Course: Calculus (3)

Lecture No: [32]

Chapter: [14]

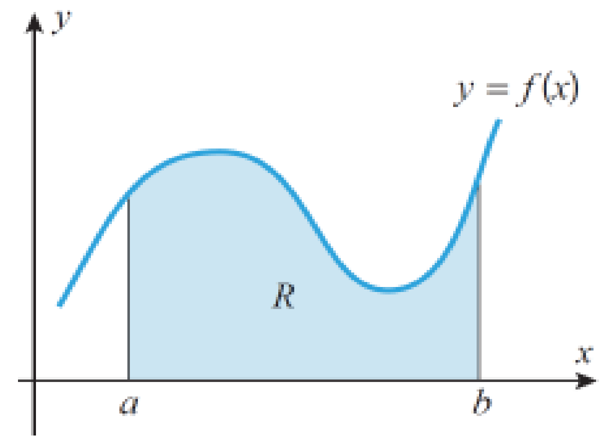
MULTIPLE INTEGRALS

Section: [14.1]

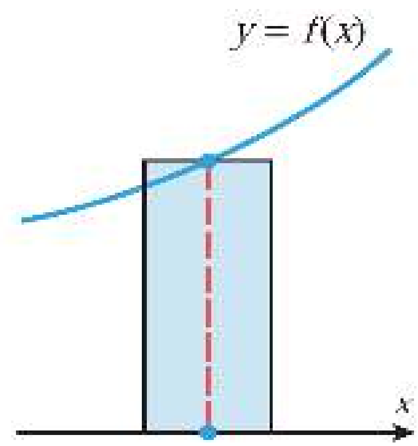
DOUBLE INTEGRALS

THE AREA PROBLEM

Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x —axis.



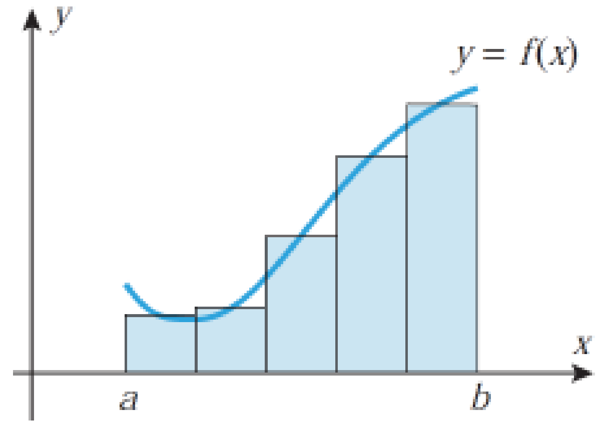
Divide the interval $[a, b]$ into n equal subintervals, and over each subinterval construct a rectangle that extends from the x —axis to any point on the curve $y = f(x)$ that is above the subinterval.

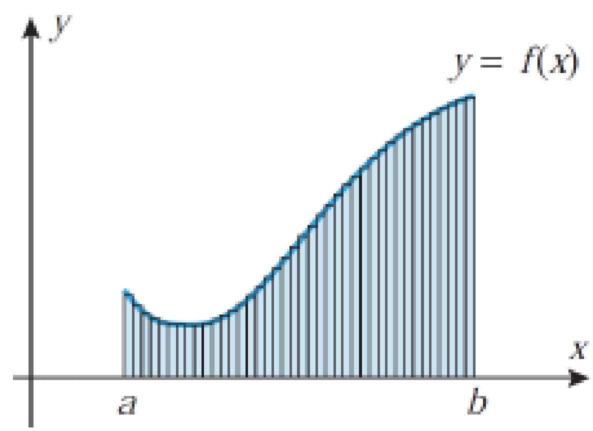


THE AREA PROBLEM

- For each n, the total area of the rectangles can be viewed as an *approximation* to the exact area under the curve over the interval $[a, b]$.
- Moreover, it is evident intuitively that *as n increases these approximations will get better and better and will approach the exact area as a limit*.
- That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n \to \infty} A_n$$



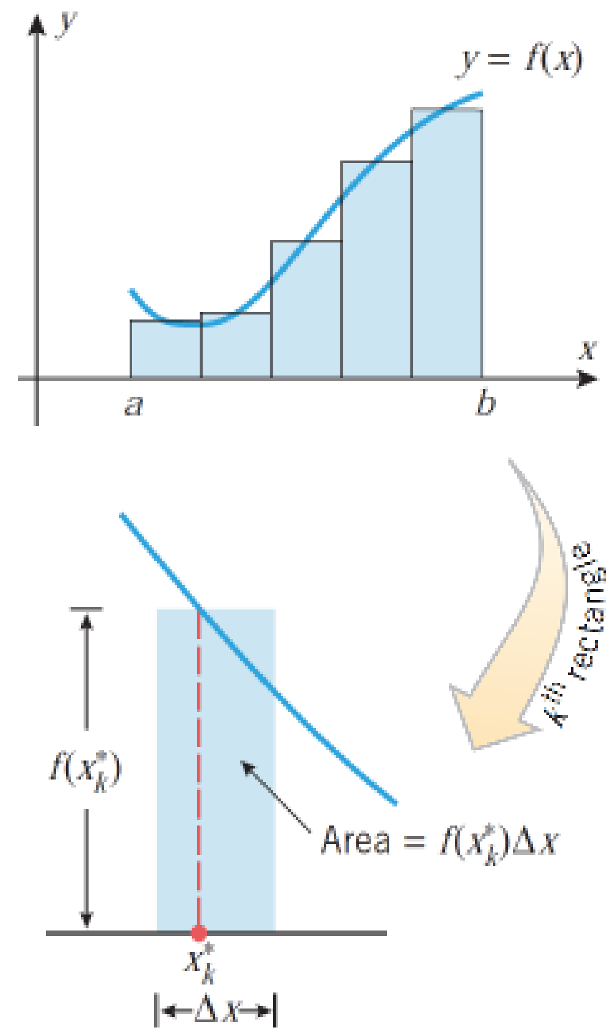


THE AREA PROBLEM

$$A \approx \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

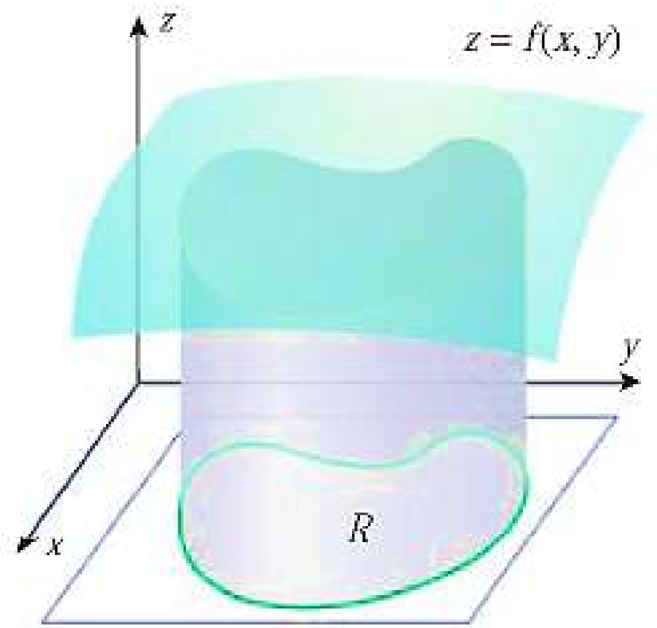
$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$



THE VOLUME PROBLEM

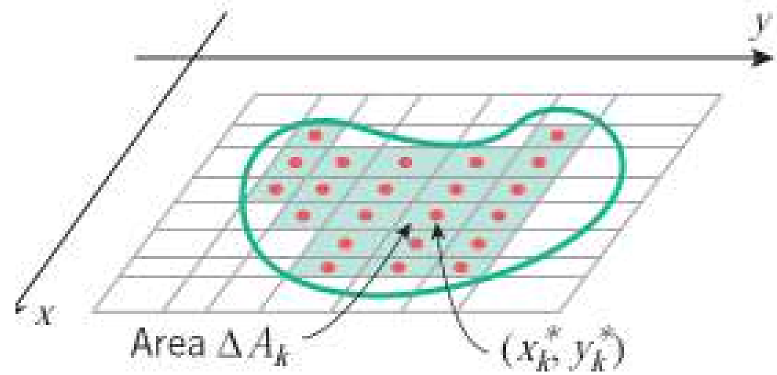
Given a function f of two variables that is continuous and nonnegative on a region R in the xy −plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R.



