

Lecture Notes for Calculus 1

Chapter 0 : Before Calculus

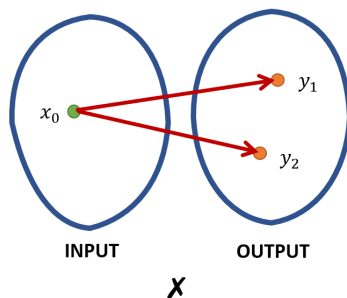
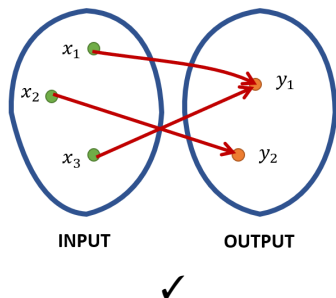
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Idea and Definition

Definition 1

A function f is a rule that associates with each input, a unique (exactly ONE) output.



Idea and Definition

Example 1

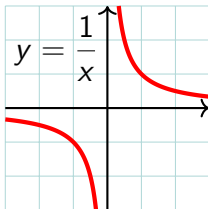
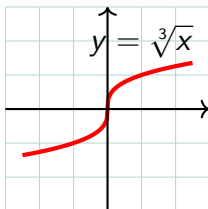
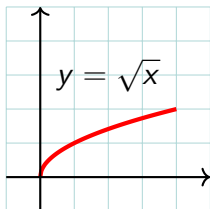
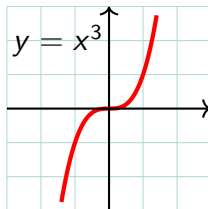
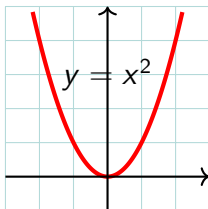
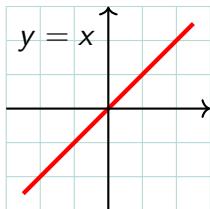
Let $f(x) = 3x^2 - 4x + 2$. Then

$$\begin{aligned} f(-1) &= 3(-1)^2 - 4(-1) + 2 \\ &= 3 + 4 + 2 = 9 \end{aligned}$$

NOTE: There are 4-ways to represent functions:

- Table of values
- Words
- Graph
- Formula

Some Graphs

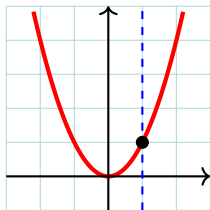


Vertical Line Test

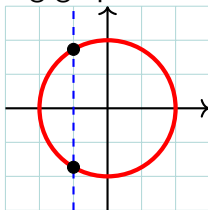
A curve in the xy -plane is the graph of some function f if and only if no vertical line intersects the curve more than once

Example 2

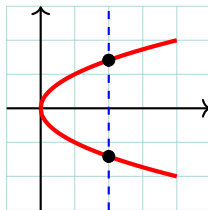
Which of the following graphs is a function?



✓



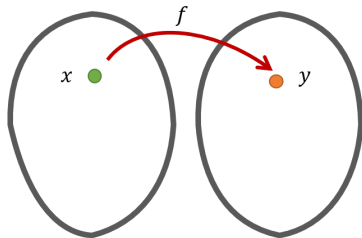
✗



✗

Domain (and Range) of Functions

- The set of all allowable inputs is the **Domain** of f .
- The set of all resulting outputs is the **Range** of f .



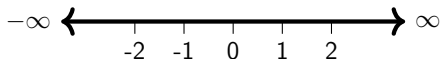
NOTE: If no domain is mentioned, we always assume the largest possible domain, called the **Natural Domain**.

Examples on Function Domain

Example 3

Find the natural domain of the function $f(x) = x^3 - 2x^2 + 1$.

$$\begin{aligned}\text{dom}(f) &= \text{all real numbers} \\ &= (-\infty, \infty) \\ &= \mathbb{R}\end{aligned}$$



NOTE: Any polynomial

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

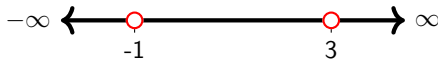
has domain $\mathbb{R}$.

Examples on Function Domain

Example 4

Find the natural domain of the function $f(x) = \frac{1}{(x+1)(x-3)}$.

$$\begin{aligned}\text{dom}(f) &= \text{all real numbers except } -1 \text{ and } 3 \\ &= \mathbb{R} - \{-1, 3\} \\ &= (-\infty, -1) \cup (-1, 3) \cup (3, \infty)\end{aligned}$$



NOTE: For rational functions, avoid division by 0.

Examples on Function Domain

Example 5

Find the natural domain of the function $f(x) = \sqrt{x-1}$.

$$\begin{aligned}\text{dom}(f) &= \text{all real numbers such that } x - 1 \geq 0 \\ &= \text{all } x \in \mathbb{R} \text{ such that } x \geq 1 \\ &= [1, \infty)\end{aligned}$$



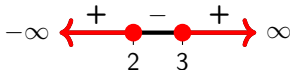
NOTE: Avoid even roots of negative numbers.

Examples on Function Domain

Example 6

Find the natural domain of the function $f(x) = \sqrt{x^2 - 5x + 6}$.

$$\begin{aligned} \text{dom}(f) &= \text{all real numbers such that } x^2 - 5x + 6 \geq 0 \\ & \qquad \qquad \qquad (x - 2)(x - 3) \geq 0 \\ &= (-\infty, 2] \cup [3, \infty) \end{aligned}$$



Examples on Function Domain

Example 7

Find the natural domain of the function $f(x) = \sqrt[3]{4 - 3x^2}$.

$$\begin{aligned}\text{dom}(f) &= \text{all real numbers} \\ &= \mathbb{R}\end{aligned}$$

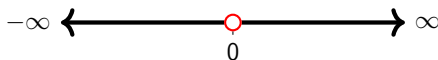
NOTE: Odd root accepts $+$, $-$, 0 but depends on the function inside it.

