 1$ is defined

2. $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)}$$

$$= \lim_{r\to 0^+} \frac{\sin(r^2)}{(r^2)}$$

$$= 1 \quad = f(0,0)$$

Course: Calculus (3)

Lecture No: [23]

Chapter: [13]

PARTIAL DERIVATIVES

Section: [13.3]

PARTIAL DERIVATIVES

PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

- How will the value of a function be affected by a change in *one* of its independent variables?
- The procedure used to determine the rate of change of a function $f(x, y)$ with respect to one of its several independent variables is called **partial differentiation**, and the result is referred to as the **partial derivative** of f with respect to the chosen independent variable.

PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

Definition of Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$, then the **first partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$ Partial derivative with respect to x

and

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$ Partial derivative with respect to y

provided the limits exist.

PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

NOTE This previous definition indicates that if $z = f(x, y)$ then:

- ✓ To find f_x you consider y constant and differentiate with respect to x.
- ✓ Similarly, to find f_y you consider x constant and differentiate with respect to y.

THE PARTIAL DERIVATIVE FUNCTIONS

Example Find $f_x(x,y)$ and $f_y(x,y)$ for $f(x,y) = 2x^3y^2 + 2y + 4x$ and use those partial derivatives to compute $f_x(1,3)$ and $f_y(1,3)$.

Keeping y fixed (*constant*) and differentiating with respect to x yields

$$f_x(x,y) = \frac{d}{dx}[2x^3y^2 + 2y + 4x] = 6x^2y^2 + 4$$

and keeping x fixed (*constant*) and differentiating with respect to y yields

$$f_y(x,y) = \frac{d}{dy}[2x^3y^2 + 2y + 4x] = 4x^3y + 2$$

Thus, $f_x(1,3) = 6(1^2)(3^2) + 4 = 58$ $f_y(1,3) = 4(1^3)(3) + 2 = 14$

PARTIAL DERIVATIVE NOTATION

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x}$$

Partial derivative with respect to x

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y}.$$

Partial derivative with respect to y

The first partials evaluated at the point (a, b) are denoted by

$$\left.\frac{\partial z}{\partial x}\right|_{(a, b)} = f_x(a, b) \quad \text{and} \quad \left.\frac{\partial z}{\partial y}\right|_{(a, b)} = f_y(a, b).$$

PARTIAL DERIVATIVE NOTATION

Example Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[x^4 \sin(xy^3)]$$

$$= x^4 \frac{\partial}{\partial x}[\sin(xy^3)] + \sin(xy^3)\frac{\partial}{\partial x}[x^4]$$

$$= x^4 y^3 \cos(xy^3) + 4x^3 \sin(xy^3)$$

PARTIAL DERIVATIVE NOTATION

Example Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^4 \sin(xy^3)$.

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[x^4 \sin(xy^3)\right]$$

$$= x^4 \frac{\partial}{\partial y}\left[\sin(xy^3)\right] = x^4 \times 3xy^2 \cos(xy^3)$$

$$= 3x^5y^2 \cos(xy^3)$$

PARTIAL DERIVATIVE NOTATION

Example Find $f_x(1, \ln 2)$ and $f_y(1, \ln 2)$ if $f(x, y) = ye^{x^2 y}$.

$$f_x = \frac{\partial}{\partial x}\left[ye^{x^2 y}\right]$$

$$= y\frac{\partial}{\partial x}\left[e^{x^2 y}\right] = y \times 2xye^{x^2 y} = 2xy^2 e^{x^2 y}$$

$$\therefore\ f_x\ (1, \ln 2) = 2(1)(\ln 2)^2 e^{(1^2)\ln 2}$$

$$= 4(\ln 2)^2$$

PARTIAL DERIVATIVE NOTATION

Example Find $f_x(1, \ln 2)$ and $f_y(1, \ln 2)$ if $f(x, y) = ye^{x^2 y}$.

$$f_y = \frac{\partial}{\partial y}\left[ye^{x^2 y}\right] = y\frac{\partial}{\partial y}\left[e^{x^2 y}\right] + e^{x^2 y}\frac{\partial}{\partial y}[y]$$

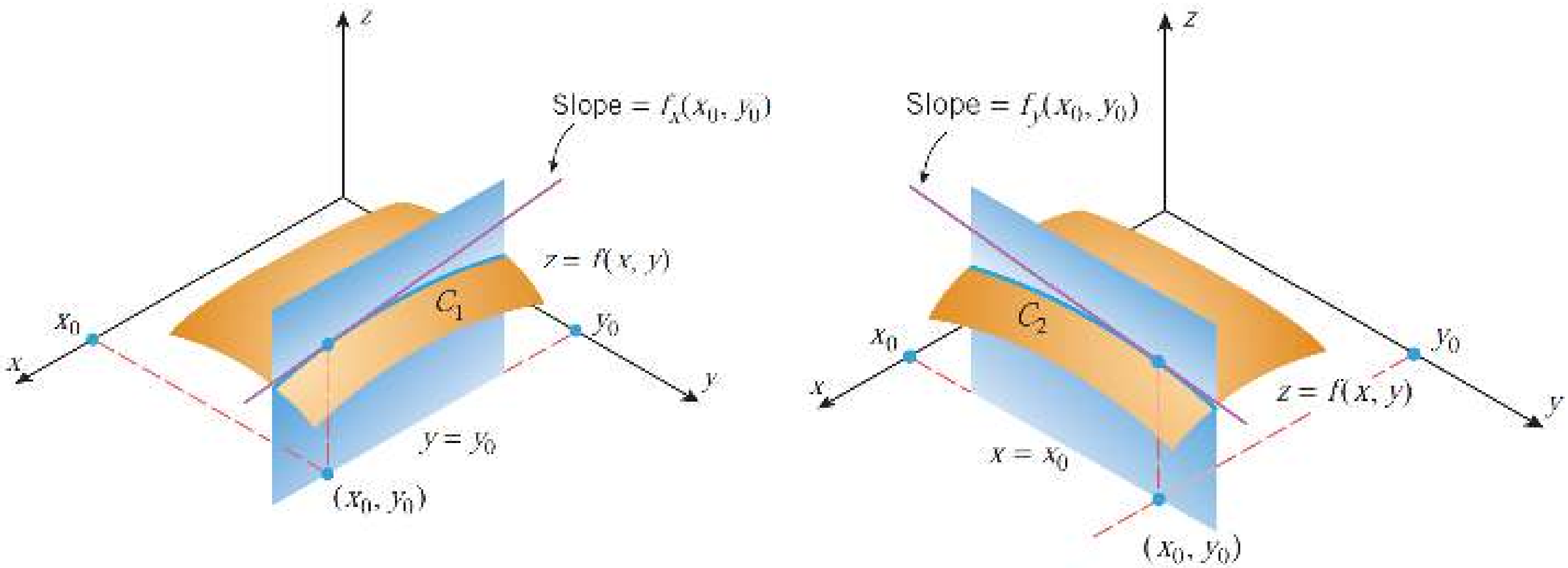
$$= yx^2 e^{x^2 y} + e^{x^2 y} \qquad = (yx^2 + 1)e^{x^2 y}$$

$$\therefore \ f_y\ (1, \ln 2) = ((1^2)\ln 2 + 1)e^{(1^2)\ln 2}$$

$$= 2\ln 2 + 2$$

PARTIAL DERIVATIVES VIEWED AS SLOPES

The values of f_x and f_y at the point (x_0, y_0, z_0) denote the slopes of the surface in the $x-$ and $y-$directions, respectively.



PARTIAL DERIVATIVES VIEWED AS SLOPES

Example Let $f(x, y) = x^2y + 5y^3$.

a) Find the slope of the surface $f(x, y)$ in the x −direction at the point $(1, -2)$.

$$\because f_x(x, y) = 2xy$$

Thus, the slope in the x −direction is $f_x(1, -2) = -4$

b) Find the slope of the surface $f(x, y)$ in the y −direction at the point $(1, -2)$.

$$\because f_y(x, y) = x^2 + 15y^2$$

Thus, the slope in the y −direction is $f_y(1, -2) = 61$

IMPLICIT PARTIAL DIFFERENTIATION

Example Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in the y −direction at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1]$$

$$2y + 2z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\left.\frac{\partial z}{\partial y}\right|_{\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)} = -\frac{1/3}{2/3}$$

$$= -\frac{1}{2}$$

PARTIAL DERIVATIVES OF FUNCTIONS WITH MORE THAN TWO VARIABLES

- For a function $w = f(x, y, z)$ of three variables, there are *three partial derivatives*:

$$\frac{\partial w}{\partial x} = f_x \quad , \quad \frac{\partial w}{\partial y} = f_y \quad , \quad \frac{\partial w}{\partial z} = f_z$$

- The partial derivative f_x is calculated by holding y and z constant and differentiating with respect to x.
- For f_y the variables x and z are held constant,
- and for f_z the variables x and y are held constant.

PARTIAL DERIVATIVES OF FUNCTIONS WITH MORE THAN TWO VARIABLES

Example If $f(x, y, z) = x^3y^2z^4 + 2xy + z$, then

$$f_x(x, y, z) = 3x^2y^2z^4 + 2y$$

$$f_y(x, y, z) = 2x^3yz^4 + 2x$$

$$f_z(x, y, z) = 4x^3y^2z^3 + 1$$

Example If $f(x, y, z, w) = \frac{x+y+z}{w}$, then $\dfrac{\partial f}{\partial w} = -\dfrac{x + y + z}{w^2}$

PARTIAL DERIVATIVES OF FUNCTIONS WITH MORE THAN TWO VARIABLES

Example If $w = \frac{x^2 - z^2}{y^2 + z^2}$, then

$$\frac{\partial w}{\partial z} = \frac{(y^2 + z^2)(-2z) - (x^2 - z^2)(2z)}{(y^2 + z^2)^2}$$

$$= \frac{-2z(x^2 + y^2)}{(y^2 + z^2)^2}$$

HIGHER-ORDER PARTIAL DERIVATIVES

- ✓ Suppose that f is a function of two variables x and y.
- ✓ Since the partial derivatives f_x and f_y are also functions of x and y, these functions may themselves have partial derivatives.
- ✓ This gives rise to four possible second-order partial derivatives of f , which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Differentiate twice with respect to x.