The procedure for finding the volume V of the solid in the figure will be similar to the limiting process used for finding areas, except that now the approximating elements will be rectangular parallelepipeds rather than rectangles.

THE VOLUME PROBLEM

We proceed as follows:



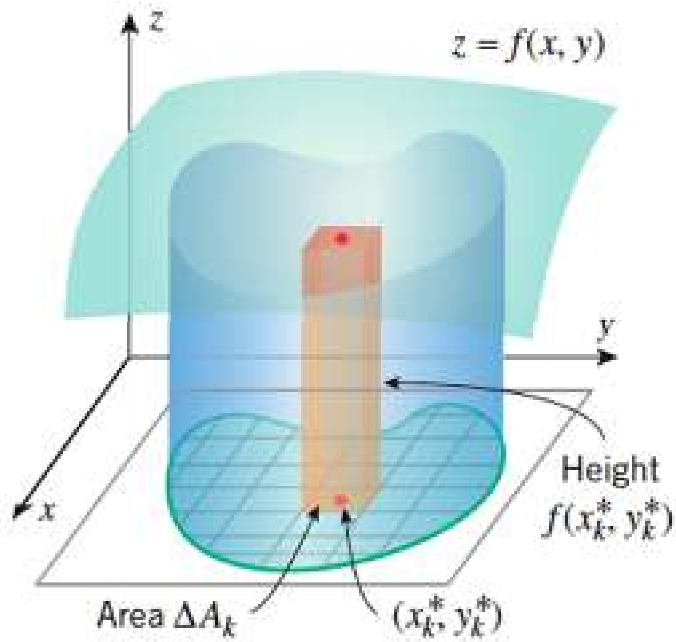
- Using lines parallel to the coordinate axes, divide the rectangle enclosing the region R into sub-rectangles, and exclude from consideration all those sub-rectangles that contain any points outside of R.

- Assume that there are n such rectangles, and denote the area of the k^{th} such rectangle by ΔA_k.

- Choose any arbitrary point in each sub-rectangle, and denote the point in the k^{th} sub-rectangle by (x_k^*, y_k^*).

THE VOLUME PROBLEM

- As shown in the figure, the product $f(x_k^*, y_k^*)\Delta A_k$ is the volume of a rectangular parallelepiped with base area ΔA_k and height $f(x_k^*, y_k^*)$.
- So the following sum can be viewed as an approximation to the volume V of the entire solid.



$$V \approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

$$\iint_R f(x,y)dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

which is called the double integral of $f(x,y)$ over R.

EVALUATING DOUBLE INTEGRALS

- The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable.
- Let us consider the reverse of this process, **partial integration**.

$$\int_a^b f(x, y)dx$$

$$\int_c^d f(x, y)dy$$

- ✓ **The partial definite integral with respect to x.**
- ✓ Is evaluated by holding y fixed and integrating with respect to x.

- ✓ **The partial definite integral with respect to y.**
- ✓ Is evaluated by holding x fixed and integrating with respect to y.

EVALUATING DOUBLE INTEGRALS

Example **(1)** $$\int_0^1 xy^2 dx = y^2 \int_0^1 x dx = \left.\frac{y^2x^2}{2}\right]_0^1 = \frac{y^2}{2}$$

(2) $$\int_0^1 xy^2 dy = x \int_0^1 y^2 dy = \left.\frac{xy^3}{3}\right]_0^1 = \frac{x}{3}$$

NOTE

- A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y.
- A partial definite integral with respect to y can be integrated with respect to x.
- This two-stage integration process is called **iterated** (or *repeated*) **integration**.

EVALUATING DOUBLE INTEGRALS

- We introduce the following notation:

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)dx\right] dy$$

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)dy\right] dx$$

- These integrals are called *iterated integrals*.

EVALUATING DOUBLE INTEGRALS

Example Evaluate $\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx$

$$\int_1^3 \int_2^4 (40 - 2xy)\,dy\,dx = \int_1^3 \left[\int_2^4 (40 - 2xy)\,dy\right] dx$$

$$= \int_1^3 (40y - xy^2)]_2^4\,dx$$

$$= \int_1^3 [(160 - 16x) - (80 - 4x)]\,dx = \int_1^3 (80 - 12x)\,dx$$

$$= 112$$

EVALUATING DOUBLE INTEGRALS

Homework Evaluate $\int_2^4 \int_1^3 (40 - 2xy)\,dx\,dy = 112$

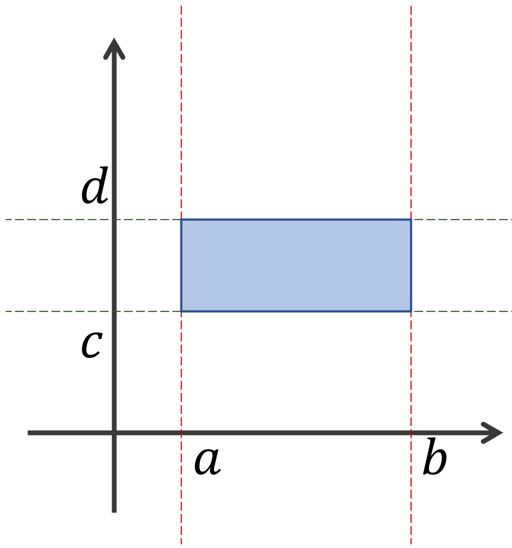
Fubini's Theorem

Let R be the rectangle defined by

$$R = \{(x, y) :\ a \le x \le b\,, c \le y \le d\}$$
$$= [a, b] \times [c, d]$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x, y)\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy$$



EVALUATING DOUBLE INTEGRALS

Example Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0,1] \times [0,2]$.

$$V = \iint_R (4 - x - y)dA = \int_0^1 \int_0^2 (4 - x - y)dydx = \int_0^1 \left[\int_0^2 (4 - x - y)dy\right] dx$$

$$= \int_0^1 \left(4y - xy - \frac{y^2}{2}\right)\Bigg]_0^2 dx$$

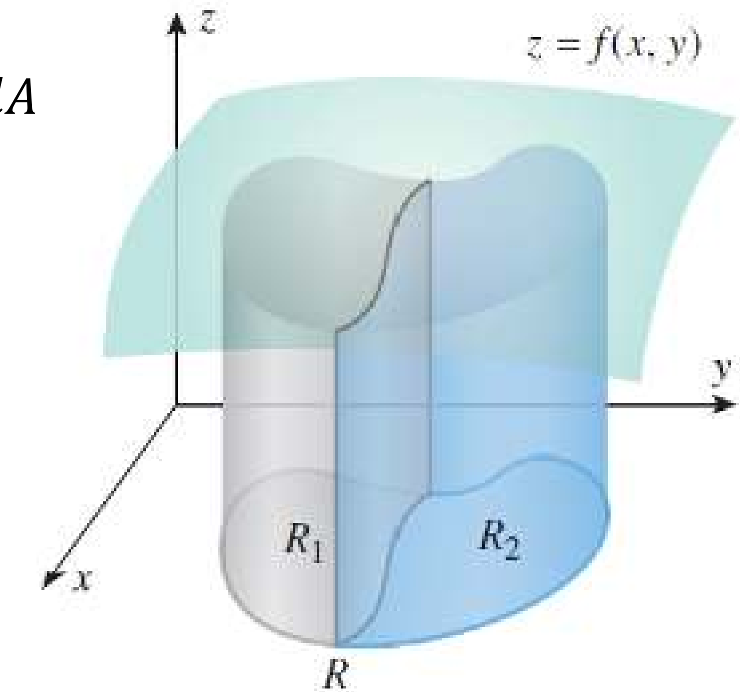
$$= \int_0^1 (6 - 2x)dx \quad = 5 \quad \boxed{= \int_0^2 \int_0^1 (4 - x - y)dxdy}$$