Examples on Function Domain

Example 8

Find the natural domain of the function $f(x) = \sqrt[3]{\frac{1}{x}}$.

$$\begin{aligned}\text{dom}(f) &= \text{all real numbers except } 0 \\ &= (-\infty, 0) \cup (0, \infty)\end{aligned}$$

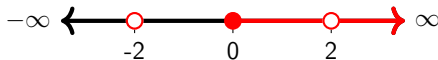


Examples on Function Domain

Example 9

Find the natural domain of the function $f(x) = \frac{\sqrt{x}}{x^2 - 4}$.

$$\begin{aligned}\text{dom}(f) &= \text{all real numbers such that } x \geq 0 \text{ except } \pm 2 \\ &= [0, 2) \cup (2, \infty)\end{aligned}$$



Examples on Function Domain

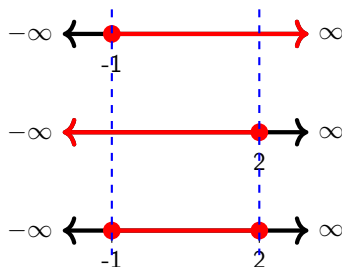
Example 10

Find the natural domain of the function $f(x) = \sqrt{x+1} - \sqrt{2-x}$.

$$\begin{aligned}\text{dom}(\sqrt{x+1}) &= x \in \mathbb{R}; x+1 \geq 0 \\ & \quad x \geq -1 \\ &= [-1, \infty)\end{aligned}$$

$$\begin{aligned}\text{dom}(\sqrt{2-x}) &= x \in \mathbb{R}; 2-x \geq 0 \\ & \quad -x \geq -2 \\ & \quad x \leq 2 \\ &= (-\infty, 2]\end{aligned}$$

$$\therefore \text{dom}(f) = [-1, \infty) \cap (-\infty, 2] = [-1, 2]$$



Think About It

Exercise 1

(1) Compare the domain of $g(x) = x$ and $f(x) = \frac{x^2 + x}{1 + x}$.

(2) Find the domain of each of the following.

a) $f(x) = \frac{\sqrt{x^2 + 2x + 4}}{x}$

b) $g(x) = \frac{x}{x^2 - x - 2}$

c) $f(x) = \frac{\sqrt{9 - x^2}}{1 - x^2}$

d) $g(x) = \frac{x - 3}{1 - \sqrt{x}}$

e) $f(x) = \frac{x - 3}{1 + \sqrt{x}}$

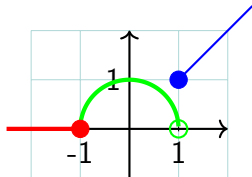
Piecewise Functions

Functions can be defined in pieces (cases).

Example 11

$$\text{Let } f(x) = \begin{cases} 0 & : x \leq -1 \\ \sqrt{1-x^2} & : -1 < x < 1 \\ x & : x \geq 1 \end{cases}$$

$$\begin{aligned} f(0) &= \sqrt{1-0^2} = 1 \\ f(-3) &= 0 \\ f(5) &= 5 \\ f(1) &= 1 \end{aligned}$$



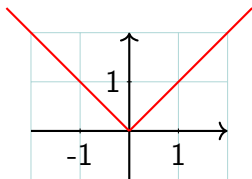
NOTE: $x = -1$ and $x = 1$ are called piecewise points.

Absolute Value

The absolute value of $x \in \mathbb{R}$ is defined by

$$|x| = \begin{cases} x & : x \geq 0 \\ -x & : x < 0 \end{cases}$$

For example, $|4| = 4$, $|0| = 0$, $|\frac{-1}{2}| = \frac{1}{2}$.



NOTE:

- (1) $|x| = a \Leftrightarrow x = \pm a$. **Ex.** $|x| = 4 \Leftrightarrow x = \pm 4$
- (2) $|x| \leq a \Leftrightarrow -a \leq x \leq a$. **Ex.** $|x| \leq 4 \Leftrightarrow -4 \leq x \leq 4$
- (3) $|x| \geq a \Leftrightarrow x \geq a$ or $x \leq -a$. **Ex.** $|x| \geq 4 \Leftrightarrow x \leq -4$ or $x \geq 4$
- (4) $\sqrt{x^2} = |x|$
- (5) $\text{dom}|g(x)| = \text{dom}(g(x))$

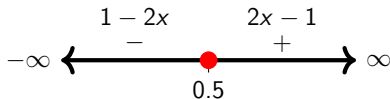
Absolute Value

Example 12

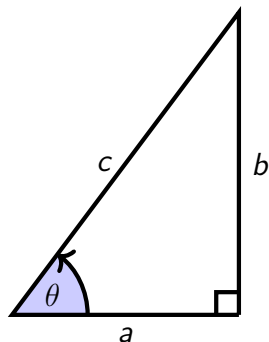
Write the function $g(x) = |2x - 1|$ as piecewise function.

$$\begin{aligned} g(x) &= \begin{cases} 2x - 1 & : 2x - 1 \geq 0 \\ -(2x - 1) & : 2x - 1 < 0 \end{cases} \\ &= \begin{cases} 2x - 1 & : x \geq 1/2 \\ -2x + 1 & : x < 1/2 \end{cases} \end{aligned}$$

$$\begin{aligned} g(x) = 0 &\Rightarrow |2x - 1| = 0 \\ &\Rightarrow 2x - 1 = 0 \\ &\Rightarrow x = 1/2 \end{aligned}$$