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to y and then with respect to x.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Differentiate first with respect to x and then with respect to y.

HIGHER-ORDER PARTIAL DERIVATIVES

- The last two cases are called *the mixed second-order partial derivatives* or the mixed second partials.
- Observe that the two notations for the mixed second partials have *opposite conventions for the order of differentiation*.
- Let f be a function of two variables. If f_{xy} and f_{yx} are continuous on some open disk, then $f_{xy} = f_{yx}$ on that disk.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to y and then with respect to x.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Differentiate first with respect to x and then with respect to y.

HIGHER-ORDER PARTIAL DERIVATIVES

Example

Find the second-order partial derivatives of

$f(x, y) = x^2y^3 + x^4y$

$f_x(x, y) = 2xy^3 + 4x^3y$

$f_y(x, y) = 3x^2y^2 + x^4$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2xy^3 + 4x^3y) = 2y^3 + 12x^2y$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 + x^4) = 6x^2y$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2xy^3 + 4x^3y) = 6xy^2 + 4x^3 = f_{yx}$$

HIGHER-ORDER PARTIAL DERIVATIVES

Third-order, fourth-order, and higher-order partial derivatives can be obtained by successive differentiation. Some possibilities are

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = f_{xxx} \qquad \frac{\partial^4 f}{\partial y^4} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y^3}\right) = f_{yyyy}$$

$$\frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = f_{xyy} \qquad \frac{\partial^4 f}{\partial y^2 \partial x^2} = \frac{\partial}{\partial y}\left(\frac{\partial^3 f}{\partial y \partial x^2}\right) = f_{xxyy}$$

Example Let $f(x, y) = y^2 e^x + y$. Find f_{xyy}.

$$f_{xyy} = \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial^2}{\partial y^2}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y^2}(y^2 e^x) = \frac{\partial}{\partial y}(2ye^x) = 2e^x$$

PARTIAL DERIVATIVES AND CONTINUITY

In contrast to the case of functions of a single variable, *the existence of partial derivatives for a multivariable function does not guarantee the continuity of the function*.

Example Let $f(x,y) = \begin{cases} -\dfrac{xy}{x^2+y^2} & : \quad (x,y) \neq (0,0) \\ 0 & : \quad (x,y) = (0,0) \end{cases}$

We previously show that $\lim_{(x,y)\to(0,0)} -\dfrac{xy}{x^2+y^2}$ does not exist.

$\therefore f(x,y)$ is discontinuous at $(0,0)$.

PARTIAL DERIVATIVES AND CONTINUITY

Example Let $f(x,y) = \begin{cases} -\dfrac{xy}{x^2+y^2} & : \quad (x,y) \neq (0,0) \\ 0 & : \quad (x,y) = (0,0) \end{cases}$

$\therefore f(x,y)$ is discontinuous at $(0,0)$.

We will have to use the definitions of the partial derivatives to determine whether f has partial derivatives at $(0,0)$, and if so, we find the values of those derivatives.

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

PARTIAL DERIVATIVES AND CONTINUITY

Example Let $f(x,y) = \begin{cases} -\dfrac{xy}{x^2+y^2} & : \quad (x,y) \neq (0,0) \\ 0 & : \quad (x,y) = (0,0) \end{cases}$

$\therefore f(x,y)$ is discontinuous at $(0,0)$.

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0-0}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0-0}{\Delta y} = 0$$

This shows that f has partial derivatives at $(0,0)$ and the values of both partial derivatives are 0 at that point.

Course: Calculus (3)

Lecture No: [25]

Chapter: [13]

PARTIAL DERIVATIVES

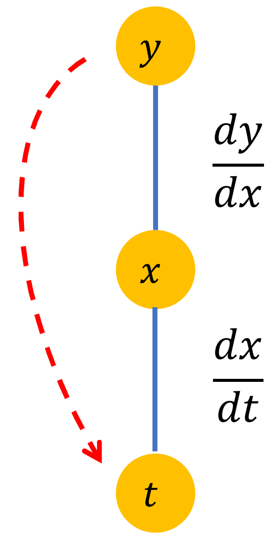
Section: [13.5]

THE CHAIN RULE

CHAIN RULES FOR DERIVATIVES

If y is a differentiable function of x and x is a differentiable function of t, then the *chain rule* for functions of *one variable* states that, under composition, y becomes a differentiable function of t with

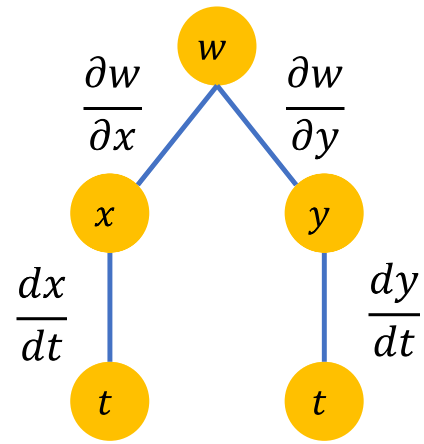
$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$



CHAIN RULES FOR DERIVATIVES

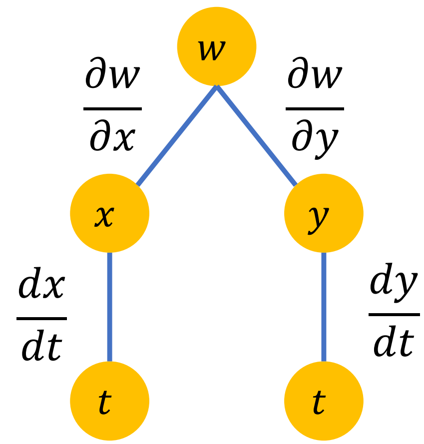
- Let $w = f(x, y)$ where f is a differentiable function of x and y.
- If $x = g(t)$ and $y = h(t)$ where g and h are differentiable functions of t then w is a differentiable function of t.
- And

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$



CHAIN RULES FOR DERIVATIVES

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$



Example Let $w = x^2y - y^2$, where $x = \sin t$ and $y = e^t$. Find $\frac{dw}{dt}$ when $t = 0$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= (2xy)(\cos t) + (x^2 - 2y)(e^t)$$

$$= (2\sin t\; e^t)(\cos t) + (\sin^2 t - 2e^t)(e^t)$$

$$\left.\frac{dw}{dt}\right|_{t=0} = -2$$

NOTE $w = e^t \sin^2 t - e^{2t}$

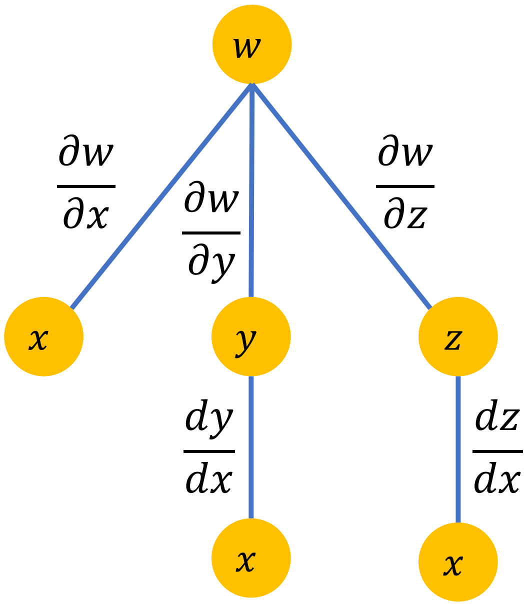
CHAIN RULES FOR DERIVATIVES

Example Let $w = xy + yz$, where $y = \sin x$ and $z = e^x$. Use an appropriate form of the chain rule to find dw/dx.

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx} + \frac{\partial w}{\partial z}\frac{dz}{dx}$$

$$= y + (x + z)(\cos x) + (y)(e^x)$$

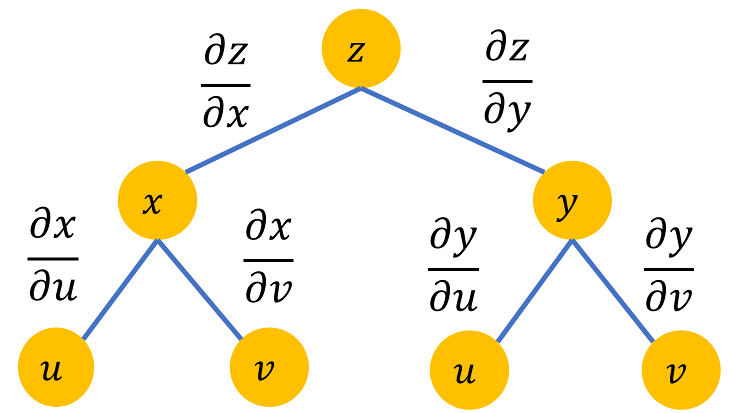
$$= (1 + e^x)\sin x + (x + e^x)\cos x$$



NOTE

$$w = x \sin x + e^x \sin x$$

CHAIN RULES FOR DERIVATIVES



Example Given that $z = e^{xy}$, $x = 2u + v$, and $y = u/v$. Find $\partial z/\partial u$ and $\partial z/\partial v$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= (ye^{xy})(2) + (xe^{xy})(1/v) = e^{xy}\left(2y + \frac{x}{v}\right) = e^{(2u+v)(u/v)}\left(1 + \frac{4u}{v}\right)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$= (ye^{xy})(1) + (xe^{xy})(-u/v^2) = e^{xy}\left(y - \frac{xu}{v^2}\right) = -\frac{2u^2}{v^2}e^{(2u+v)(u/v)}$$