PROPERTIES OF DOUBLE INTEGRALS

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA \qquad (c \text{ constant})$$

$$\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$$

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$



PROPERTIES OF DOUBLE INTEGRALS

NOTE If $R = [a,b] \times [c,d]$ is a rectangular region, and $f(x,y) = g(x)h(y)$, then

$$\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dA = \left[\int_a^b g(x)\,dx\right]\left[\int_c^d h(y)\,dy\right]$$

Example $\displaystyle \int_0^1 \int_0^2 e^{x+y}\,dx\,dy = \int_0^1 \int_0^2 e^x e^y\,dx\,dy$

$$= \left(\int_0^2 e^x\,dx\right)\left(\int_0^1 e^y\,dy\right) = (e^2 - 1)(e - 1)$$

EXERCISE SET 14.1 QUESTION 33

Homework Evaluate the integral by choosing a convenient order of integration:

$$\frac{1}{3\pi} = \iint_R x\cos(xy)\cos^2(\pi x)\,dA \quad ; \quad R = \left[0, \tfrac{1}{2}\right] \times [0, \pi]$$

Course: Calculus (3)

Lecture No: [33]

Chapter: [14]

MULTIPLE INTEGRALS

Section: [14.2]

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

In this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals

Example $$\int_0^1 \int_{-x}^{x^2} y^2 x \, dy dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x \, dy \right] dx = \int_0^1 \frac{xy^3}{3} \bigg]_{-x}^{x^2} dx$$

$$= \int_0^1 \left(\frac{x^7}{3} + \frac{x^4}{3} \right) dx = \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \bigg]_0^1 = \frac{13}{120}$$

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Example $$\int_0^{\pi/3}\int_0^{\cos y} x\sin y\,dx\,dy = \int_0^{\pi/3}\left[\int_0^{\cos y} x\sin y\,dx\right]dy$$

$$= \int_0^{\pi/3} \frac{x^2\sin y}{2}\Bigg]_0^{\cos y} dy$$

$$= \int_0^{\pi/3} \frac{1}{2}\cos^2 y\sin y\,dy = -\frac{1}{2}\int_1^{1/2} t^2 \sin y\frac{dt}{\sin y}$$

$$= \frac{1}{2}\int_{1/2}^{1} t^2 dt = \frac{t^3}{6}\Bigg]_{1/2}^{1} = \frac{7}{48}$$

By Substitution

Let $t = \cos y$

$$\frac{dt}{dy} = -\sin y$$

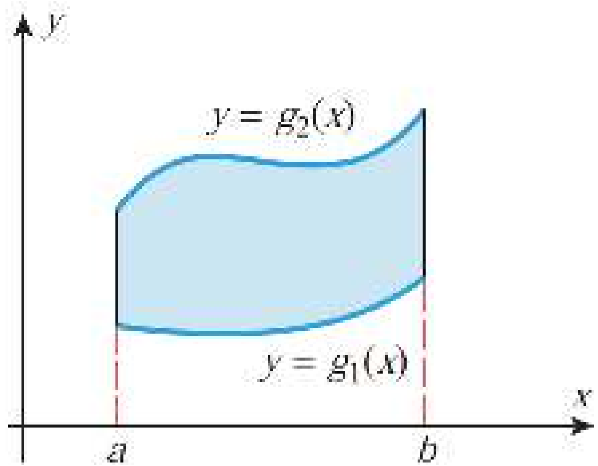
$$dy = -\frac{dt}{\sin y}$$

$y = \pi/3$	$t = 1/2$
$y = 0$	$t = 1$

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

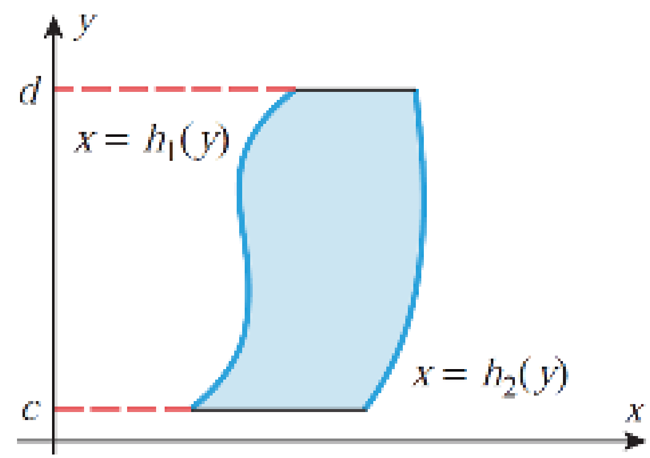
Type I Region

is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$.



Type II Region

is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$



DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

1) If R is a **type I region** on which $f(x, y)$ is continuous, then

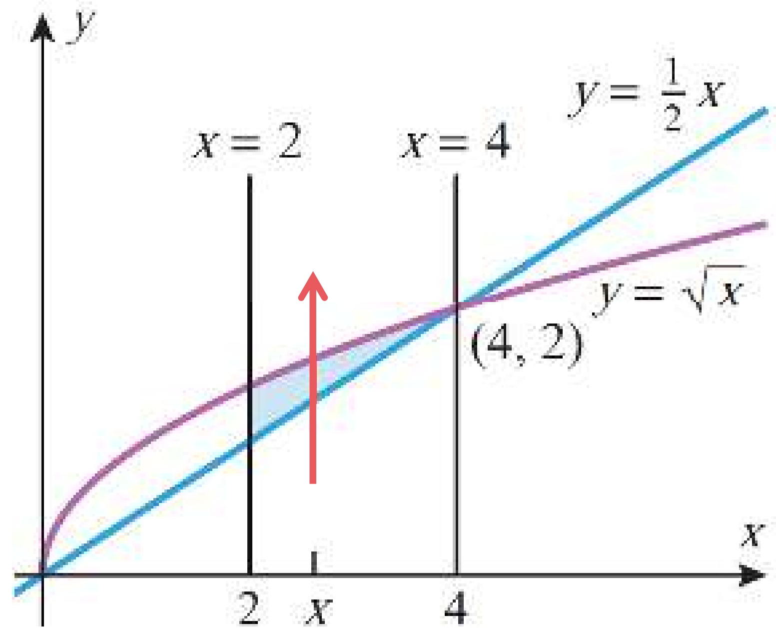
$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

2) If R is a **type II region** on which $f(x, y)$ is continuous, then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$$

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\iint\limits_R xy\,dA$ over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$ and $x = 4$.



Type I Region

$$\iint\limits_R xy\,dA = \int\int xy\,dy\,dx = \int_2^4 \left[\int_{x/2}^{\sqrt{x}} xy\,dy\right] dx$$

$$= \int_2^4 \frac{xy^2}{2}\bigg]_{x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx = \frac{11}{6}$$