Trigonometric Ratios (Right Triangle)



a : adjacent

b : opposite

c : hypotenuse

$$\sin \theta = \frac{b}{c}, \quad \cos \theta = \frac{a}{c}, \quad \tan \theta = \frac{b}{a}$$

$$\sec \theta = \frac{c}{a}, \quad \csc \theta = \frac{c}{b}, \quad \cot \theta = \frac{a}{b}$$

NOTE:

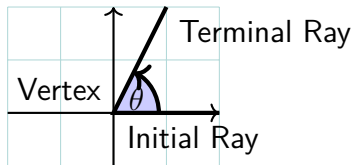
$$\sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

Angle Measures and Standard Position

We measure angles either by degrees or by radians, where

$$\pi \text{ rad} = 180^\circ$$

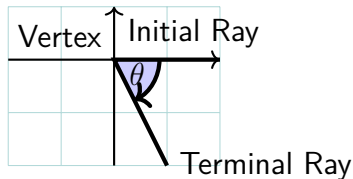


Counterclockwise
 θ is positive

Example 13

$$90^\circ = 90 \times \frac{\pi}{180} = \frac{\pi}{2}$$

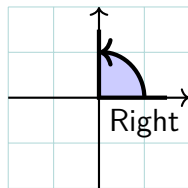
$$\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180}{\pi} = 120^\circ$$



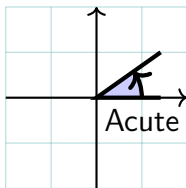
Clockwise
 θ is negative

Angle Measures and Standard Position

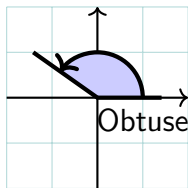
Example 14



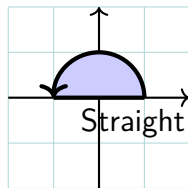
$$90^\circ = \pi/2$$



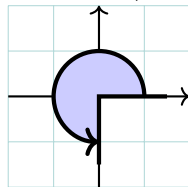
$$30^\circ = \pi/6$$



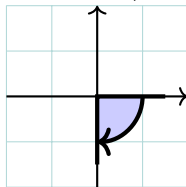
$$150^\circ = 5\pi/6$$



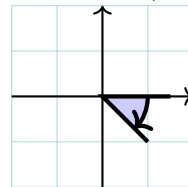
$$180^\circ = \pi$$



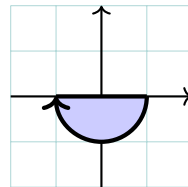
$$270^\circ = 3\pi/2$$



$$-90^\circ = -\pi/2$$



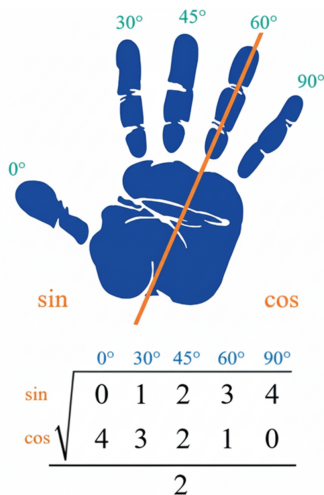
$$-45^\circ = -\pi/4$$



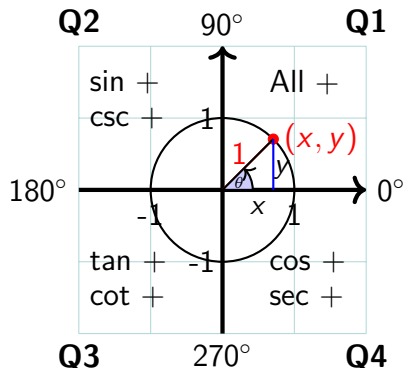
$$-540^\circ = -3\pi$$

Some Trigonometric Values

θ deg	0°	30°	45°	60°	90°
θ rad	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin \theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos \theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan \theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	$\times$
$\csc \theta$	$\times$	2	$\sqrt{2}$	$2/\sqrt{3}$	1
$\sec \theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	$\times$
$\cot \theta$	$\times$	$\sqrt{3}$	1	$1/\sqrt{3}$	0



Trigonometric Values in Coordinate Plane



$$(x, y) = (\cos \theta, \sin \theta)$$

4 Quadrants

θ	Point	sin	cos
0°	(1, 0)	0	1
90°	(0, 1)	1	0
180°	(-1, 0)	0	-1
270°	(0, -1)	-1	0

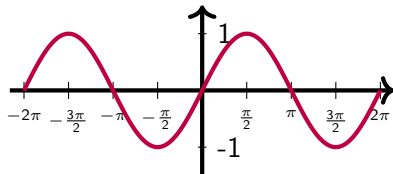
Trigonometric Functions

① $f(x) = \sin x$

Domain : $\mathbb{R}$

Range : $[-1, 1]$

Period : 2π



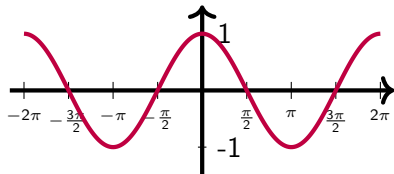
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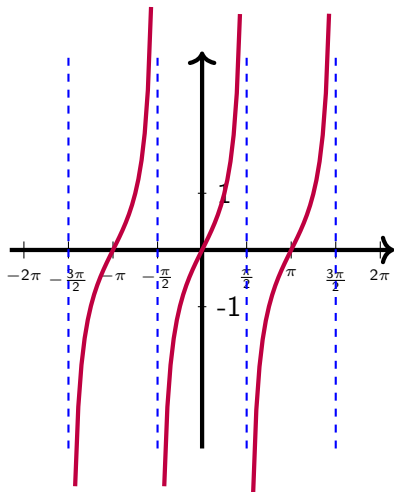
Trigonometric Functions