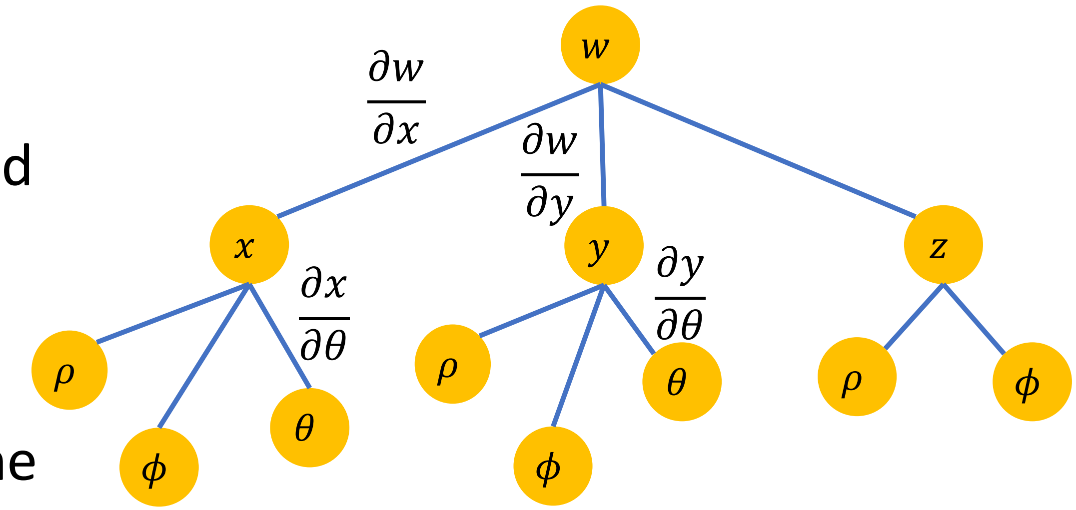
CHAIN RULES FOR DERIVATIVES

Example

Given that $w = x^2 + y^2 - z^2$, and

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Use appropriate forms of the chain rule to find $\partial w / \partial \theta$.

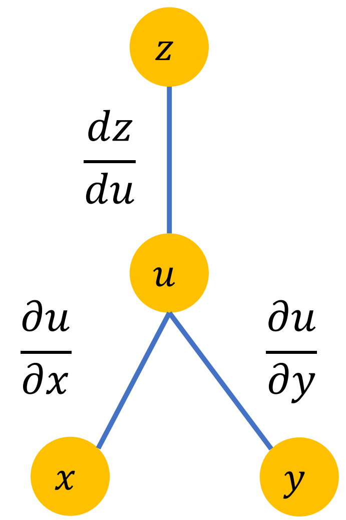


$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = (2x)(-\rho \sin\phi \sin\theta) + (2y)(\rho \sin\phi \cos\theta)$$

$$= 0$$

This result is explained by the fact that w does not vary with θ.

CHAIN RULES FOR DERIVATIVES



Example Let f be a differentiable function of one variable and let $z = f(x+2y)$. Show that

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

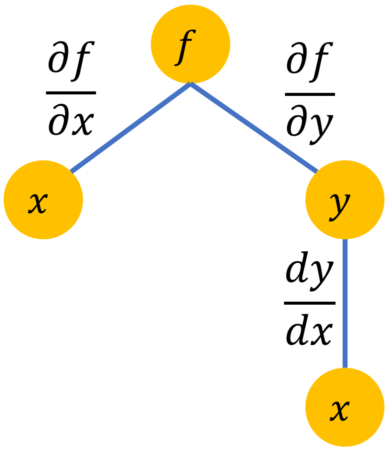
Let $u = x + 2y$

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}(1) = \frac{dz}{du}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = \frac{dz}{du}(2) = 2\frac{dz}{du}$$

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 2\frac{dz}{du} - 2\frac{dz}{du} = 0$$

IMPLICIT DIFFERENTIATION



Consider the special case where $f(x, y)$ is a function of x and y and y is a differentiable function of x.

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

Now, suppose that $f(x, y) = c$. Then

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

IMPLICIT DIFFERENTIATION

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Example Given that $x^3 + y^2x - 3 = 0$, find $\frac{dy}{dx}$

$$\underbrace{x^3 + y^2x}_{f(x,y)} = 3$$

$$\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$$

Course: Calculus (3)

Lecture No: [27]

Chapter: [13]

PARTIAL DERIVATIVES

Section: [13.6]

DIRECTIONAL DERIVATIVES AND GRADIENTS

DIRECTIONAL DERIVATIVES

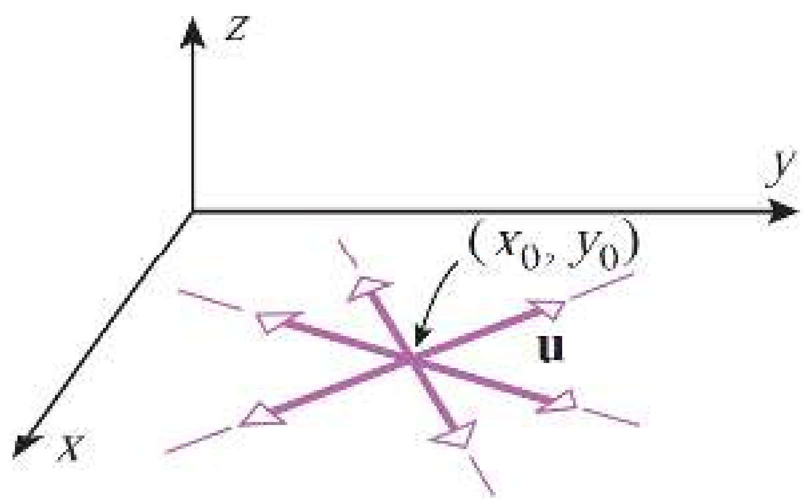
- In this section we extend the concept of a partial derivative to the more general notion of a directional derivative.
- You will see that $f_x(x, y)$ and $f_y(x, y)$ can be used to find the slope in any direction.
- To determine the slope at a point on a surface, you will define a new type of derivative called a *directional derivative*.

DIRECTIONAL DERIVATIVES

- To do this is to use a unit vector

$$\mathrm{u} = u_1\mathrm{i} + u_2\mathrm{j}$$

that has its initial point at (x_0, y_0) and points in the desired direction.



DIRECTIONAL DERIVATIVES

If $f(x, y)$ is a function of x and y, and if $\mathrm{u} = u_1\mathrm{i} + u_2\mathrm{j}$ is a unit vector, then the directional derivative of f in the direction of u at (x_0, y_0) is denoted by $D_{\mathrm{u}}f(x_0, y_0)$ and is defined by

$$D_{\mathrm{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$$

DIRECTIONAL DERIVATIVES

Example Find the directional derivative of $f(x,y) = e^{xy}$ at $(-2,0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive x −axis.

$f_x(x,y) = ye^{xy}$

$f_x(-2,0) = 0$

$f_y(x,y) = xe^{xy}$

$f_y(-2,0) = -2$

$\mathrm{u} = \cos\dfrac{\pi}{3}\mathrm{i} + \sin\dfrac{\pi}{3}\mathrm{j}$

$\mathrm{u} = \dfrac{1}{2}\mathrm{i} + \dfrac{\sqrt{3}}{2}\mathrm{j}$

$$D_{\mathrm{u}}f(-2,0) = f_x(-2,0)u_1 + f_y(-2,0)u_2$$

$$= (0)\left(\frac{1}{2}\right) + (-2)\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

DIRECTIONAL DERIVATIVES

Example Find the directional derivative of $f(x,y,z) = x^2y - yz^3 + z$ at $(1,-2,0)$ in the direction of the vector $\mathrm{a} = 2\mathrm{i} + \mathrm{j} - 2\mathrm{k}$.

$$f_x(x,y,z) = 2xy$$

$$f_y(x,y,z) = x^2 - z^3$$

$$f_z(x,y,z) = -3yz^2 + 1$$

$$\mathrm{u} = \frac{\mathrm{a}}{\|\mathrm{a}\|} = \frac{2\mathrm{i} + \mathrm{j} - 2\mathrm{k}}{\sqrt{2^2 + 1^2 + (-2)^2}}$$

$$= \frac{2}{3}\mathrm{i} + \frac{1}{3}\mathrm{j} - \frac{2}{3}\mathrm{k}$$

$$f_x(1,-2,0) = -4$$

$$f_y(1,-2,0) = 1$$

$$f_z(1,-2,0) = 1$$

DIRECTIONAL DERIVATIVES

Example Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at $(1, -2, 0)$ in the direction of the vector $\mathrm{a} = 2\mathrm{i} + \mathrm{j} - 2\mathrm{k}$.

$f_x(1, -2, 0) = -4 \qquad f_y(1, -2, 0) = 1 \qquad f_z(1, -2, 0) = 1$

$$\mathrm{u} = \frac{\mathrm{a}}{\|\mathrm{a}\|} = \frac{2}{3}\mathrm{i} + \frac{1}{3}\mathrm{j} - \frac{2}{3}\mathrm{k}$$

$$D_{\mathrm{u}}f(1, -2, 0) = f_x(1, -2, 0)u_1 + f_y(1, -2, 0)u_2 + f_z(1, -2, 0)u_3$$

$$= (-4)\left(\frac{2}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)\left(\frac{-2}{3}\right) = -3$$

THE GRADIENT

(a) If f is a function of x and y, then the ***gradient of f*** is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

(b) If f is a function of x, y, and z, then the ***gradient of f*** is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

NOTE

$$\begin{aligned} D_{\mathbf{u}} f(x_0, y_0) &= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \\ &= \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \langle u_1, u_2\rangle \\ &= \nabla f \cdot \mathbf{u} \end{aligned}$$

PROPERTIES OF THE GRADIENT

Let f be a function of either two variables or three variables and let P denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that f is differentiable at P.

a) If $\nabla f = 0$ at P, then all directional derivatives of f at P are zero.

b) If $\nabla f \neq 0$ at P, then among all possible directional derivatives of f at P, the derivative in the direction of ∇f at P has the largest value. The value of this largest directional derivative is $\|\nabla f\|$ at P.

c) If $\nabla f \neq 0$ at P, then among all possible directional derivatives of f at P, the derivative in the opposite direction of ∇f at P has the smallest value. The value of this smallest directional derivative is $-\|\nabla f\|$ at P.

