Philadelphia University
Department of Basic Sciences and Mathematics

Second Exam | Problem Solving | Answers

1. Imagine that I have three counters X, Y and Z. They are colored red, white and blue, but not necessarily in this order. One, but only one, of the following statements is true:

   X is red

   Y is not red

   Z is not blue

   Can you work out the colors of the counters?

   Solution: We know that only one of the three statements is true. If we assume that each statement is true in turn we can see if that leads to a contradiction. Let's try.

   If "X is red" is true then "Y is not red" is false, forcing Y is be red — a contradiction since only one counter can be red. Now the second statement.

   If "Y is not red" is true then "Z is not blue" and "X is red" are both false. That forces Z to be blue and since X cannot be red, none of the counters is red — that's also a contradiction, since one of them must be red.

   If "Z is not blue" is true then "Y is not red" is false, making Y a red counter. The statement "X is red" is also false, leading to X being blue and Z being white as the solution.

2. A watermelon weighs 500 pounds. It turns out that 99% of the weight of the watermelon is due to water in the watermelon. After the watermelon has sat in a drying room for a while, it turns out that it is only 98% water by weight. How much does it weigh now?

   Solution: 99% of 500 pounds is 495 pounds of water. So we have 5 pounds that are not water. Once the watermelon has dried out for a while, these 5 pounds represent a 2% of the total weight of the watermelon (since water is 98% of the weight of the watermelon). If 5 pounds is 2%, then the watermelon now weighs 250 pounds.

MR. FERAS AWAD
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3. Given a 3 litre jug and a 5 litre jug what is the best possible way that **minimize water wastage** to measure 73 litres into a container ?

   Solution: Since $5 \times 5 + 16 \times 3 = 73$, then we can measure out 73 litres if we use the 5 litre jug $5 + 3m$ times and the 3 litre jug $16 - 5m$ times.

   Minimizing Water Usage : This will be done if no water is wasted, which requires that $5 + 3m \geq 0$ and $16 - 5m \geq 0$. Hence we need $m \geq -1$ and $m \leq 3$. We then have the following table.

m	-1	0	1	2	3
use of 3 litre jug	21	16	11	6	1
use of 5 litre jug	2	5	8	11	14

   So if we fill the 3 litre jug $16 - 5m$ times and the 5 litre jug $5 + 3m$ times for $m = -1, 0, 1, 2, 3$ and dump the contents in the container, we will produce 73 litres of water. Here then there are 5 best possible ways because we waste no water with any of them.

4. I knock on the door of some old friends whom I have not seen in years. Their daughter Mary answers the door. Mary says "Oh, we are so glad to see you. I'll go into the back room and get my sibling." What is the probability that the sibling is a boy?

   Solution: We assume that a couple has the same likelihood of conceiving a boy or a girl. Write B for boy and G for girl. Then the possible combinations we have are BB, BG, GB, GG. The case BB is ruled out, since they have at least a girl. Thus, the combinations that remain are BG, GB, GG, each with probability 1/3. Now, we see that in two of these combinations there is a boy. Therefore, the probability that her sibling is a boy is 2/3.

5. Examine the equations and then determine the pattern and prove the identity.

$$\begin{aligned} 1 &= 1 \\ 2 + 3 + 4 &= 1 + 8 \\ 5 + 6 + 7 + 8 + 9 &= 8 + 27 \\ 10 + 11 + 12 + \cdots + 16 &= 27 + 64 \end{aligned}$$

Solution: The general pattern is

$$(n^2+1)+(n^2+2)+\cdots+(n+1)^2 = n^3+(n+1)^3.$$

To prove this formula, we will use the summation formula:

$$\sum_{k=M+1}^{N} k = (M+1)+(M+2)+\cdots+N = \frac{(M+N+1)(N-M)}{2}.$$

Therefore,

$$\begin{aligned}(n^2+1)+(n^2+2)+\cdots+(n+1)^2 &= \frac{\left[n^2+1+(n+1)^2\right]\left[(n+1)^2-n^2\right]}{2}\\ &= 1+3n+3n^2+2n^3\\ &= (n+1)^3+n^3\end{aligned}$$