③ $f(x) = \tan x$

Domain : $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

Range : $\mathbb{R}$

Period : π



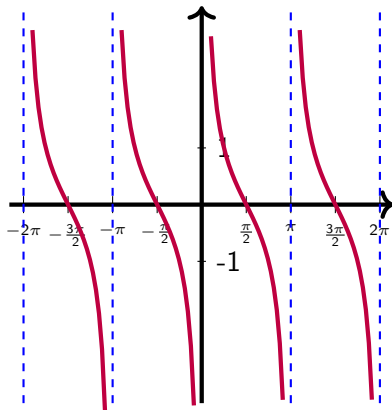
Trigonometric Functions

④ $f(x) = \cot x$

Domain : $x \neq 0, \pm\pi, \pm2\pi, \dots$

Range : $\mathbb{R}$

Period : π



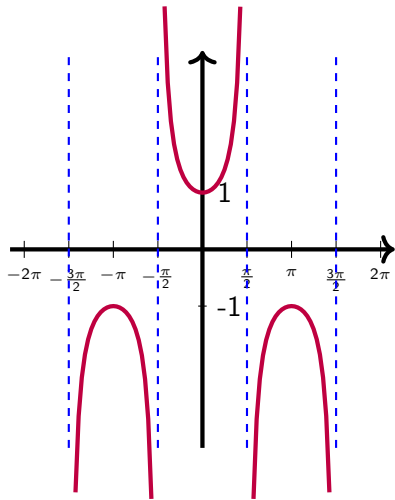
Trigonometric Functions

⑤ $f(x) = \sec x$

Domain : $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$

Range : $(-\infty, -1] \cup [1, \infty)$

Period : 2π



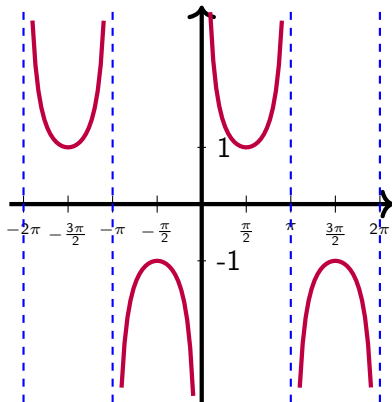
Trigonometric Functions

6 $f(x) = \csc x$

Domain : $x \neq 0, \pm\pi, \pm2\pi, \dots$

Range : $(-\infty, -1] \cup [1, \infty)$

Period : 2π



Trigonometric Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$= 2 \cos^2 \theta - 1$$

$$= 1 - 2 \sin^2 \theta$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

$$\textbf{Ex: } \sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$\tan\left(-\frac{\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$$

Reference Angle

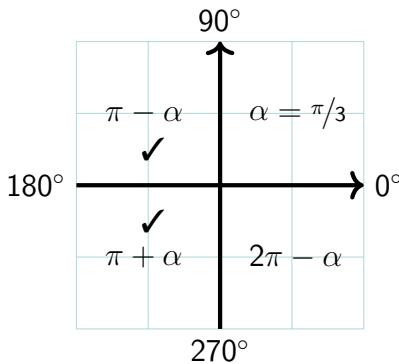
Example 15

Find all angles θ such that $\cos \theta = -1/2$.

$$\theta_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \pm 2n\pi$$

$$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \pm 2n\pi$$

where $n = 0, 1, 2, \dots$



Reference Angle

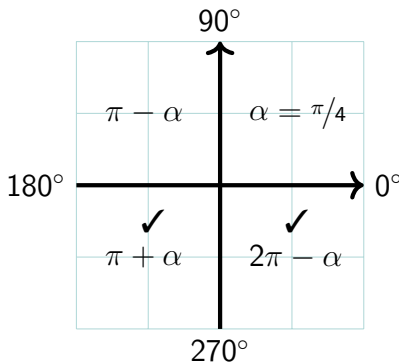
Example 16

Find all angles θ such that $\sin \theta = -1/\sqrt{2}$.

$$\theta_1 = \pi + \frac{\pi}{4} = \frac{5\pi}{4} \pm 2n\pi$$

$$\theta_2 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} \pm 2n\pi$$

where $n = 0, 1, 2, \dots$



Reference Angle

Example 17

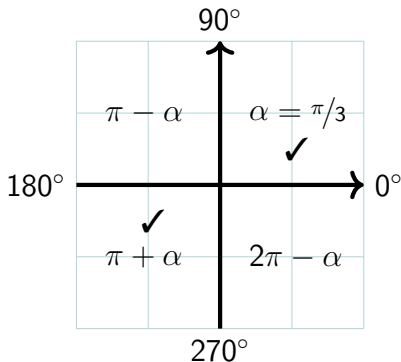
Find all angles θ such that $\tan \theta = \sqrt{3}$.

$$\theta_1 = \frac{\pi}{3} \pm 2n\pi$$

$$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \pm 2n\pi$$

$$\therefore \theta = \frac{\pi}{3} \pm n\pi$$

where $n = 0, 1, 2, \dots$



Reference Angle

Example 18

Find all angles θ such that $\sin \theta = 0$.

$$\theta_1 = 0 \pm 2n\pi$$

$$\theta_2 = \pi \pm 2n\pi$$

$$\therefore \theta = 0 \pm n\pi = \pm n\pi$$

where $n = 0, 1, 2, \dots$

Reference Angle

Example 19

Find all angles θ such that $\cos \theta = 0$.

$$\begin{aligned}\theta_1 &= \frac{\pi}{2} \pm 2n\pi \\ \theta_2 &= \frac{3\pi}{2} \pm 2n\pi \\ \therefore \theta &= \frac{\pi}{2} \pm n\pi\end{aligned}$$

where $n = 0, 1, 2, \dots$

Reference Angle

Example 20

Find all angles θ such that $\cos \theta = -1$.