PROPERTIES OF THE GRADIENT

Example Let $f(x,y) = x^2e^y$. Find the maximum value of a directional derivative at $(-2,0)$, and find the unit vector in the direction in which the maximum value occurs.

$$\nabla f(x,y) = f_x(x,y)\mathrm{i} + f_y(x,y)\mathrm{j} = 2xe^y\mathrm{i} + x^2e^y\mathrm{j}$$

$$\nabla f(-2,0) = -4\mathrm{i} + 4\mathrm{j}$$

So, the maximum value of the directional derivative is

$$\|\nabla f(-2,0)\| = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$$

PROPERTIES OF THE GRADIENT

Example Let $f(x, y) = x^2 e^y$. Find the maximum value of a directional derivative at $(-2,0)$, and find the unit vector in the direction in which the maximum value occurs.

So, the maximum value of the directional derivative is

$$\|\nabla f(-2,0)\| = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$$

This maximum occurs in the direction of $\nabla f(-2,0)$.

The unit vector in this direction is

$$\mathbf{u} = \frac{\nabla f(-2, 0)}{\|\nabla f(-2, 0)\|} = \frac{1}{4\sqrt{2}}(-4\mathbf{i} + 4\mathbf{j}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

Course: Calculus (3) Lecture No: [28]

Chapter: [13]

PARTIAL DERIVATIVES

Section: [13.7]

TANGENT PLANES AND NORMAL VECTORS

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $\boldsymbol{F(x, y, z) = c}$

In this section we will discuss "How do we find equations of tangent planes to surfaces in three-dimensional space?"

- So far, you have represented surfaces in space primarily by equations of the form $z = f(x, y)$.
- In the development to follow, however, it is convenient to use the more general representation $F(x, y, z) = 0$.

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $F(x, y, z) = c$

- For a surface S given by $z = f(x, y)$ you can convert to the general form by defining F as

$$F(x, y, z) = f(x, y) - z$$

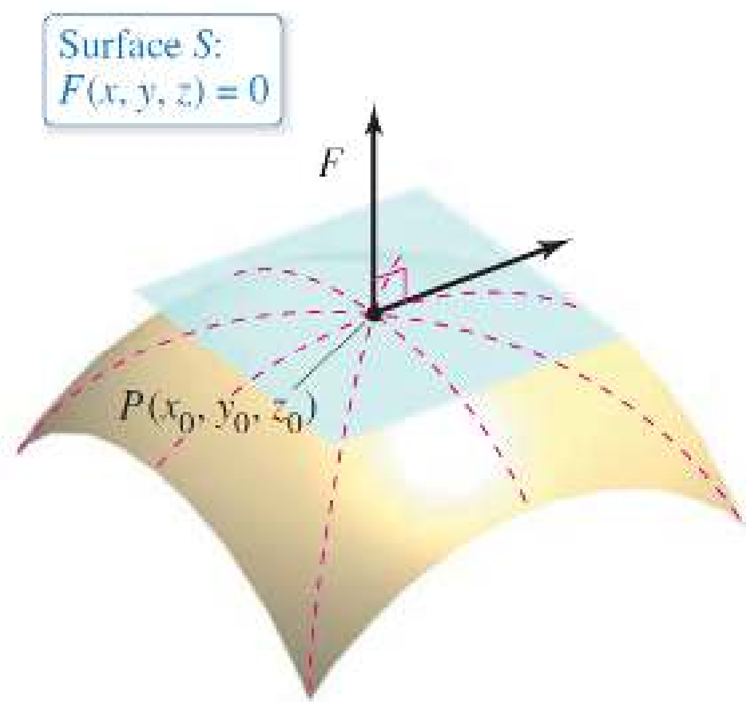
- Because $f(x, y) - z = 0$, you can consider S to be the level surface of F given by $F(x, y, z) = 0$.
- In the process of finding a *normal line* to a surface, you are also able to solve the problem of finding a **tangent plane** to the surface.

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $\boldsymbol{F(x, y, z) = c}$

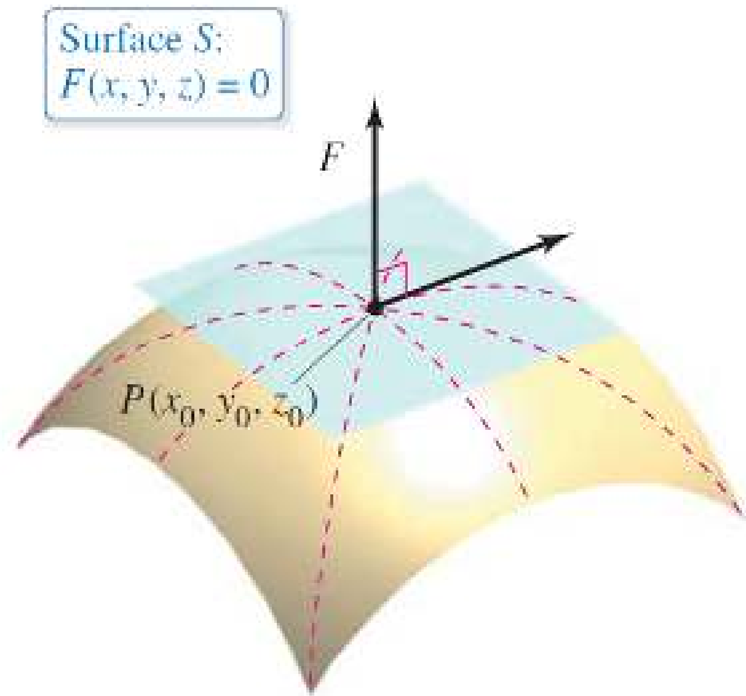
- Let S be a surface given by $F(x, y, z) = 0$ and let $P(x_0, y_0, z_0)$ be a point on S.
- Let C be a curve on S through P that is defined by the vector-valued function

$$\mathrm{r}(t) = x(t)\mathrm{i} + y(t)\mathrm{j} + z(t)\mathrm{k}$$

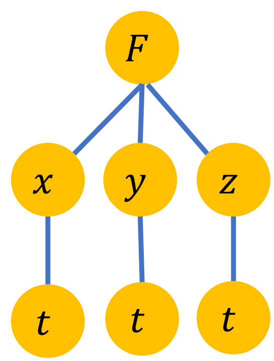
- Then, for all t, $F\big(x(t), y(t), z(t)\big) = 0$.



TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $\boldsymbol{F(x,y,z)=c}$



- If F is differentiable and $x'(t)$, $y'(t)$ and $z'(t)$ all exist, then it follows from the Chain Rule that



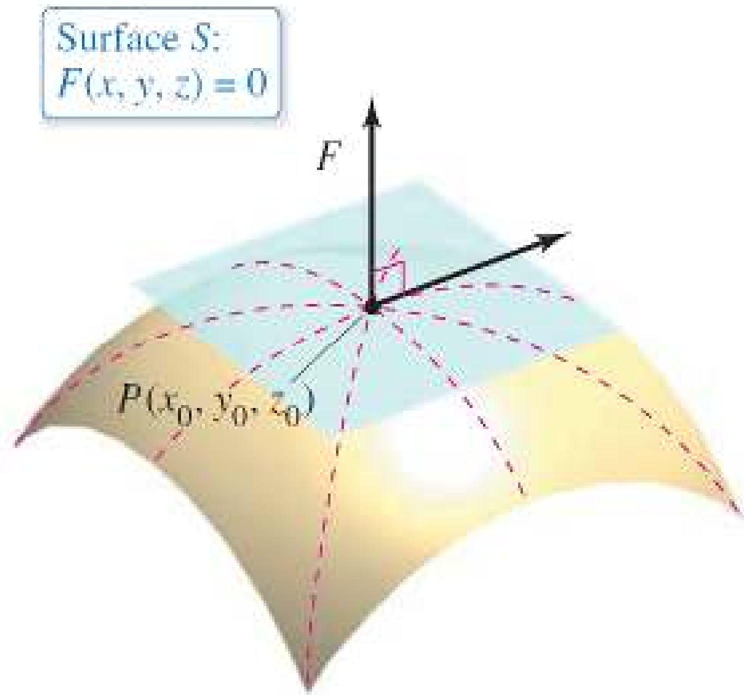
$$0 = F'(t_0)$$
$$= F_x(x_0, y_0, z_0)x'(t_0) + F_y(x_0, y_0, z_0)y'(t_0) + F_z(x_0, y_0, z_0)z'(t_0)$$

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $\boldsymbol{F(x, y, z) = c}$

$$0 = F'(t_0)$$

$$= F_x(x_0, y_0, z_0)x'(t_0) + F_y(x_0, y_0, z_0)y'(t_0) + F_z(x_0, y_0, z_0)z'(t_0)$$

$$= \underbrace{\nabla F(x_0, y_0, z_0)}_{\text{Gradient}} \cdot \underbrace{\mathrm{r}'(t_0)}_{\text{Tangent Vector}}$$



This result means that all tangent lines on S lie in a plane that is normal to $\nabla F(x_0, y_0, z_0)$ and contains P.

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $F(x, y, z) = c$

Definitions of Tangent Plane and Normal Line

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that

$$\nabla F(x_0, y_0, z_0) \neq \mathbf{0}.$$

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S at P.**
2. The line through P having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S at P.**

Equation of Tangent Plane

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given by $F(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES $\boldsymbol{F(x,y,z)=c}$

Example Find an equation of the tangent plane to the hyperboloid $z^2 - 2x^2 - 2y^2 = 12$ at the point $(1,-1,4)$.

$$z^2 - 2x^2 - 2y^2 - 12 = 0$$

$$F(x,y,z) = z^2 - 2x^2 - 2y^2 - 12$$

$$F_x(x,y,z) = -4x$$

$$F_y(x,y,z) = -4y$$

$$F_z(x,y,z) = 2z$$

$$F_x(1,-1,4) = -4$$

$$F_y(1,-1,4) = 4$$

$$F_z(1,-1,4) = 8$$

So, an equation of the tangent plane at $(1,-1,4)$ is

$$-4(x-1) + 4(y+1) + 8(z-4) = 0$$

$$-4x + 4y + 8z = 24$$

$$x - y - 2z + 6 = 0$$

TANGENT PLANES AND NORMAL VECTORS TO LEVEL SURFACES

$F(x,y,z) = c$

Example Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2 y$ at the point $(2,1,4)$.

