Philadelphia University
Department of Basic Sciences and Mathematics

First Exam | Problem Solving | Answers

1. (4 points) Use induction to show that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

   Proof : Let $P(n): \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

   1. The statement $P(1)$ is true since $\frac{1}{1\cdot 2}=\frac{1}{1+1}$.
   2. Suppose that $P(j)$ is true.
   3. Want to show that $P(j+1): \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}$. This can be done as follows.

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n+1)(n+2)} &= \frac{n}{n+1}+\frac{1}{(n+1)(n+2)}\\ &= \frac{n(n+2)+1}{(n+1)(n+2)}\\ &= \frac{n^2+2n+1}{(n+1)(n+2)}\\ &= \frac{(n+1)^2}{(n+1)(n+2)}=\frac{n+1}{n+2}\end{aligned}$$

2. (3 points) How many zeros end the number $8^{15}\times 5^{12}$?

   Solution : The number $8^{15}\times 5^{12}$ ends with 12 zeros since

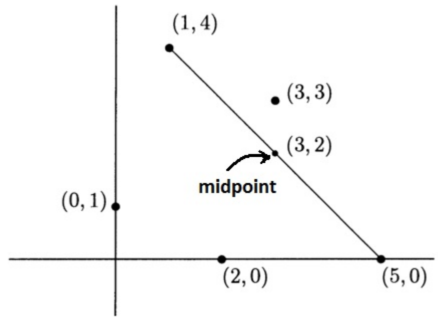
$$8^{15}\times 5^{12}=2^{45}\times 5^{12}=2^{33}\times 2^{12}\times 5^{12}=2^{33}\times 10^{12}$$

3. (3 points) Find all pairs of integers m,n such that $m\times n=m+n$.

   Solution : Write the equation as $n(m-1)=m$. This means that $m-1$ divides m which can only happen if $m=0$ or $m=2$. If $m=0$ then $n=0$, and If $m=2$ then $n=2$. Thus, the only solutions are $m=n=0$ or $m=n=2$.

4. (4 points) A *lattice* point in the plane is an ordered pair $p=(x,y)$ with **integer** coordinates x and y. Given five lattice points p_1,p_2,p_3,p_4,p_5 in the plane, show that the midpoint of the line segment p_ip_j determined by some two distinct lattice points p_i and p_j is also a lattice point.

MR. FERAS AWAD
NOVEMBER 10, 2015



Solution : There are four possibilities for the parities of an ordered pair of integers : (even, even), (even, odd), (odd, even), and (odd, odd). Therefore, using the pigeonhole principle, in any set of five lattice points, there exists two points which agree in parity in both coordinates. Such points determine a lattice midpoint, as the midpoint of the line segment joining them is given by the ordered pair of coordinates averages, and if two numbers agree in parity, then their sum is even and their average is an integer.

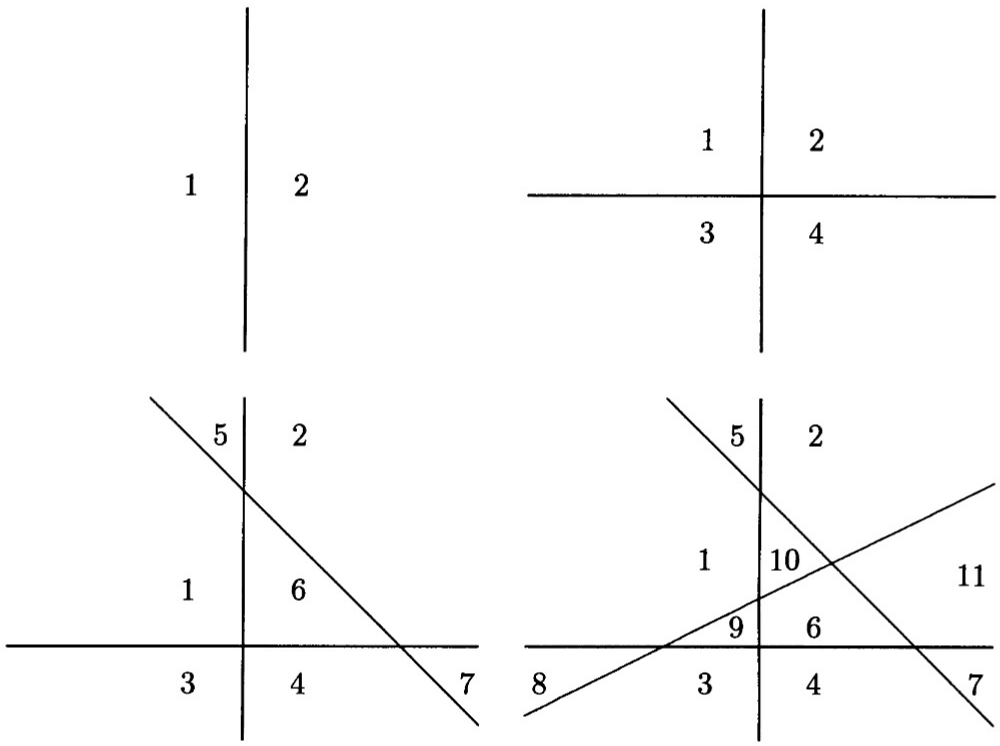
5. (6 points) Suppose that there are n lines in the plane, no two parallel and no three intersecting at a point. Into how many regions is the plane divided by these lines?

   Solution : Let $f(n)$ be the number of regions into which the plane is divided by n lines. With some simple sketches, we see that $f(1) = 2$, $f(2) = 4$, $f(3) = 7$, and $f(4) = 11$. The successive differences of these numbers are $2, 3, 4, \cdots$. Thus it appears that $f(n) - f(n-1) = n$, or, equivalently,

$$f(n) = f(n-1) + n, \text{ for } n > 2.$$

   The formula $f(n) = f(n-1) - n$ is a recurrence relation for $f(n)$. Although this is good, we might also want an explicit formula for $f(n)$, i.e., one that allows us to compute $f(n)$ immediately for any given value of n. Due to its particularly simple form, our recurrence relation can be solved by "working backwards":

$$\begin{aligned} f(n) &= f(n-1) + n \\ &= f(n-2) + (n-1) + n \\ &= f(n-3) + (n-2) + (n-1) + n \\ &\vdots \\ &= f(1) + (2 + 3 + 4 + \cdots + n) = 2 + \frac{n(n+1)}{2} - 1 = \frac{n^2 + n + 2}{2} \end{aligned}$$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