$$\theta = \pi \pm 2n\pi = (1 \pm 2n)\pi \text{ where } n = 0, 1, 2, \dots$$

Example 21

Find all angles θ such that $\sin \theta = 1$.

$$\theta = \frac{\pi}{2} \pm 2n\pi \text{ where } n = 0, 1, 2, \dots$$

Reference Angle

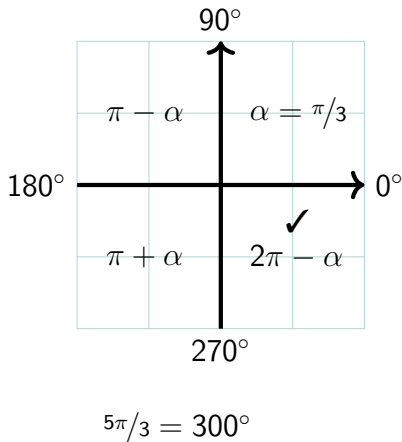
Example 22

Find the value of $\sin\left(\frac{5\pi}{3}\right)$.

$$2\pi - \alpha = \frac{5\pi}{3}$$

$$\alpha = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

$$\therefore \sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$



Reference Angle

Example 23

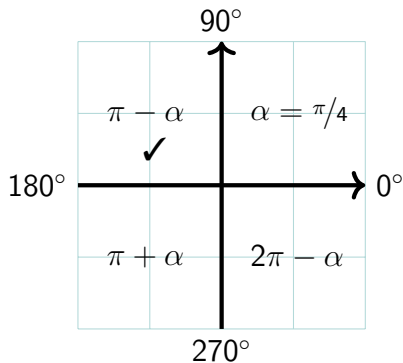
Find the value of

$$\tan\left(\frac{-3\pi}{4}\right) = -\tan\left(\frac{3\pi}{4}\right)$$

$$\pi - \alpha = \frac{3\pi}{4}$$

$$\alpha = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

$$\begin{aligned}\therefore \tan\left(\frac{-3\pi}{4}\right) &= -\tan\left(\frac{3\pi}{4}\right) \\ &= -(-)\tan\left(\frac{\pi}{4}\right) \\ &= 1\end{aligned}$$



$$3\pi/4 = 135^\circ$$

Reference Angle

Example 24

Find the domain of $f(x) = \tan x = \frac{\sin x}{\cos x}$

$$\begin{aligned}\text{dom}(\tan x) &= \mathbb{R} - \{ \cos x = 0 \} \\ &= \mathbb{R} - \{ x = \pi/2 \pm n\pi \} = \text{dom}(\sec x)\end{aligned}$$

Example 25

Find the domain of $f(x) = \csc x = \frac{1}{\sin x}$

$$\begin{aligned}\text{dom}(\csc x) &= \mathbb{R} - \{ \sin x = 0 \} \\ &= \mathbb{R} - \{ x = \pm n\pi \} = \text{dom}(\cot x)\end{aligned}$$

Operations on Functions

Let f and g be functions, then

$$\left(f \overset{\times}{\underset{-}{+}} g\right)(x) = f(x) \overset{\times}{\underset{-}{+}} g(x) ; x \in \text{dom}(f) \cap \text{dom}(g)$$

$$(f \div g)(x) = f(x) \div g(x) ; x \in \text{dom}(f) \cap \text{dom}(g) - \{g(x) = 0\}$$

Example 26

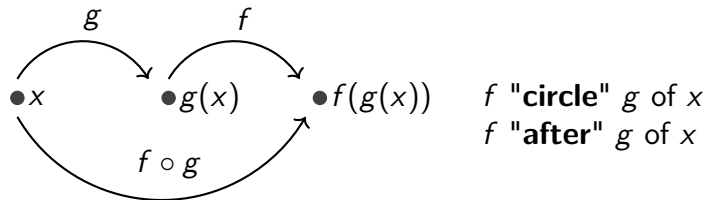
Let $f(x) = \sqrt{x} \rightarrow \text{dom}(f) = [0, \infty)$. Find $(f \cdot g)(x)$ and its domain.

$$g(x) = \sqrt{x} \rightarrow \text{dom}(g) = [0, \infty)$$

$$(f \cdot g)(x) = \sqrt{x} \cdot \sqrt{x} = (\sqrt{x})^2 = x$$

$$\text{dom}(f \cdot g) = [0, \infty) \cap [0, \infty) = [0, \infty)$$

Composition of Functions



Definition 2

$$(f \circ g)(x) = f(g(x))$$

$\text{dom}(f \circ g) = \text{set of all } x \in \text{dom}(g) \text{ such that } g(x) \in \text{dom}(f)$

NOTE: In general, $f \circ g \neq g \circ f$.

Composition of Functions

Example 27

Given that $f(1) = 1$, $f(-1) = 0$, $g(-1) = 1$ and $g(0) = 0$. Find $(f \circ g)(-1)$.

$$(f \circ g)(-1) = f(g(-1)) = f(1) = 1$$

Example 28

Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. then

$$(1) \quad (f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 + 3 = x + 3$$

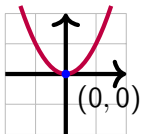
$$\text{dom}(f \circ g) = [0, \infty) \cap \mathbb{R} = [0, \infty)$$

$$(2) \quad (g \circ f)(x) = g(f(x)) = g(x^2 + 3) = \sqrt{x^2 + 3}$$

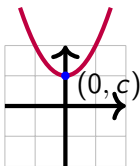
$$\text{dom}(g \circ f) = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$$

How Operations Affect Functions Graphs (Translation)