$z - x^2 y = 0$ $\quad F(x,y,z) = z - x^2 y$

$\nabla F(x,y,z) = -2xy\mathrm{i} - x^2\mathrm{j} + \mathrm{k}$

$\nabla F(2,1,4) = -4\mathrm{i} - 4\mathrm{j} + \mathrm{k}$

So, the tangent plane has equation

$$-4(x-2) - 4(y-1) + (z-4) = 0$$

$$-4x - 4y + z + 8 = 0$$

And the normal line has parametric equations:

$$x = 2 - 4t$$
$$y = 1 - 4t$$
$$z = 4 + t$$

Course: Calculus (3) Lecture No: [28]

Chapter: [13]

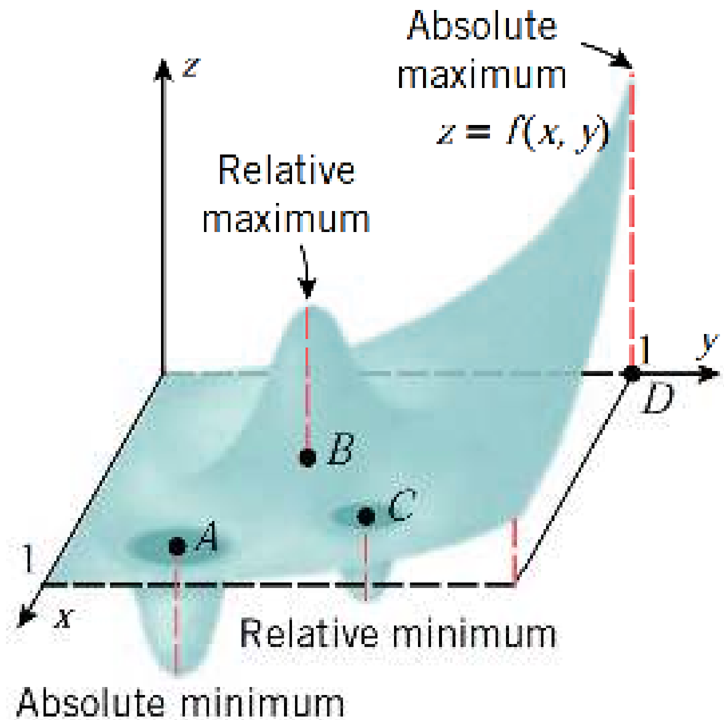
PARTIAL DERIVATIVES

Section: [13.8]

MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

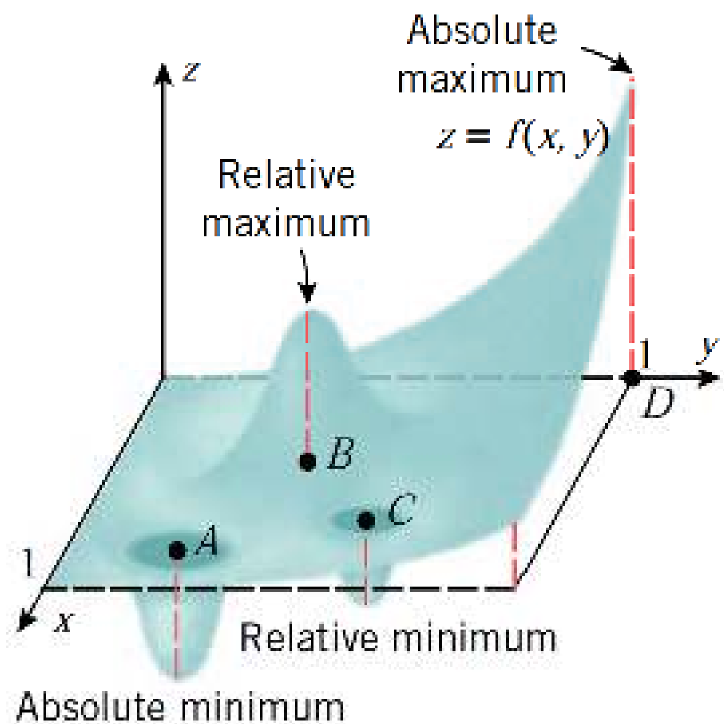
EXTREMA

- A function f of two variables is said to have a *relative maximum* at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk.
- And f is said to have an *absolute maximum* at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.



EXTREMA

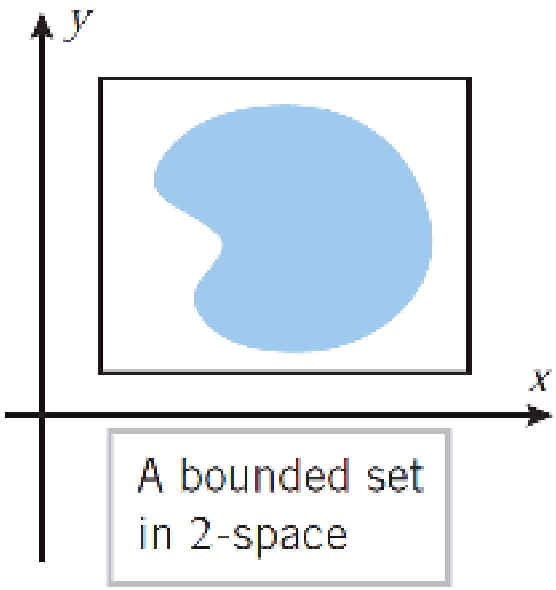
- A function f of two variables is said to have a *relative minimum* at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk.
- And f is said to have an *absolute minimum* at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f.

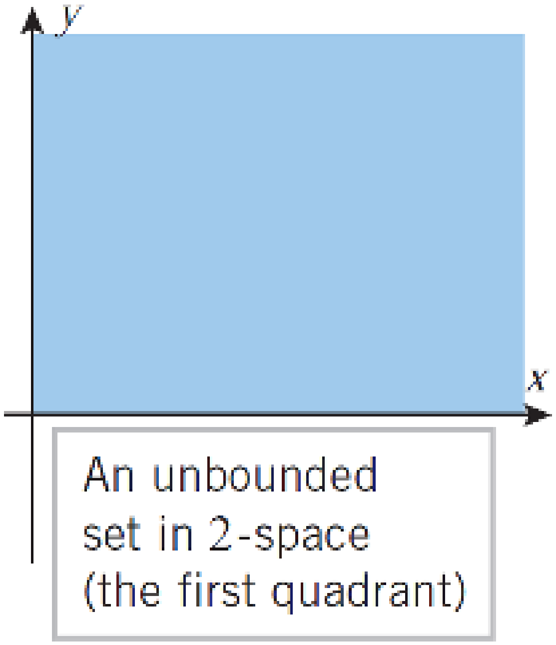


BOUNDED SETS

A set of points in 2 −space is called *bounded* if the entire set can be contained within some rectangle.

And is called *unbounded* if there is no rectangle that contains all the points of the set.





THE EXTREME-VALUE THEOREM

If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an **absolute maximum** and an **absolute minimum** on R.

NOTE If any of the conditions in the Extreme-Value Theorem *fail to hold*, then **there is no guarantee** that an absolute maximum or absolute minimum exists on the region R.

FINDING RELATIVE EXTREMA

Definition of Critical Point

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$
2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

NOTE If f is differentiable and

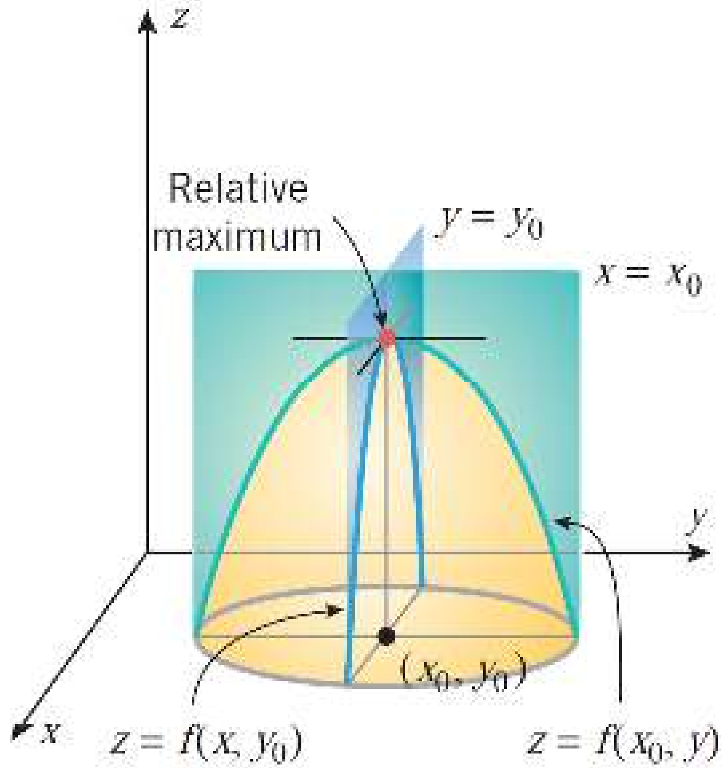
$$\nabla f(x_0, y_0) = f_x(x_0, y_0)\mathrm{i} + f_y(x_0, y_0)\mathrm{j} = 0\mathrm{i} + 0\mathrm{j} = 0$$

then every directional derivative at (x_0, y_0) must be 0.

FINDING RELATIVE EXTREMA

Relative Extrema Occur Only at Critical Points

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.