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

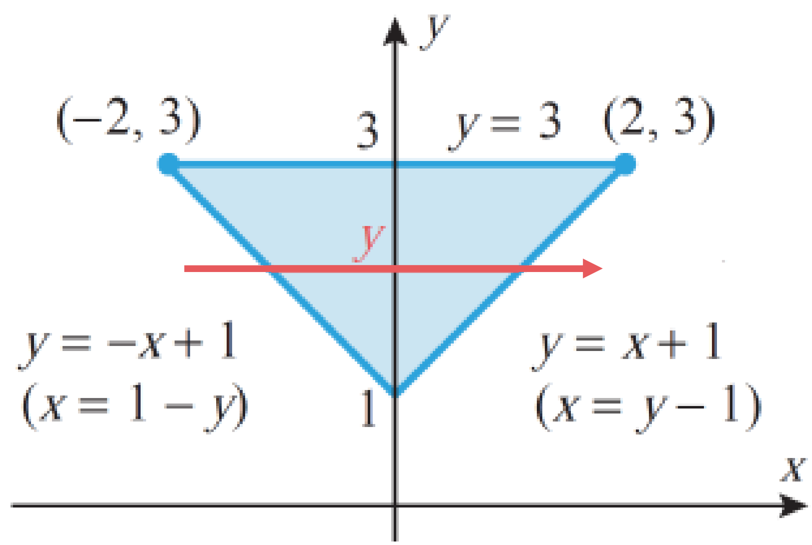
Example Evaluate $\iint\limits_R (2x - y^2)dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

$$\iint\limits_R (2x - y^2)dA = \int \int (2x - y^2)dxdy$$

$$= \int_1^3 x^2 - xy^2]_{1-y}^{y-1} dy$$

$$= \int_1^3 (2y^2 - 2y^3)dy = -\frac{68}{3}$$

Type II Region

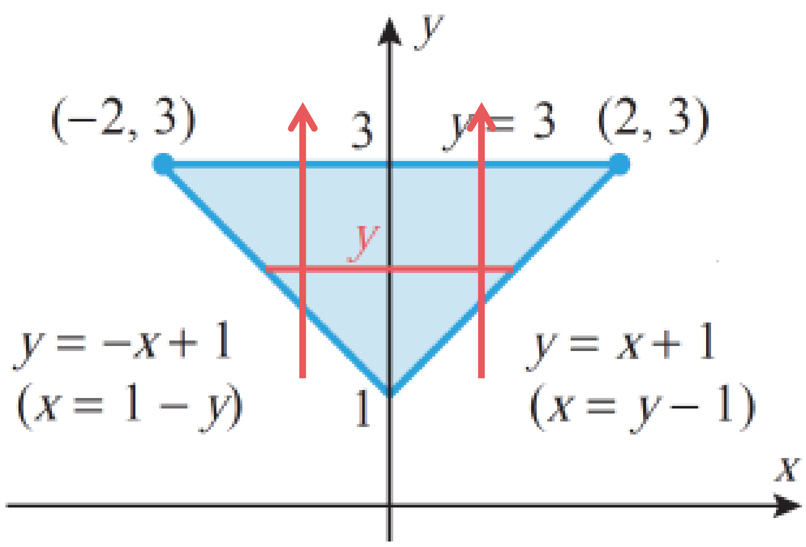


DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\iint\limits_R (2x - y^2)\,dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

$$\iint\limits_R (2x - y^2)\,dA$$

Type I Region



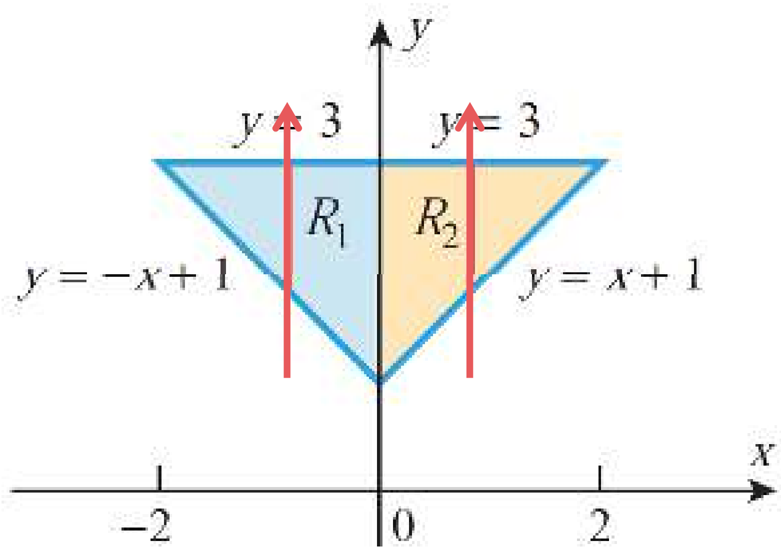
DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\iint\limits_R (2x - y^2)dA$ over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

$$\iint\limits_R (2x - y^2)dA = \iint\limits_{R_1} (2x - y^2)dA + \iint\limits_{R_2} (2x - y^2)dA$$

$$= \int_{-2}^{0} \int_{-x+1}^{3} (2x - y^2)dydx + \int_{0}^{2} \int_{x+1}^{3} (2x - y^2)dydx$$

Type I Region

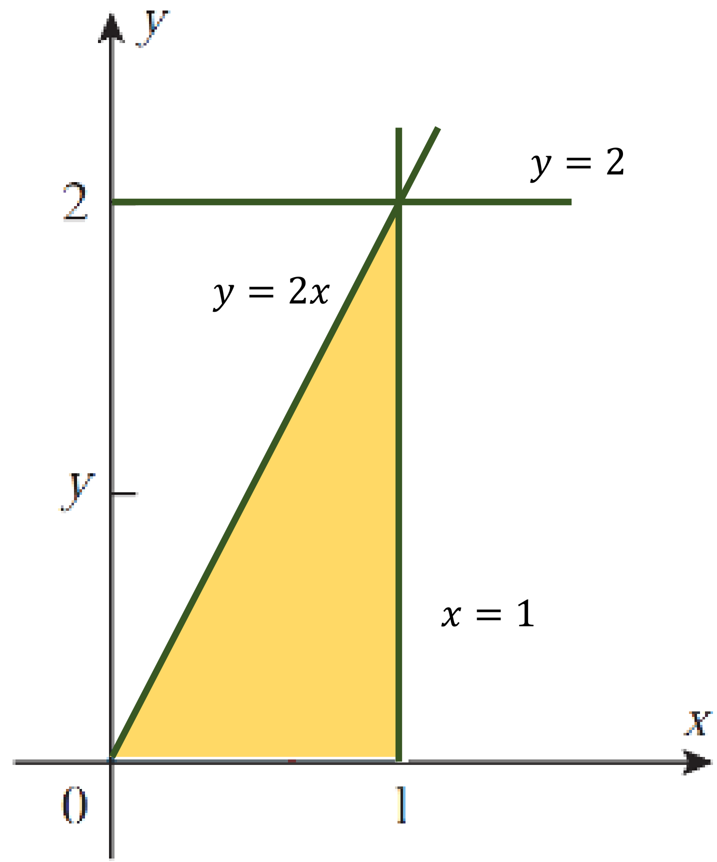


DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Example Evaluate $\int_0^2 \int_{\boxed{y/2}}^1 e^{x^2}\,dx\,dy$ $\rightarrow$ $y = 2x$

Since there is no elementary antiderivative of e^{x^2}, the integral cannot be evaluated by performing the x −integration first.

We will try to evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.



DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

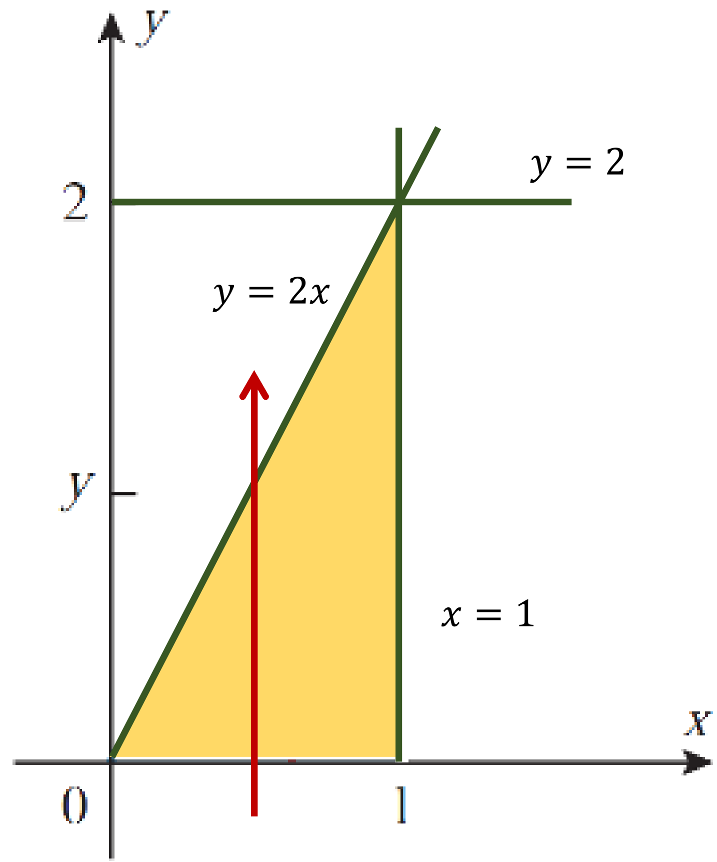
Example Evaluate $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \int\int e^{x^2}\,dy\,dx = \int_0^1 \left[\int_0^{2x} e^{x^2}\,dy\right] dx$$