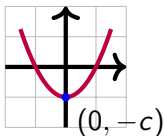
Example
 $f(x) = x^2$



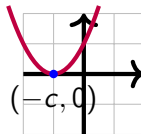
$f(x) + c$
up
 $x^2 + c$



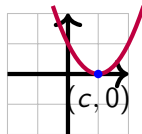
$f(x) - c$
down
 $x^2 - c$



$f(x + c)$
left
 $(x + c)^2$



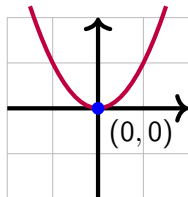
$f(x - c)$
right
 $(x - c)^2$



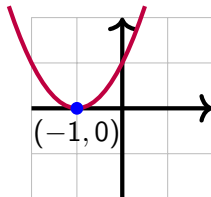
How Operations Affect Functions Graphs (Translation)

Example 29

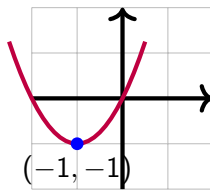
Draw the graph of $f(x) = (x + 1)^2 - 1$.



$$x^2$$



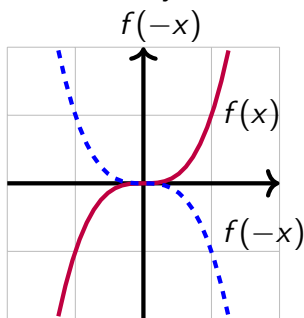
$$(x + 1)^2$$



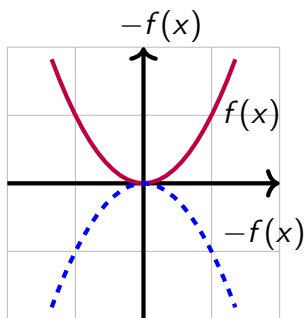
$$(x + 1)^2 - 1$$

How Operations Affect Functions Graphs (Reflection)

About y -axis



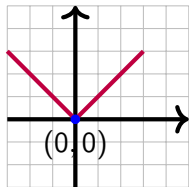
About x -axis



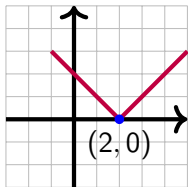
How Operations Affect Functions Graphs (Reflection)

Example 30

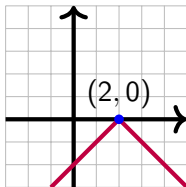
Draw the graph of $f(x) = 4 - |x - 2|$



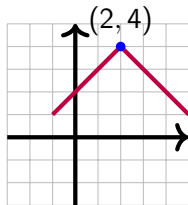
$$|x|$$



$$|x - 2|$$



$$-|x - 2|$$



$$4 - |x - 2|$$

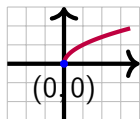
How Operations Affect Functions Graphs (Reflection)

NOTE: Follow this order of transformations:

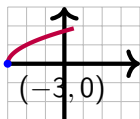
- (1) Horizontal shifts: $f(x \pm c)$
- (2) Reflection: $f(-x)$ and/or $-f(x)$
- (3) Vertical shifts: $f(x) \pm c$

Example 31

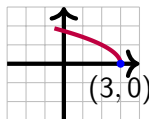
Draw the graph of $f(x) = 2 - \sqrt{3 - x}$



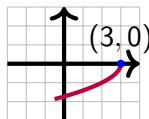
$$\sqrt{x}$$



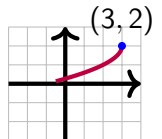
$$\sqrt{x+3}$$



$$\sqrt{-x+3}$$



$$-\sqrt{3-x}$$

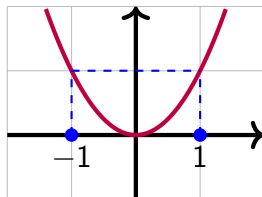
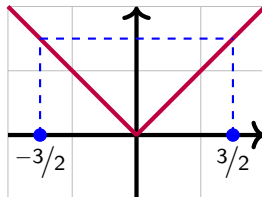
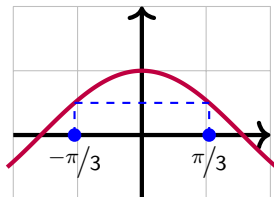


$$2 - \sqrt{3-x}$$

Functions with Symmetric Graphs

Definition 3

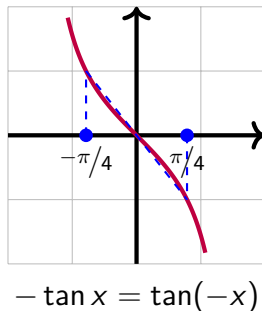
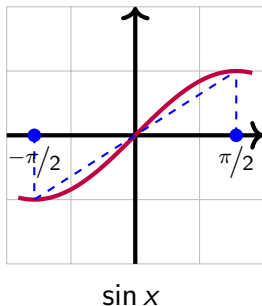
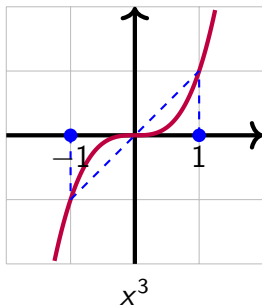
$f(x)$ is an **even function** if $f(-x) = f(x)$. In this case, f is symmetric about the y -axis.


 x^2

 $|x|$

 $\cos x$

Functions with Symmetric Graphs

Definition 4

$f(x)$ is an **odd function** if $f(-x) = -f(x)$. In this case, f is symmetric about the origin.



Think About It

Exercise 2

1. If $f(x)$ is an **odd** function that passes through the point $(-3, 4)$, then which of the following points must also lie on the graph of f ?

(a) $(3, 4)$

(c) $(-3, -4)$

(b) $(3, -4)$

(d) $(4, -3)$

2. If $f(x)$ is an **even** function that passes through the point $(-3, 4)$, then which of the following points must also lie on the graph of f ?

