Philadelphia University
Department of Basic Sciences and Mathematics

Final Exam | Problem Solving | Answers

Question ONE : (3 points) Find all integer solutions of $\left(x^2-3x+1\right)^{x+1}=1$.

Solution: The left side is 1 if

1. $x^2-3x+1=1$, and this occurs when $x=0$, $x=3$.
2. $x^2-3x+1=-1$, and $x+1$ is even. This can be happen when $x=1$.
3. $x+1=0$ and $x^2-3x+1\neq 0$ which implies $x=-1$.

$\therefore \qquad x=-1,0,1,3.$

Question TWO : (4 points) Let N is a 4−digit number comprised of the digits 1,2,3,4,5 used at most once each. How many such numbers are there which are multiples of 12 ?

Solution: Since $12\big|N$ and $3\times 4=12$ with $\gcd(3,4)=1$, then $3\big|N$ and $4\big|N$. So, to be divisible by 4, N should be of the form

$$**12 \quad , \quad **24 \quad , \quad **32 \quad , \quad **52$$

Since N is divisible by 3 it can only be

$$4512 \quad , \quad 5412 \quad , \quad 1524 \quad , \quad 5124 \quad , \quad 1452 \quad , \quad 4152$$

Question THREE : (4 points) Let $M=AB4$ and $N=4AB$. Further, N is as much bigger then 400 as M is smaller than 400. What are the numbers M and N ?

Solution: Since $N>400$ and $M<400$ by the same amount then $N-x=400$ and $M+x=400$, which implies $M+N=800$. So,

$$\left.\begin{array}{cccc} & A & B & 4 \\ + & 4 & A & B \\ \hline & 8 & 0 & 0 \end{array}\right\} \Rightarrow A=3 \text{ and } B=6.$$

Question FOUR : (3 points) Ten people sit at a round table. The sum of 10 JD is to be distributed among them so that each person receives the average of what each of his two neighbors receives. In how many different ways can this be achieved?

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Solution: There is one obvious way of distributing the money so that it complies with the conditions of the problem, and that is to divide it equally among all ten people. We show that this is the only way.

Regardless of how we distribute the money, there is always some body who gets the least share. Let us refer to him by P. There might be more than one such person, in which case we pick one and call him P. In fact we demonstrate that all of them must get the same amount as P. Since P has received the average share of his two neighbors, if one of them has a bigger share than P, then the other should have a smaller share. But P was supposed to have received the least share. So both neighbors must have the same share as P. Continuing the same way, we conclude that everybody gets an equal share.

Notice that the amount of money or the number of people are irrelevant for this problem. The key factor is that people are sitting at a round table so that each person has two neighbors.

Question FIVE : (3 points) A function f from the real numbers to the real numbers satisfies the equation $f(x+y) = f(x)+f(y)$. If it is known that $f(1) = 1$, then what is $f\left(\frac{1}{2}\right)$?

Solution: We have

$$1 = f(1) = f\left(\frac{1}{2}+\frac{1}{2}\right) = f\left(\frac{1}{2}\right)+f\left(\frac{1}{2}\right) = 2f\left(\frac{1}{2}\right) \Rightarrow f\left(\frac{1}{2}\right) = \frac{1}{2}$$

Question SIX : (3 points) A herd of cattle invades a barn dance. Suddenly the barn is running amok with both cattle and people. A quick count reveals 120 heads and 300 feet. How many cattle are there, and how many people?

Solution: We write p for the number of people and c for the number of cattle. Then the total number of cattle plus people is $c+p$ while the total number of feet is $4c+2p$ (since a head of cattle has four feet while a person has two). Thus

$$\begin{aligned} c+p &= 120 \\ 4c+2p &= 200 \end{aligned}$$

We solve this system to find that $c = 30$ and $p = 90$.

Question SEVEN : (4 points) A hat contains 1000 slips of paper, each with a different positive integer from 1 to 1000 written on it. Four slips of paper are drawn blind from the hat. What is the probability that the numbers on them occur in increasing order?

Solution: The 4 numbers can be drawn in $4! = 24$ different orderings, and each ordering is equally likely to happen. Therefore the probability that they are drawn in increasing order is 1/24.

Question EIGHT : (5 points) Examine the equations and then determine the pattern and prove the identity.

$$\begin{aligned} 1 &= 1 \\ 3+5 &= 8 \\ 7+9+11 &= 27 \\ 13+15+17+19 &= 64 \\ 21+23+25+27+29 &= 125 \end{aligned}$$

Solution: The pattern is

$$\left[n^2-n+1\right]+\left[n^2-n+3\right]+\left[n^2-n+5\right]+\cdots+\left[n^2-n+(2n-1)\right]=n^3$$

Note that the left hand side has n summands. Thus it can be written as

$$\begin{aligned} &\left[n^2-n+1\right]+\left[n^2-n+3\right]+\cdots+\left[n^2-n+(2n-1)\right] \\ &\qquad = n\cdot n^2-n\cdot n+\left[1+3+5+\cdots+(2n-1)\right] \\ &\qquad = n^3-n^2+n^2 \\ &\qquad = n^3 \end{aligned}$$

Question NINE : (4 points) Evaluate

$$\frac{(4\times 7+2)(6\times 9+2)(8\times 11+2)\cdots(100\times 103+2)}{(5\times 8+2)(7\times 10+2)(9\times 12+2)\cdots(99\times 102+2)}$$

Solution: Note that $n(n+3)+2 = n^2+3n+2 = (n+1)(n+2)$. So,

$$\begin{aligned}\frac{(4\times 7+2)(6\times 9+2)(8\times 11+2)\cdots(100\times 103+2)}{(5\times 8+2)(7\times 10+2)(9\times 12+2)\cdots(99\times 102+2)} &= \frac{(5\times \cancel{6})\cdot(\cancel{7}\times \cancel{8})\cdot(\cancel{9}\times \cancel{10})\cdots(\cancel{101}\times 102)}{(\cancel{6}\times \cancel{7})\cdot(\cancel{8}\times \cancel{9})\cdot(\cancel{10}\times \cancel{11})\cdots(\cancel{100}\times \cancel{101})}\\ &= 5\times 102\\ &= 510\end{aligned}$$

Question TEN : (3 points) Prove that in a group of 13 people at least two have their birthday in the same month.

Solution: Using the pigeonhole principle: let the pigeonholes are the 12 months of the year, and the pigeons are the 13 birthdays. So, there must be at least two birthdays in the same month.

Question ELEVEN : (4 points) Let $x \in \mathbb{R}$, and let

$$A = \frac{-1+3x}{1+x} - \frac{\sqrt{|x|-2}+\sqrt{2-|x|}}{|2-x|}$$

Prove that A is an integer, and find its value.

Solution: Note that $\sqrt{|x|-2}$ is defined if $|x|-2 \geq 0$. Also, $\sqrt{2-|x|}$ is defined if $2-|x| \geq 0$. This implies that $|x| = 2$, and then $x = \pm 2$. Since $x = 2$ makes $|2-x| = 0$, and then A is undefined, it should be $x = -2$. Substitute $x = -2$ to obtain $A = 7 \in \mathbb{Z}$.