$$= \int_0^1 \left. e^{x^2} y\right]_0^{2x} dx$$

By Substitution
Let $t = x^2$

$$= \int_0^1 2xe^{x^2}\,dx = \int_0^1 e^t\,dt = e - 1$$

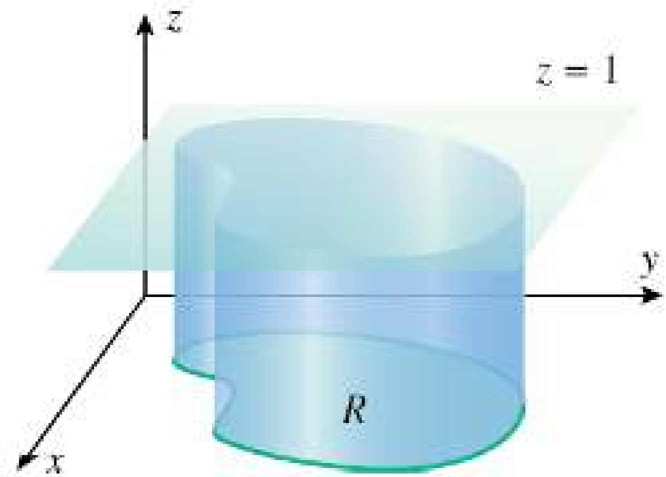


AREA CALCULATED AS A DOUBLE INTEGRAL

Example

Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA$$



Cylinder with base R and height 1

AREA CALCULATED AS A DOUBLE INTEGRAL

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA$$

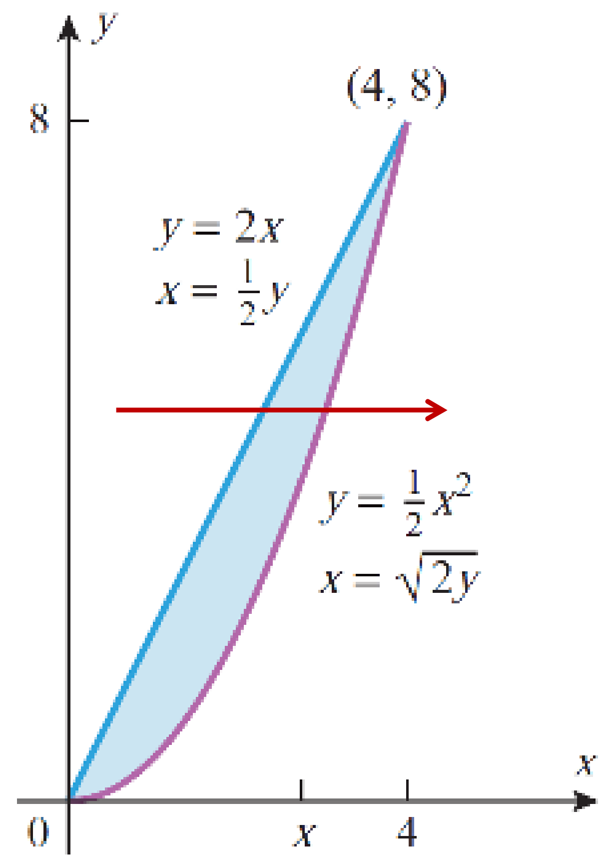
Example Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

$$\text{Area of } R = \iint_R dA$$

(Type II Region)

$$= \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 x\Big]_{y/2}^{\sqrt{2y}} dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{y}{2}\right) dy = \frac{16}{3}$$



AREA CALCULATED AS A DOUBLE INTEGRAL

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA$$

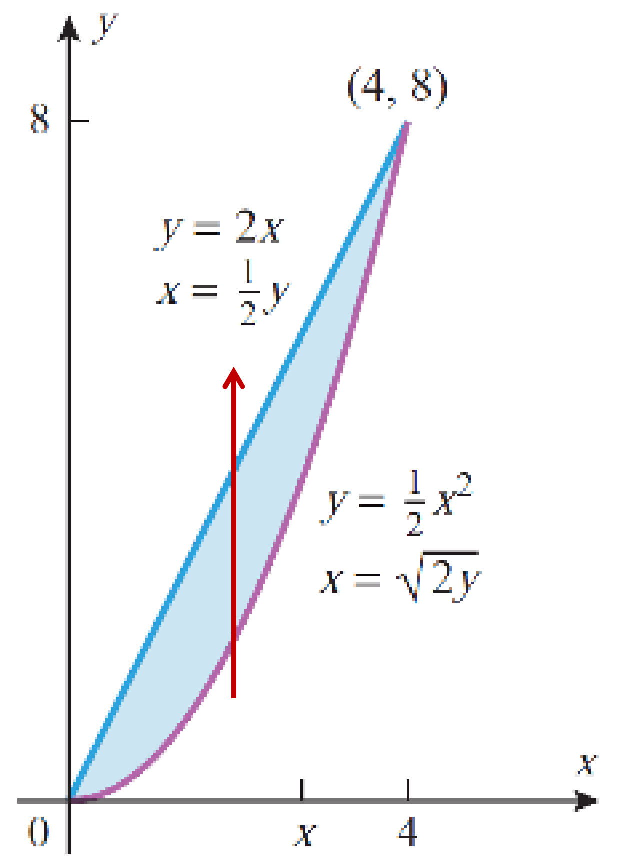
Example Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

$$\text{Area of } R = \iint_R dA$$

(Type I Region)

$$= \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 y]_{x^2/2}^{2x}\, dx$$

$$= \int_0^4 \left(2x - \frac{x^2}{2}\right) dx = \frac{16}{3}$$

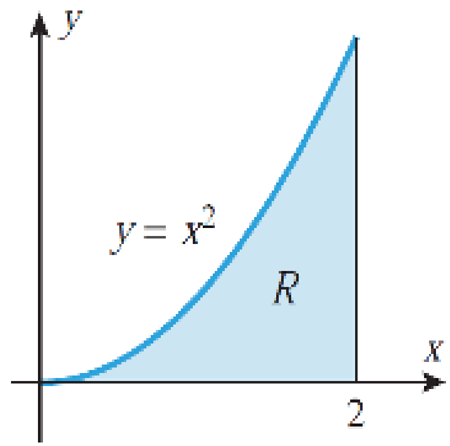


EXERCISE SET 14.2

9. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration.

(a) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dy\, dx$

(b) $\displaystyle\iint_R f(x, y)\, dA = \int_{\square}^{\square} \int_{\square}^{\square} f(x, y)\, dx\, dy$



Course: Calculus (3)

Lecture No: [35]

Chapter: [14]

MULTIPLE INTEGRALS

Section: [14.3]

DOUBLE INTEGRALS IN POLAR COORDINATES

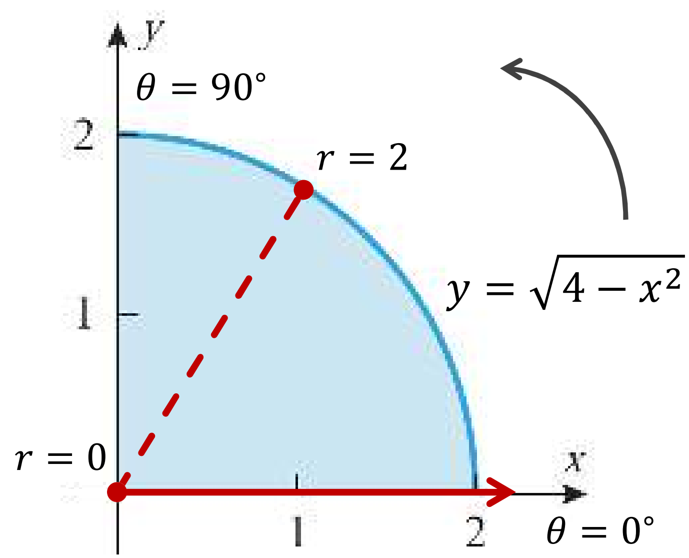
SIMPLE POLAR REGIONS

- Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates.
- This is usually true if the region is bounded by any curve whose equation is simpler in polar coordinates than in rectangular coordinates.
- **Example:** Consider the quarter-disk $x^2 + y^2 = 4$ in the first quadrant shown below.