(a) $(3, 4)$

(c) $(-3, -4)$

(b) $(3, -4)$

(d) $(4, 3)$

Functions with Symmetric Graphs

NOTE:

- (1) Some functions are Odd, some are Even, and some neither Odd nor Even like $f(x) = x + 2 \cos x$.
- (2)
- ▶ $\text{Odd} \pm \text{Odd} = \text{Odd}$
 - ▶ $\text{Even} \pm \text{Even} = \text{Even}$
 - ▶ $\text{Odd} \times \text{Odd} = \text{Even}$, $\text{Odd} \div \text{Odd} = \text{Even}$
 - ▶ $\text{Even} \times \text{Even} = \text{Even}$, $\text{Even} \div \text{Even} = \text{Even}$
 - ▶ $\text{Odd} \times \text{Even} = \text{Odd}$, $\text{Odd} \div \text{Even} = \text{Odd}$

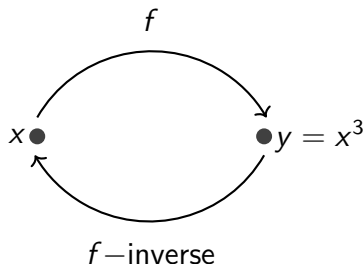
Idea of Inverse Functions

When two functions "UNDO" each other.

The inverse function is denoted by f^{-1} and read f -inverse.

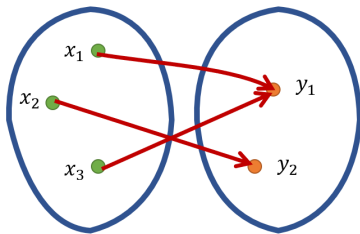
NOTE:

- * $f^{-1} \neq \frac{1}{f}$
- * If $y = x^3$, then $x = \sqrt[3]{y}$
- * The inputs of f , are the outputs of f^{-1} and vice-versa.

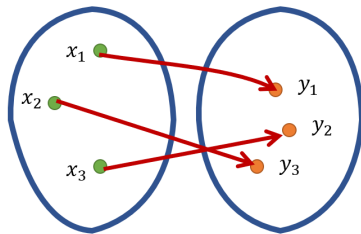


$$\begin{aligned}\text{domain } (f^{-1}) &= \text{range } (f) \\ \text{range } (f^{-1}) &= \text{domain } (f)\end{aligned}$$

When Do the Inverse Exist?



Function but NOT 1 – 1



Function and 1 – 1

A function f has inverse $\Leftrightarrow$ it is one-to-one (1 – 1)

$\Leftrightarrow x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

$\Leftrightarrow$ The graph intersects any horizontal line at most once (*Horizontal Line Test*)

When Do the Inverse Exist?

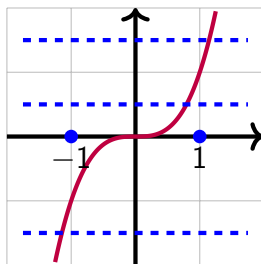
Example 32

Let $f(x) = x^3$.

- * The domain of f is $\mathbb{R}$.
- * If we take $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$, then

$$x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$$

- * So, x^3 is 1-1 on $\mathbb{R}$.

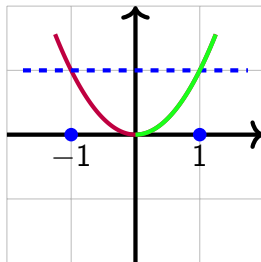


When Do the Inverse Exist?

Example 33

Let $f(x) = x^2$.

- * The domain of f is $\mathbb{R}$.
- * Note that $-1 \neq 1$, but $(-1)^2 = (1)^2$.
- * So, x^2 is **NOT** 1-1 on $\mathbb{R}$.
- * But, x^2 is 1-1 on $[0, \infty)$.



Example 34

$f(x) = |x|$ is not one-to-one on $\mathbb{R}$. **Why?**

Definition of Inverse Function

Definition 5

The functions f , f^{-1} are inverses provided both

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x ; x \in \text{dom}(f^{-1})$$

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x ; x \in \text{dom}(f)$$

Example 35

Let $f(x) = 2x^3 + 5x + 3$. Find x if $f^{-1}(x) = 1$.

$$\begin{aligned} f^{-1}(x) = 1 &\Rightarrow f(f^{-1}(x)) = f(1) \\ &\Rightarrow x = f(1) = 10 \end{aligned}$$

Finding the Inverse Function

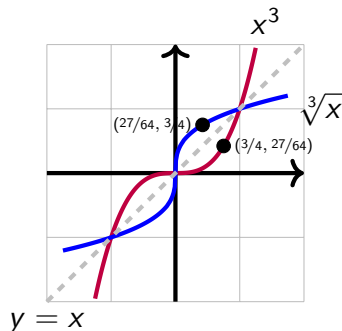
To find the inverse function of $f(x)$:

- (1) Write $y = f(x)$.
- (2) Solve the equation for x as a function of y .
- (3) Replace x by $f^{-1}(x)$, and y by x .