FINDING RELATIVE EXTREMA

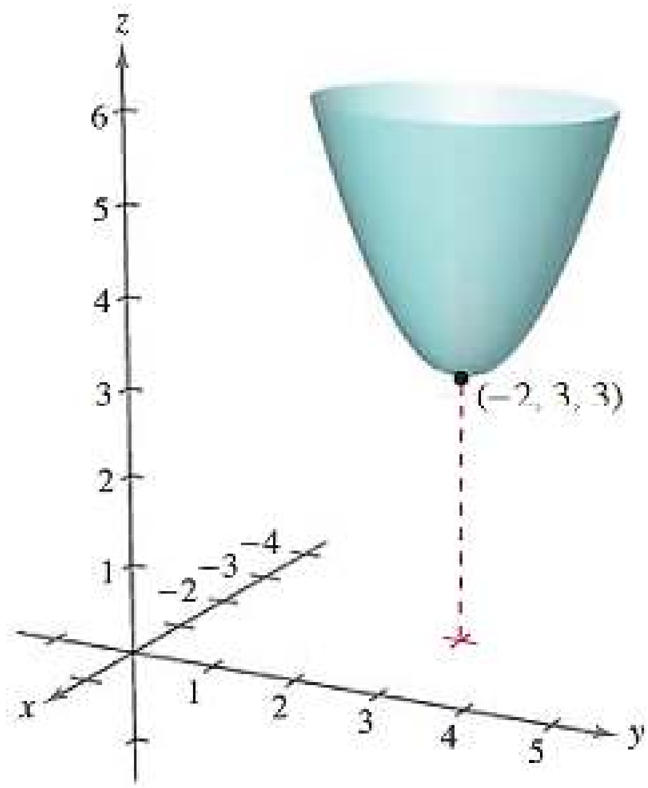
Example Find the critical value(s) of $f(x, y) = 2x^2 + y^2 + 8x - 6y + 20$.

$f_x(x, y) = 4x + 8 = 0$ $\quad x = -2$

$f_y(x, y) = 2y - 6 = 0$ $\quad y = 3$

The critical point is $(-2,3)$.

From the figure, f has a relative minimum at $(-2,3)$, and the value of this relative minimum is $f(-2,3) = 3$.



FINDING RELATIVE EXTREMA

Example Find the critical value(s) of $f(x, y) = 1 - (x^2 + y^2)^{1/3}$.

$$f_x(x, y) = 0 - \frac{1}{3}(x^2 + y^2)^{-2/3}(2x) = \frac{-2x}{3(x^2 + y^2)^{2/3}}$$

FINDING RELATIVE EXTREMA

Example Find the critical value(s) of $f(x, y) = 1 - (x^2 + y^2)^{1/3}$.

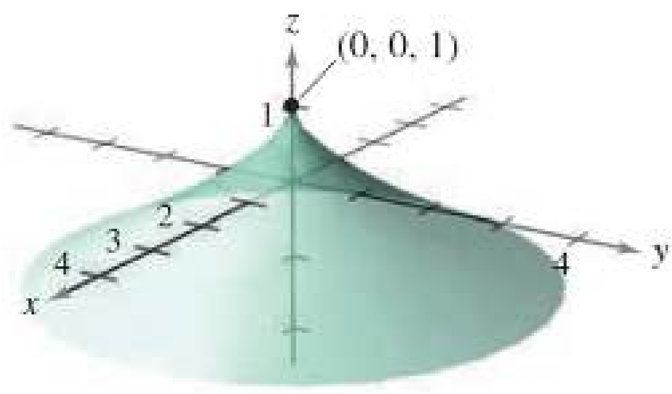
$$f_x(x, y) = \frac{-2x}{3(x^2 + y^2)^{2/3}}$$

$$f_y(x, y) = \frac{-2y}{3(x^2 + y^2)^{2/3}}$$

Both partial derivatives exist for all points in the xy −plane except for $(0,0)$.

The partial derivatives cannot both be 0 unless both x and y are 0.

The *only* critical point is $(0,0)$.



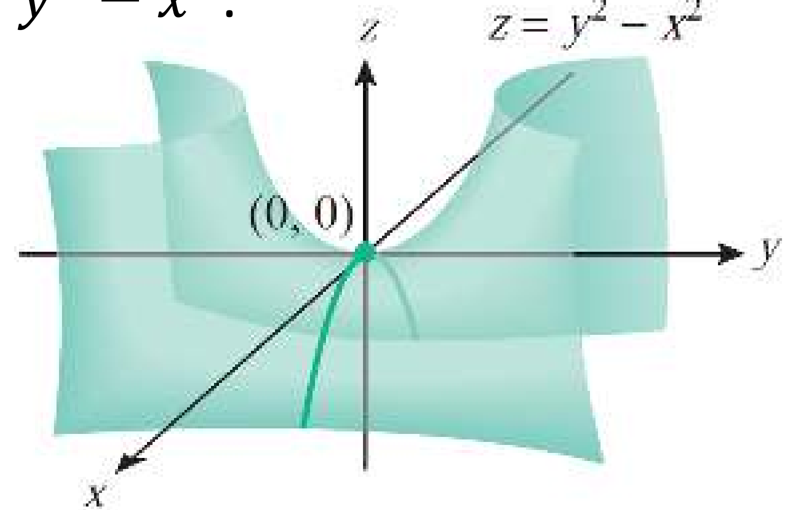
From the figure, f has a relative maximum at $(0,0)$, and the value of this relative minimum is $f(0,0) = 1$.

FINDING RELATIVE EXTREMA

Example Find the critical value(s) of $f(x, y) = y^2 - x^2$.

$f_x(x, y) = -2x = 0 \qquad x = 0$

$f_y(x, y) = 2y \quad = 0 \qquad y = 0$



The critical point is $(0,0)$.

- The function f has neither a relative maximum nor a relative minimum at $(0,0)$.
- The point $(0,0)$ is called a *saddle point* (نقطة سرج) of f.

THE SECOND PARTIALS TEST

13.8.6 THEOREM (*The Second Partials Test*) *Let f be a function of two variables with continuous second-order partial derivatives in some disk centered at a critical point (x_0, y_0), and let*

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

(*a*) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at (x_0, y_0).*

(*b*) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at (x_0, y_0).*

(*c*) *If $D < 0$, then f has a saddle point at (x_0, y_0).*

(*d*) *If $D = 0$, then no conclusion can be drawn.*

NOTE

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}$$

THE SECOND PARTIALS TEST

Example $f(x,y) = 2x^2 + y^2 + 8x - 6y + 20.$

The critical point is $(-2,3)$.

$f_x(x,y) = 4x + 8$

$f_y(x,y) = 2y - 6$

$f_{xx}(x,y) = 4$ | $f_{xx}(-2,3) = 4 > 0$

$f_{yy}(x,y) = 2$ | $f_{yy}(-2,3) = 2$

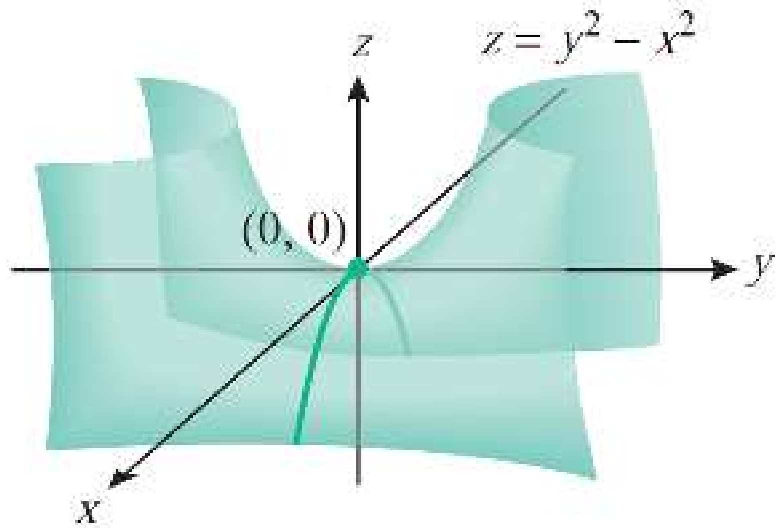
$f_{xy}(x,y) = 0$ | $f_{xy}(-2,3) = 0$

$$D = f_{xx}(-2,3)f_{yy}(-2,3) - f_{xy}^2(-2,3) = (4)(2) - (0)^2 = 8 > 0$$

f has a relative minimum at $(-2,3)$ by the second partial test, and the value of this relative minimum is $f(-2,3) = 3$.

THE SECOND PARTIALS TEST

Example $f(x,y) = y^2 - x^2.$



The critical point is $(0,0)$.

$f_x(x,y) = -2x$ $\qquad f_{xx}(0,0) = -2$

$f_y(x,y) = 2y$ $\qquad f_{yy}(0,0) = 2$

$f_{xy}(0,0) = 0$

$$D = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = (-2)(2) - (0)^2 = -4 \quad < 0$$

f has a saddle point at $(0,0)$ by the second partial test.

THE SECOND PARTIALS TEST

Example Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

$f_x(x, y) = 4y - 4x^3 = 0$ $y = x^3$ $x = (x^3)^3 = x^9$

$f_y(x, y) = 4x - 4y^3 = 0$ $x = y^3$ $x^9 - x = 0$ $x(x^8 - 1) = 0$

x	$y = x^3$
-1	-1
0	0
1	1

$f_{xx}(x, y) = -12x^2$

$f_{yy}(x, y) = -12y^2$

$f_{xy}(x, y) = 4$

THE SECOND PARTIALS TEST

Example Locate all relative extrema and saddle points of

$$f(x, y) = 4xy - x^4 - y^4$$

x	$y = x^3$
-1	-1
0	0
1	1

$$f_{xx}(x, y) = -12x^2$$
$$f_{yy}(x, y) = -12y^2$$
$$f_{xy}(x, y) = 4$$

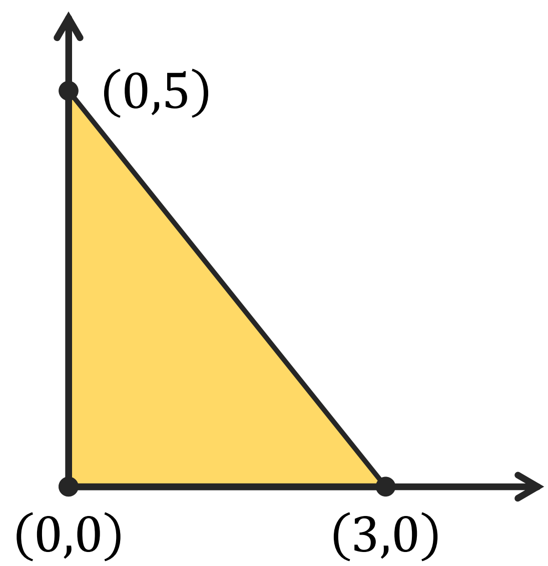
Critical Point	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - [f_{xy}]^2$	Type
$(-1,-1)$	-12	-12	4	128	Local Max
$(0,0)$	0	0	4	-16	Saddle
$(1,1)$	-12	-12	4	128	Local Max