Rectangular Coordinates

$$0 \leq x \leq 2$$

$$0 \leq y \leq \sqrt{4 - x^2}$$



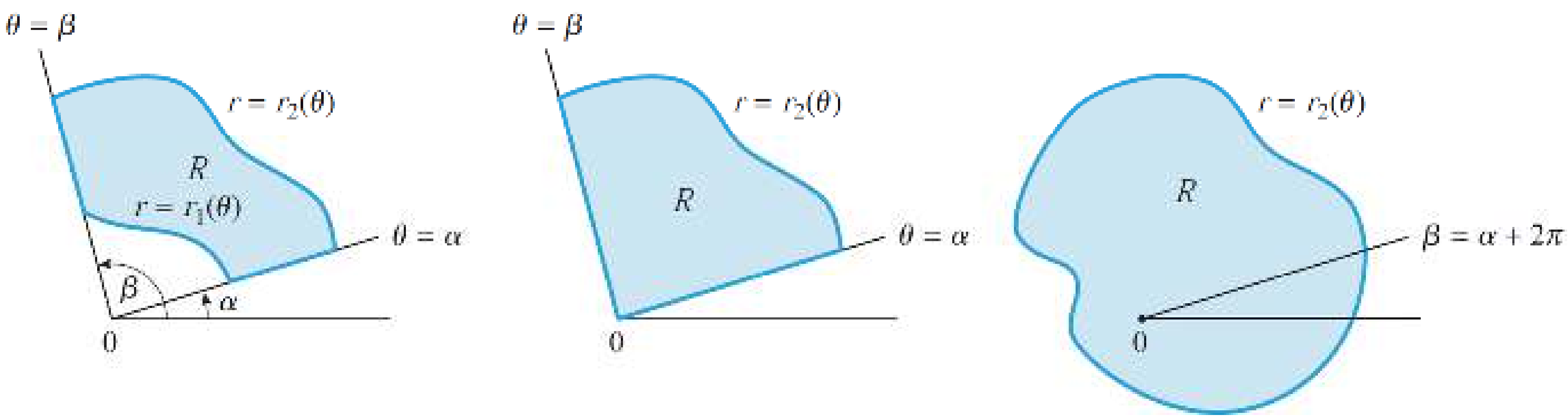
Polar Coordinates

$$0 \leq r \leq 2$$

$$0 \leq \theta \leq \pi/2$$

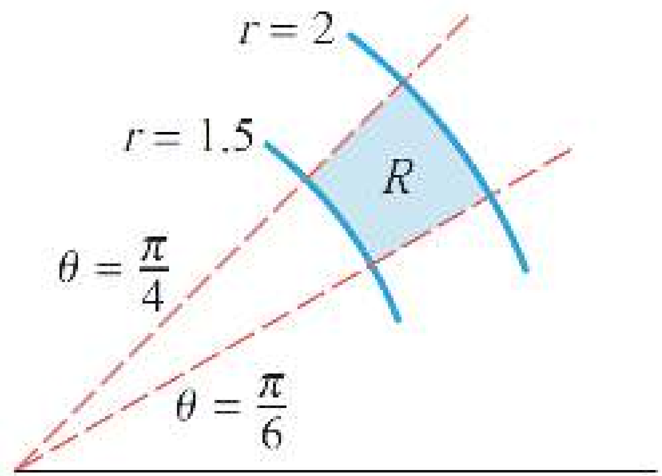
SIMPLE POLAR REGIONS

- Double integrals whose integrands involve $x^2 + y^2$ also tend to be easier to evaluate in polar coordinates because this sum simplifies to r^2 when the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$ are applied.
- The figure below shows a region R in a polar coordinate system that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$.
- If the functions $r_1(\theta)$ and $r_2(\theta)$ are continuous and their graphs do not cross, then the region R is called a *simple polar region*.



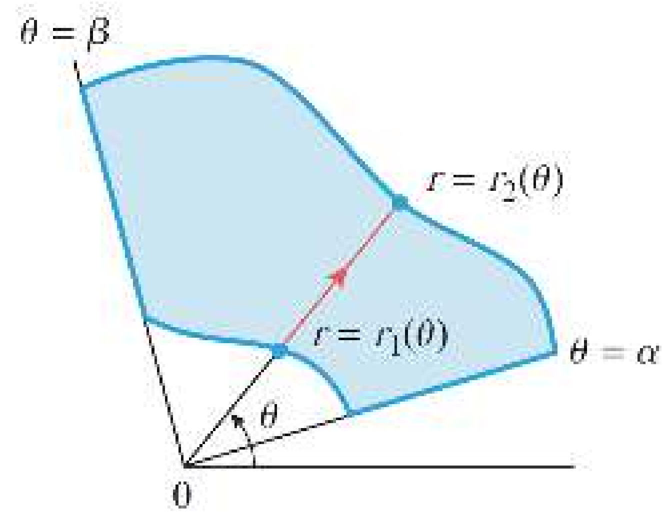
DOUBLE INTEGRALS IN POLAR COORDINATES

NOTE A **polar rectangle** is a simple polar region for which the bounding polar curves are circular arcs.



Theorem If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$, and if $f(r, \theta)$ is continuous on R, then

$$\iint_R f(r,\theta)dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r dr d\theta$$



DOUBLE INTEGRALS IN POLAR COORDINATES

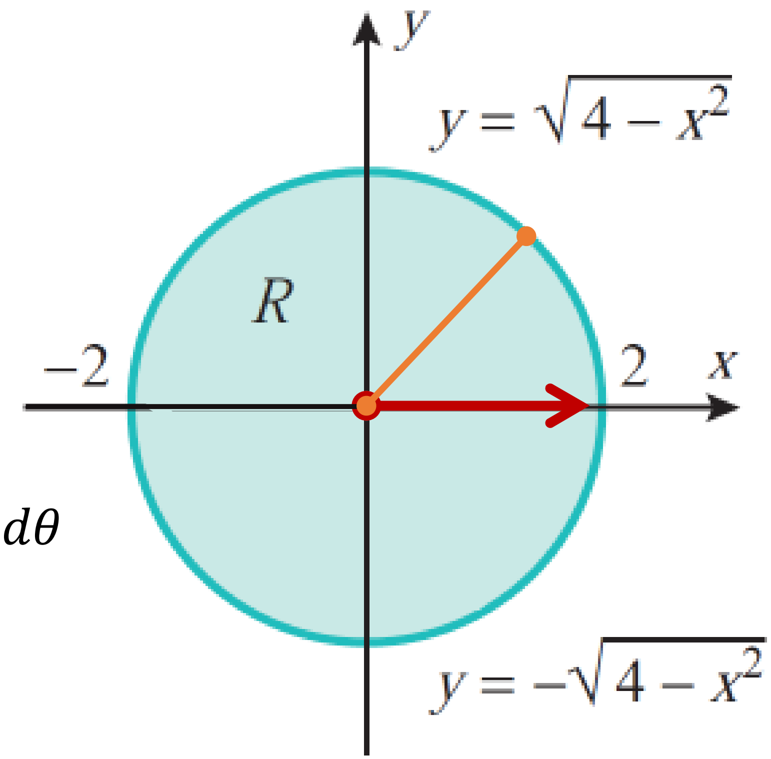
Example Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the plane $y + z = 4$.

$$V = \iint_R (4 - y)dA = \int \int (4 - r\sin\theta)\, r dr d\theta$$

$$= \int_0^{2\pi} \left[\int_0^2 (4r - r^2 \sin\theta)\, dr \right] d\theta$$

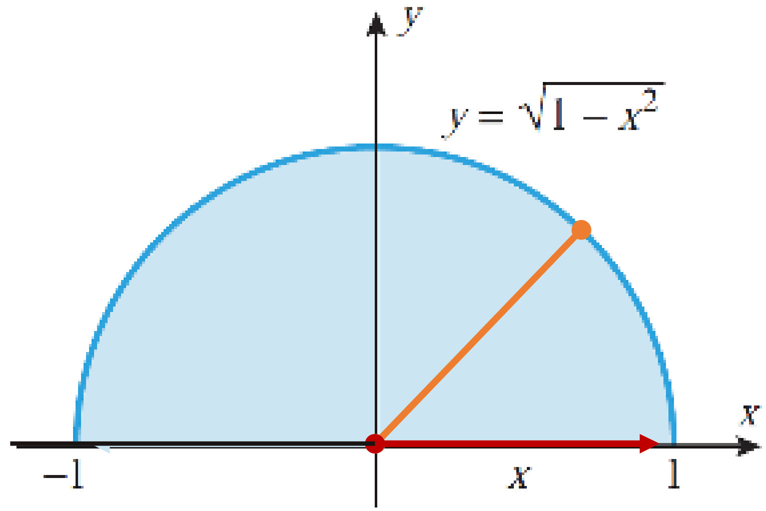
$$= \int_0^{2\pi} \left(2r^2 - \frac{1}{3} r^3 \sin\theta \right)\Big]_0^2 d\theta = \int_0^{2\pi} \left(8 - \frac{8}{3}\sin\theta \right) d\theta$$

$$= \left(8\theta + \frac{8}{3}\cos\theta \right)\Big]_0^{2\pi} = 16\pi$$



DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dydx$

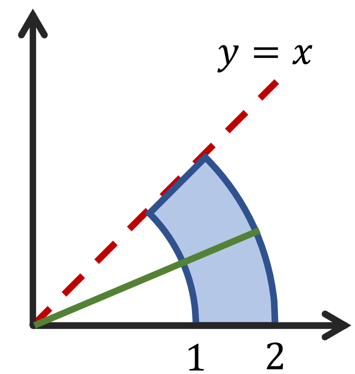


$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\,dydx = \int\int (r^2)^{3/2}\,rdrd\theta$$

$$= \int_{0}^{\pi}\int_{0}^{1} r^4\,drd\theta = \int_{0}^{\pi}\frac{1}{5}\,d\theta = \frac{\pi}{5}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\iint\limits_R \frac{1}{1+x^2+y^2}\,dA$ where R is the region in the first quadrant bounded by $y=0$, $y=x$, $x^2+y^2=1$ and $x^2+y^2=4$.



$$\tan\theta = \frac{y}{x} = \frac{x}{x} = 1$$

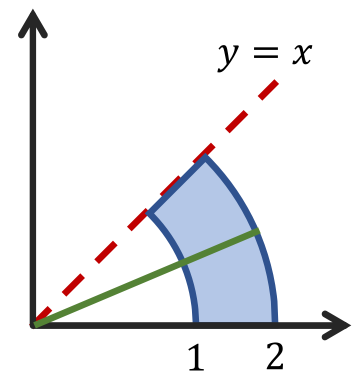
$$\theta = \frac{\pi}{4}$$

$$\iint\limits_R \frac{1}{1+x^2+y^2}\,dA = \int\int \frac{1}{1+r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\left[\int_1^2 \frac{r}{1+r^2}\,dr\right]d\theta$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\iint\limits_R \frac{1}{1+x^2+y^2}\,dA$ where R is the region in the first quadrant bounded by $y = 0$, $y = x$, $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



$$\tan\theta = \frac{y}{x} = \frac{x}{x} = 1$$

$$\theta = \frac{\pi}{4}$$

$$\iint\limits_R \frac{1}{1+x^2+y^2}\,dA = \int \int \frac{1}{1+r^2}\, r\,dr\,d\theta$$

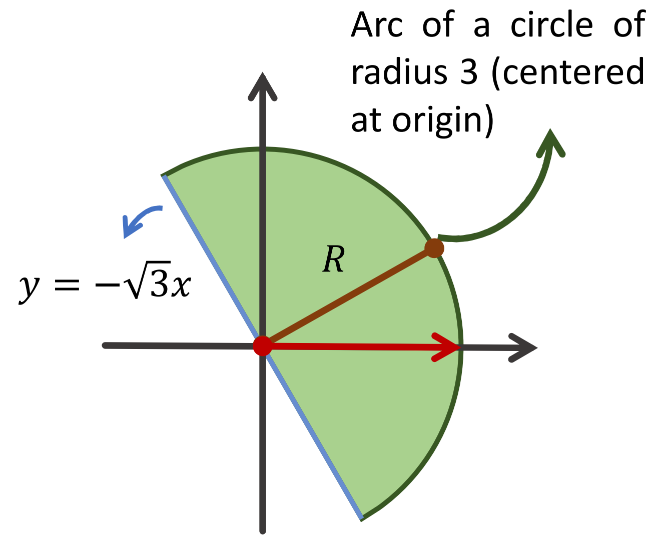
$$= \int_0^{\pi/4} \left[\frac{1}{2} \int_1^2 \frac{2r}{1+r^2}\,dr \right] d\theta = \int_0^{\pi/4} \frac{1}{2} \ln|1+r^2| \Big]_1^2 \, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} \ln\left(\frac{5}{2}\right) d\theta = \frac{\pi}{8} \ln\left(\frac{5}{2}\right)$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Use a double-integral to show that the area of the region R shown is $\frac{9\pi}{2}$.

$$\text{Area of } R = \iint_R dA = \int \int r dr d\theta$$



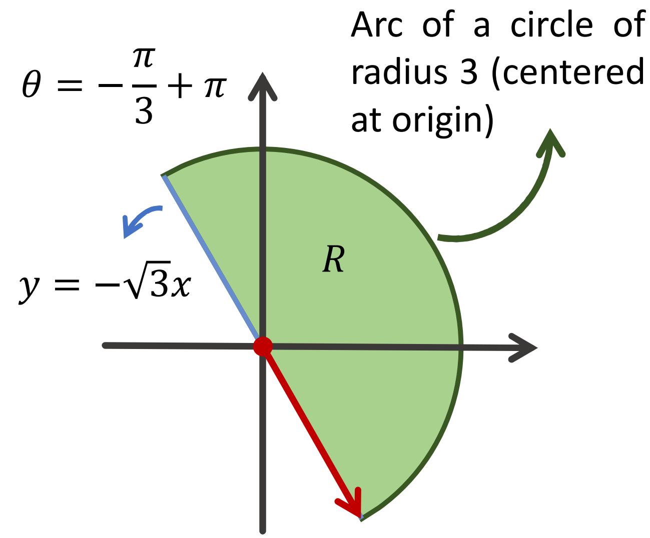
DOUBLE INTEGRALS IN POLAR COORDINATES

Example Use a double-integral to show that the area of the region R shown is $\frac{9\pi}{2}$.

$$\text{Area of } R = \iint\limits_R dA = \int \int_0^3 r dr d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} \left[\int_0^3 r dr \right] d\theta = \int_{-\pi/3}^{2\pi/3} \frac{r^2}{2} \Bigg]_0^3 d\theta$$

$$= \int_{-\pi/3}^{2\pi/3} \frac{9}{2} d\theta = \frac{9}{2}\theta \Bigg]_{-\pi/3}^{2\pi/3} = \frac{9\pi}{2}$$