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

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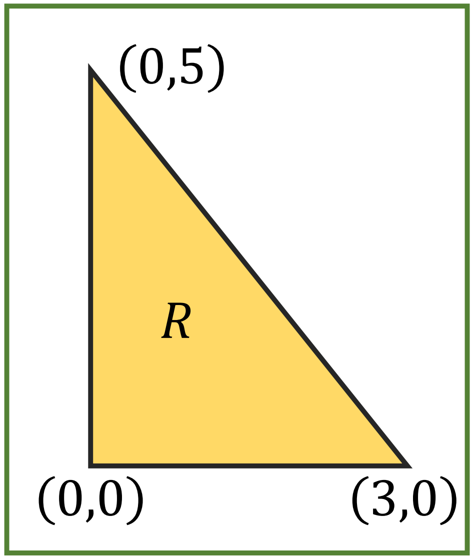


Bounded

1. Inside the region R.

$$\left.\begin{aligned} f_x(x,y) &= 3y - 6 = 0 \\ f_y(x,y) &= 3x - 3 = 0 \end{aligned}\right\}$$

$(1,2)$ is critical point

Saddle Point

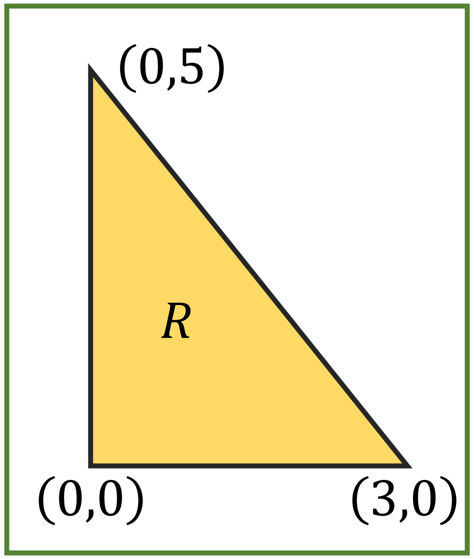
$$D = f_{xx}(1,2)f_{yy}(1,2) - f_{xy}^2(1,2)$$
$$= (0)(0) - (3)^2 = -9 < 0$$

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



Bounded

2. On the line through the points $(0, 0)$ and $(3, 0)$.

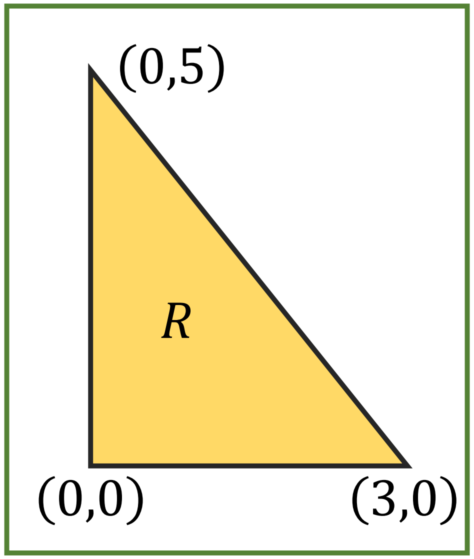
$y = 0$ $\quad f(x, 0) = -6x + 7$

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



Bounded

2. On the line through the points $(0,0)$ and $(3,0)$.

$y = 0$ $\quad u(x) = -6x + 7 \quad ; \quad x \in [0,3]$

Since $u'(x) = -6 < 0$ $\quad u(x)$ *decreases* on $[0,3]$

$(0,0)$ **MAX**

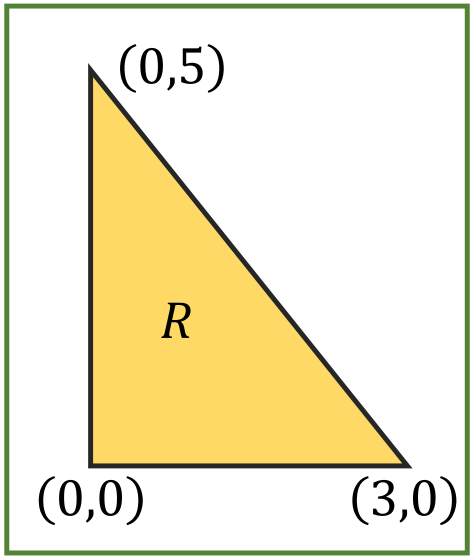
$(3,0)$ **MIN**

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



Bounded

3. On the line through the points $(0, 0)$ and $(0, 5)$.

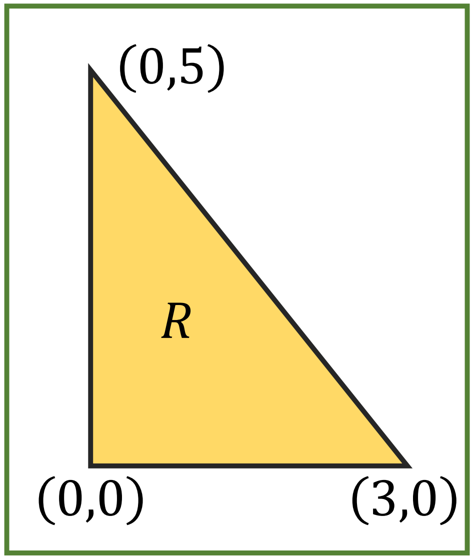
$x = 0$ $\quad f(0, y) = -3y + 7$

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



Bounded

3. On the line through the points $(0, 0)$ and $(0, 5)$.

$x = 0$ $\quad w(y) = -3y + 7 \quad ; \quad y \in [0,5]$

Since $w'(y) = -3 < 0$ $\quad w(y)$ *decreases* on $[0,5]$

$(0,0)$ **MAX**

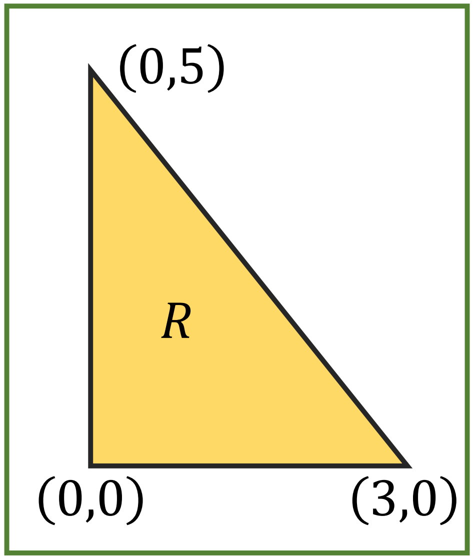
$(0,5)$ **MIN**

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



Bounded

4. On the line through the points $(3,0)$ and $(0,5)$.

$$m = \frac{5-0}{0-3} = -\frac{5}{3}$$

$$y - y_0 = m(x - x_0)$$

$$y - 0 = -\frac{5}{3}(x - 3)$$

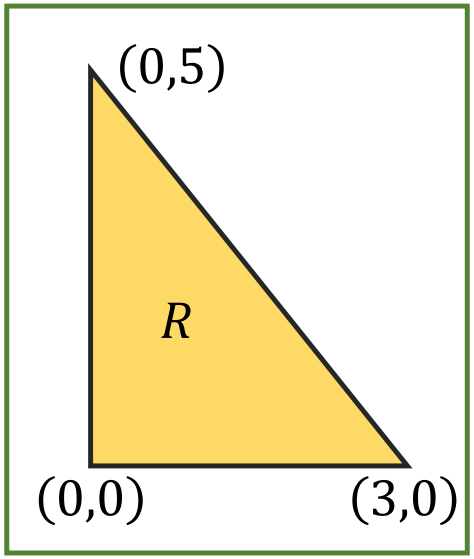
$$y = -\frac{5}{3}x + 5$$

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



Bounded

4. On the line through the points $(3,0)$ and $(0,5)$.

$$y = -\frac{5}{3}x + 5$$

$$f\left(x, -\frac{5}{3}x + 5\right) = 3x\left(-\frac{5}{3}x + 5\right) - 6x - 3\left(-\frac{5}{3}x + 5\right) + 7$$

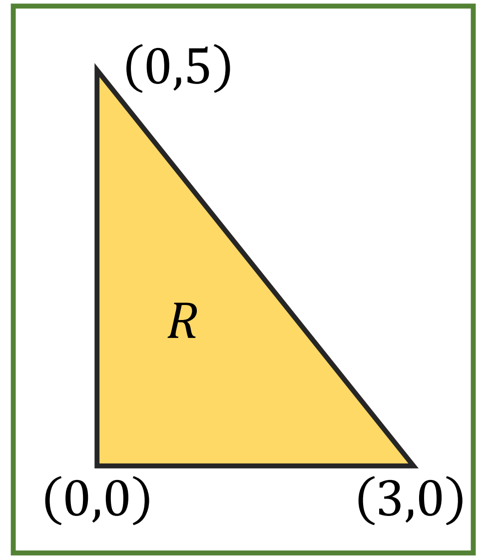
$$= -5x^2 + 14x - 8$$

FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



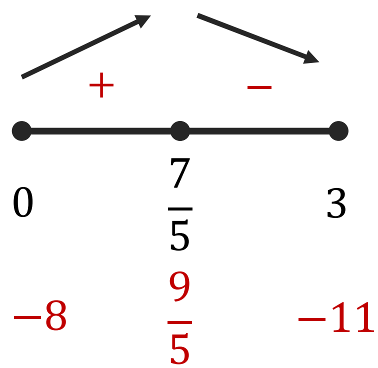
Bounded

4. On the line through the points $(3,0)$ and $(0,5)$.

$$y = -\frac{5}{3}x + 5 \qquad g(x) = -5x^2 + 14x - 8 \quad ; \quad x \in [0,3]$$

$$g'(x) = -10x + 14 = 0$$

$$x = \frac{7}{5}$$

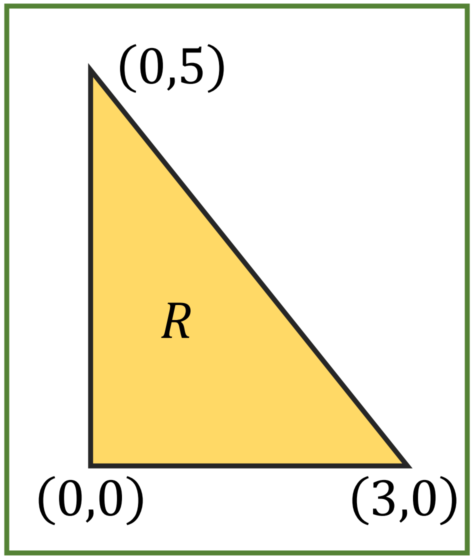


FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x, y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.