$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}x}{x} = -\sqrt{3}$$

$$\theta = -\frac{\pi}{3}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Evaluate $\int_0^\infty e^{-x^2}dx = I$

$$I^2 = \left(\int_0^\infty e^{-x^2}dx\right)^2 = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-x^2}dx\right)$$

$$= \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)$$

$$= \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}dxdy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$$

DOUBLE INTEGRALS IN POLAR COORDINATES

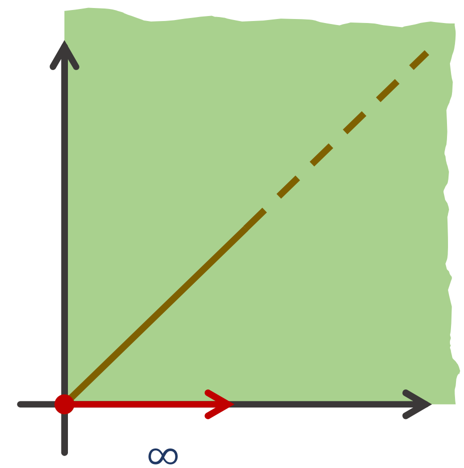
Example Evaluate $\int_0^{\infty} e^{-x^2}dx = I$

$$I^2 = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dxdy = \int \int e^{-r^2} r dr d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{\infty} r\, e^{-r^2}dr\right]d\theta$$

By substitution. Let $t = r^2$.

$$= \int_0^{\pi/2}\left[\int_0^{\infty} \frac{1}{2}e^{-t}dt\right]d\theta = \int_0^{\pi/2} \frac{-1}{2}e^{-t}\Big]_0^{\infty} d\theta = \int_0^{\pi/2} \frac{1}{2}d\theta = \frac{\pi}{4}$$

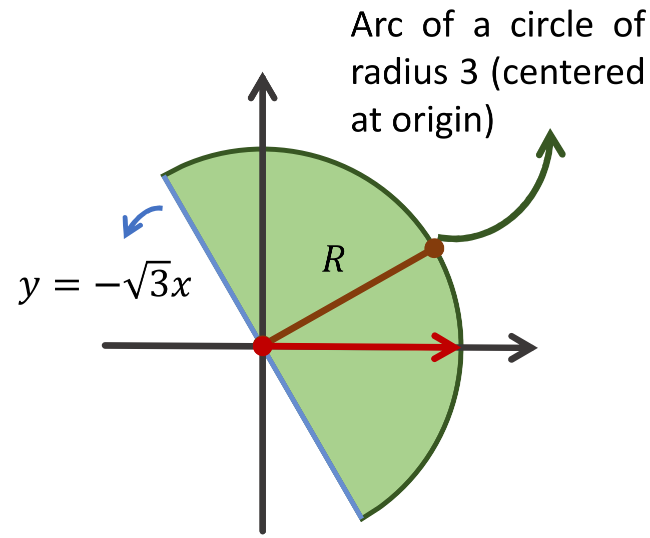


$$\therefore \boldsymbol{I = \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Example Use a double-integral to show that the area of the region R shown is $\frac{9\pi}{2}$.

$$\text{Area of } R = \iint_R dA = \int \int r\,dr\,d\theta$$



DOUBLE INTEGRALS IN POLAR COORDINATES

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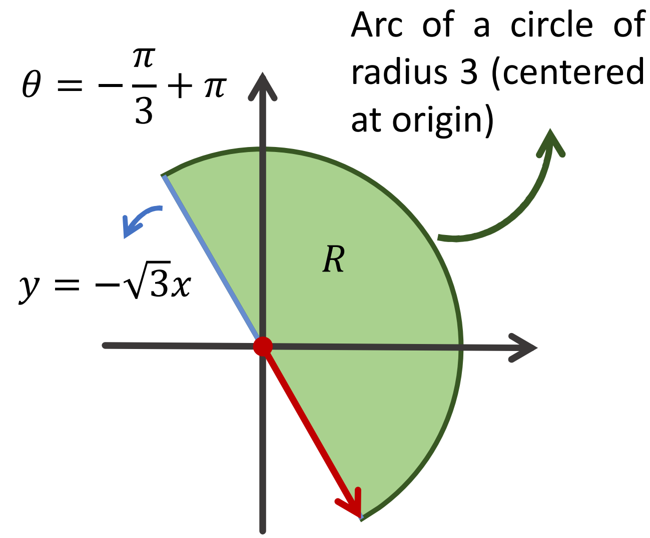

$$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}x}{x} = -\sqrt{3}$$

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DOUBLE INTEGRALS IN POLAR COORDINATES

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$$I^2 = \left(\int_0^\infty e^{-x^2}dx\right)^2 = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-x^2}dx\right)$$

$$= \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)$$

$$= \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}dxdy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$$

DOUBLE INTEGRALS IN POLAR COORDINATES

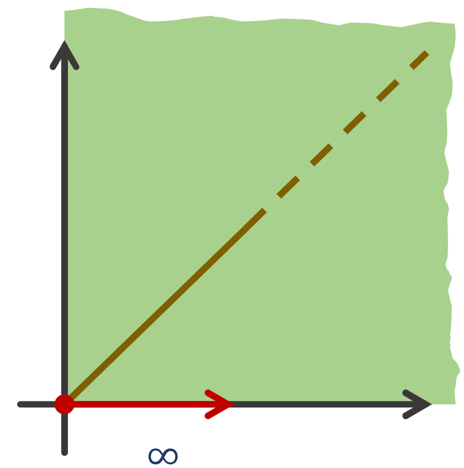
Example Evaluate $\int_0^\infty e^{-x^2}dx = I$

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int \int e^{-r^2} r dr d\theta$$

$$= \int_0^{\pi/2} \left[\int_0^\infty r\, e^{-r^2} dr \right] d\theta$$

By substitution. Let $t = r^2$.

$$= \int_0^{\pi/2} \left[\int_0^\infty \frac{1}{2} e^{-t} dt \right] d\theta = \int_0^{\pi/2} \frac{-1}{2} e^{-t} \Big]_0^\infty d\theta = \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4}$$



$$\therefore \boldsymbol{I} = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Course: Calculus (3)

Lecture No: [36]

Chapter: [14]

MULTIPLE INTEGRALS

Section: [14.5]

Triple Integral [Iterated Method]

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Let G be the rectangular box defined by the inequalities

$$a \leq x \leq b \quad , \quad c \leq y \leq d \quad , \quad k \leq z \leq \ell$$

If f is continuous on the region G, then

$$\iiint_G f(x, y, z)dV = \int_a^b \int_c^d \int_k^\ell f(x, y, z)dzdydx$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Example Evaluate the triple integral $\iiint_G 12xy^2z^3 dV$ over the rectangular box $G = [-1,2] \times [0,3] \times [0,2]$

$$\iiint_G 12xy^2z^3 dV = \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3 dzdydx = \int_{-1}^{2}\int_{0}^{3}\left[\int_{0}^{2} 12xy^2z^3 dz\right] dydx$$

$$= \int_{-1}^{2}\int_{0}^{3} 48xy^2 dydx = \int_{-1}^{2} 432x dx = 648$$

$$\iiint_G 12xy^2z^3 dV = 12\left[\int_{-1}^{2} x dx\right]\left[\int_{0}^{3} y^2 dy\right]\left[\int_{0}^{2} z^3 dz\right] = 648$$

EVALUATING TRIPLE INTEGRALS OVER MORE GENERAL REGIONS

Example Evaluate $\displaystyle\int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} z\,dzdxdy$

$$\int_0^1\int_0^y\int_0^{\sqrt{1-y^2}} z\,dzdxdy = \int_0^1\int_0^y \frac{1}{2}z^2\Big]_0^{\sqrt{1-y^2}} dxdy = \int_0^1\int_0^y \frac{1}{2}(1-y^2)dxdy$$

$$= \int_0^1 \frac{1}{2}(1-y^2)x\Big]_0^y dy = \int_0^1 \frac{1}{2}(1-y^2)ydy$$

$$= \frac{1}{2}\int_0^1 (y-y^3)dy = \frac{1}{8}$$