Bounded

4. On the line through the points $(3, 0)$ and $(0, 5)$.

$$y = -\frac{5}{3}x + 5 \qquad g(x) = -5x^2 + 14x - 8 \quad ; \quad x \in [0,3]$$

$$g'(x) = -10x + 14 = 0$$

$$x = \frac{7}{5}$$

$\left(\frac{7}{5}, \frac{8}{3}\right)$ MAX

$(3,0)$ MIN

$(0,5)$ MIN

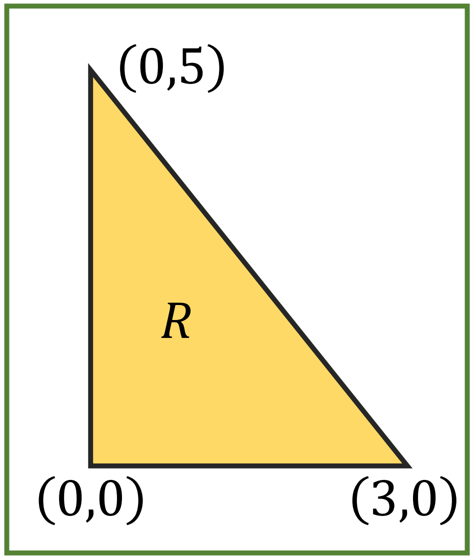
FINDING ABSOLUTE EXTREMA ON CLOSED AND BOUNDED SETS

Example Find the absolute maximum and minimum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.

Point	$f(x,y)$	Type	
$(1,2)$	1	Saddle	
$(0,0)$	7	MAX	Absolute
$(3,0)$	-11	MIN	Absolute
$(0,5)$	-8	MIN	Relative
$\left(\frac{7}{5},\frac{8}{3}\right)$	$\frac{9}{5}$	MAX	Relative



Bounded

Course: Calculus (3)

Lecture No: [30]

Chapter: [13]

PARTIAL DERIVATIVES

Section: [13.9]

LAGRANGE MULTIPLIERS

EXTREMUM PROBLEMS WITH CONSTRAINTS

- In this section we will study a powerful new method for maximizing or minimizing a function *subject to constraints on the variables*.
- This method will help us to solve certain optimization problems that are difficult or impossible to solve using the methods studied in the last section.
- We wish to:

  Find extrema of the function $z = f(x, y)$ subject to a constraint given by $g(x, y) = c$.

EXTREMUM PROBLEMS WITH CONSTRAINTS

Lagrange's Theorem

Let f and g have continuous first partial derivatives such that f has an extremum at a point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$. If $\nabla g(x_0, y_0) \neq \mathbf{0}$, then there is a real number λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

NOTE The scalar λ is called a **Lagrange multiplier**.

EXTREMUM PROBLEMS WITH CONSTRAINTS

Method of Lagrange Multipliers

Let f and g satisfy the hypothesis of Lagrange's Theorem, and let f have a minimum or maximum subject to the constraint $g(x, y) = c$. To find the minimum or maximum of f, use these steps.

1. Simultaneously solve the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = c$ by solving the following system of equations.

$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = c$$

2. Evaluate f at each solution point obtained in the first step. The greatest value yields the maximum of f subject to the constraint $g(x, y) = c$, and the least value yields the minimum of f subject to the constraint $g(x, y) = c$.

EXTREMUM PROBLEMS WITH CONSTRAINTS

$g(x, y) = x + y - 3$

Example At what point(s) on the line $x + y = 3$ does

$$f(x, y) = 9 - x^2 - y^2$$

have an absolute maximum, and what is that maximum?

$f_x(x, y) = \lambda g_x(x, y)$ $\qquad -2x = \lambda$

$f_y(x, y) = \lambda g_y(x, y)$ $\qquad -2y = \lambda$

$g(x, y) = 0$ $\qquad x + y - 3 = 0$

$-2x = -2y$

EXTREMUM PROBLEMS WITH CONSTRAINTS

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$$f_x(x, y) = \lambda g_x(x, y)$$
$$f_y(x, y) = \lambda g_y(x, y)$$
$$g(x, y) = 0$$

$$-2x = \lambda$$
$$-2y = \lambda$$
$$x + y - 3 = 0$$

$$x = y$$

$$2x - 3 = 0$$

$$x = \frac{3}{2} \quad y = \frac{3}{2}$$

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example At what point(s) on the line $x + y = 3$ does

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have an absolute maximum, and what is that maximum?

$x = \frac{3}{2}$ $y = \frac{3}{2}$

- Subject to the constraint $x + y = 3$, the function f has absolute maximum at $\left(\frac{3}{2}, \frac{3}{2}\right)$.
- The value of the absolute maximum is $f\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{9}{2}$.

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y) = x - 3y - 1$$

subject to the constraint $x^2 + 3y^2 = 16$.

$g(x, y) = x^2 + 3y^2 - 16$

$f_x(x, y) = \lambda g_x(x, y)$ $\quad 1 = 2\lambda x$

$f_y(x, y) = \lambda g_y(x, y)$ $\quad -3 = 6\lambda y$

$g(x, y) = 0$ $\quad x^2 + 3y^2 - 16 = 0$

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Use Lagrange multipliers to find the maximum and minimum values of

$$f(x,y) = x - 3y - 1$$

subject to the constraint $x^2 + 3y^2 = 16$.

$1 = 2\lambda x$

$\div$

$-3 = 6\lambda y$

$$\frac{1}{-3} = \frac{x}{3y}$$

$-x = y$

$x^2 + 3y^2 - 16 = 0$

$4x^2 - 16 = 0$

$x = 2 \rightarrow y = -2$

$x = -2 \rightarrow y = 2$

$f(2,-2) = 7$ **MAX**

$f(-2,2) = -9$ **MIN**

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Find three positive numbers whose sum is 48 and such that their product is as large as possible.

Let the three numbers x, y and z.

Constraint: $x + y + z = 48$

Function: $f(x, y, z) = xyz$

Find the maximum value of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 48$.

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Find the maximum value of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 48$.

$$g(x,y,z) = x + y + z - 48$$

$$f_x(x,y,z) = \lambda g_x(x,y,z) \qquad yz = \lambda$$

$$f_y(x,y,z) = \lambda g_y(x,y,z) \qquad xz = \lambda$$

$$\left.\begin{matrix} yz = \lambda \\ xz = \lambda \end{matrix}\right\} \frac{y}{x} = 1$$

$$f_z(x,y,z) = \lambda g_z(x,y,z) \qquad xy = \lambda$$

$$g(x,y,z) = 0 \qquad x + y + z - 48 = 0$$

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Find the maximum value of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 48$.

$$g(x, y, z) = x + y + z - 48$$

$$f_x(x, y, z) = \lambda g_x(x, y, z) \qquad yz = \lambda$$

$$f_y(x, y, z) = \lambda g_y(x, y, z) \qquad xz = \lambda$$

$$\left.\begin{array}{l} yz = \lambda \\ xz = \lambda \end{array}\right\} \; y = x$$

$$f_z(x, y, z) = \lambda g_z(x, y, z) \qquad xy = \lambda$$

$$g(x, y, z) = 0 \qquad x + y + z - 48 = 0$$

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Find the maximum value of $f(x, y, z) = xyz$ subject to the constraint $x + y + z = 48$.

$$\to g(x, y, z) = x + y + z - 48$$

$$f_x(x, y, z) = \lambda g_x(x, y, z) \qquad yz = \lambda$$

$$f_y(x, y, z) = \lambda g_y(x, y, z) \qquad xz = \lambda$$

$$f_z(x, y, z) = \lambda g_z(x, y, z) \qquad xy = \lambda$$

$$g(x, y, z) = 0 \qquad x + y + z - 48 = 0$$

$yz = \lambda$, $xz = \lambda$ $\Rightarrow$ $y = x$

$xz = \lambda$, $xy = \lambda$ $\Rightarrow$ $\dfrac{z}{y} = 1$

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Find the maximum value of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 48$.

$$g(x,y,z) = x + y + z - 48$$

$$f_x(x,y,z) = \lambda g_x(x,y,z) \qquad yz = \lambda$$

$$f_y(x,y,z) = \lambda g_y(x,y,z) \qquad xz = \lambda$$

$$f_z(x,y,z) = \lambda g_z(x,y,z) \qquad xy = \lambda$$

$$\left.\begin{matrix} xz = \lambda \\ xy = \lambda \end{matrix}\right\} \quad y = x, \quad y = z$$

$$g(x,y,z) = 0 \qquad x + y + z - 48 = 0$$

EXTREMUM PROBLEMS WITH CONSTRAINTS

Example Find the maximum value of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 48$.

$$g(x,y,z) = x + y + z - 48$$

$$f_x(x,y,z) = \lambda g_x(x,y,z) \qquad yz = \lambda$$

$$f_y(x,y,z) = \lambda g_y(x,y,z) \qquad xz = \lambda$$

$$f_z(x,y,z) = \lambda g_z(x,y,z) \qquad xy = \lambda$$

$$g(x,y,z) = 0 \qquad x + y + z - 48 = 0$$

$$\left.\begin{array}{l} y = x \\ y = z \end{array}\right\} \quad x = y = z$$

$$3x - 48 = 0$$

$$x = 16 \qquad y = 16 \qquad z = 16$$

$$f(16,16,16) = 16^3 = 4096$$