Example 36

Find $f^{-1}(x)$ given that $f(x) = x^3$.

$$\begin{aligned}
 y &= x^3 \\
 \sqrt[3]{y} &= \sqrt[3]{x^3} \\
 x &= \sqrt[3]{y} \\
 f^{-1}(x) &= \sqrt[3]{x}
 \end{aligned}$$

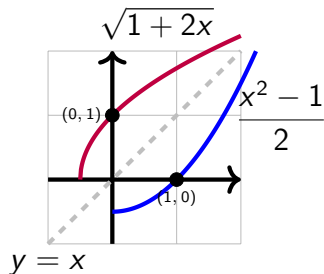


Finding the Inverse Function

Example 37

Find $f^{-1}(x)$ given that
 $f(x) = \sqrt{1 + 2x}$.

$$\begin{aligned}
 y &= \sqrt{1 + 2x} \Rightarrow y^2 = (\sqrt{1 + 2x})^2 \\
 &\Rightarrow y^2 = 1 + 2x \\
 &\Rightarrow x = \frac{y^2 - 1}{2} \\
 &\Rightarrow f^{-1}(x) = \frac{1}{2} (x^2 - 1)
 \end{aligned}$$



Finding the Inverse Function

Example 38

Find $f^{-1}(x)$ given that $f(x) = \frac{x+1}{x-1}$.

$$y = \frac{x+1}{x-1} \Rightarrow (x-1)y = x+1$$

$$\Rightarrow xy - y = x + 1$$

$$\Rightarrow xy - x = y + 1$$

$$\Rightarrow x(y-1) = y+1$$

$$\Rightarrow x = \frac{y+1}{y-1} \Rightarrow f^{-1}(x) = \frac{x+1}{x-1}$$

Finding the Inverse Function

Example 39

Let $f(x) = x^2 - 2x - 3$.

- (1) Draw the graph of f by shifting the graph of x^2 .

$$\begin{aligned}(x^2 - 2x) - 3 &= (x^2 - 2x + 1) - 1 - 3 \\ &= (x - 1)^2 - 4\end{aligned}$$

- (2) Find $f^{-1}(x)$.

$$y = (x - 1)^2 - 4$$

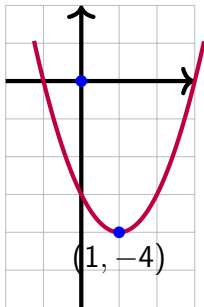
$$x = \sqrt{y + 4} + 1$$

$$f^{-1}(x) = \sqrt{x + 4} + 1$$

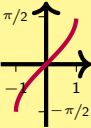
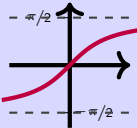
- (3) What is the range of f .

$$\text{range of } f = [-4, \infty)$$

$$= \text{domain of } f^{-1}$$



Inverse Trigonometric Functions

f^{-1}	Domain	Range	Graph	Note
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$		$\arcsin x$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$		$\arccos x$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$		$\arctan x$

Inverse Trigonometric Functions

Example 40

Find the domain of $f(x) = \sin^{-1}(x - 1)$.

$$\begin{aligned}
 -1 \leq x - 1 \leq 1 &\Rightarrow \underset{+1}{-1} \leq \underset{+1}{x-1} \leq \underset{+1}{1} \\
 &\Rightarrow 0 \leq x \leq 2 \Rightarrow \text{dom}(f) = [0, 2]
 \end{aligned}$$

Example 41

Find the exact value.

$$\begin{array}{ll}
 (1) \sin^{-1}(1/2) = \pi/6 \in [-\pi/2, \pi/2] & (3) \tan^{-1}(1) = \pi/4 \in (-\pi/2, \pi/2) \\
 (2) \cos^{-1}(1/2) = \pi/3 \in [0, \pi] & (4) \cos^{-1}(2) \text{ undefined}
 \end{array}$$

Inverse Trigonometric Functions

NOTE:

$$\begin{aligned}
 * \sin^{-1}(-x) &= -\sin^{-1} x \\
 * \tan^{-1}(-x) &= -\tan^{-1} x \\
 * \cos^{-1}(-x) &= \pi - \cos^{-1} x
 \end{aligned}$$

$$\begin{aligned}
 * \sin^{-1}(x) + \cos^{-1}(x) &= \frac{\pi}{2} \\
 * \tan^{-1}(x) &\neq \frac{\sin^{-1}(x)}{\cos^{-1}(x)}
 \end{aligned}$$

Example 42

Find the exact value.

$$(1) \sin^{-1}(-1/2) = -\sin^{-1}(1/2) = -\pi/6 \in [-\pi/2, \pi/2]$$

$$(2) \tan^{-1}(-1) = -\tan^{-1}(1) = -\pi/4 \in (-\pi/2, \pi/2)$$

$$(3) \cos^{-1}(-1/2) = \pi - \cos^{-1}(1/2) = \pi - \pi/3 = 2\pi/3 \in [0, \pi]$$

Inverse Trigonometric Functions

Example 43

Find the exact value.

(1) $\sin(\sin^{-1}(1/4)) = 1/4$

Note that $\text{dom}(\sin^{-1}) = [-1, 1]$ and $1/4 \in [-1, 1]$

Also, $\sin$ and $\sin^{-1}$ are inverses and cancel each other

(2) $\tan(\tan^{-1}(-17/9)) = -17/9$

Note that $\text{dom}(\tan^{-1}) = \mathbb{R}$ and $-17/9 \in \mathbb{R}$

Also, $\tan$ and $\tan^{-1}$ are inverses and cancel each other

(3) $\cos(\cos^{-1}(-2/3)) = -2/3$

Note that $\text{dom}(\cos^{-1}) = [-1, 1]$ and $-2/3 \in [-1, 1]$

Also, $\cos$ and $\cos^{-1}$ are inverses and cancel each other

Inverse Trigonometric Functions

NOTE: $\sin^{-1}(\sin x) = \begin{cases} \pi - x & : \pi/2 \leq x \leq 3\pi/2 \\ x - 2n\pi & : x \geq 3\pi/2 \end{cases}$

$$\cos^{-1}(\cos x) = 2n\pi - x ; \text{ if } x \geq \pi$$

Example 44

Find the exact value.

$$\begin{aligned} (1) \quad \sin^{-1}(\sin(5\pi/3)) &= \sin^{-1}(\sin(5\pi/3 - 2\pi)) = \sin^{-1}(\sin(-\pi/3)) \\ &= -\pi/3 \end{aligned}$$

$$\begin{aligned} (2) \quad \sin^{-1}(\sin(4\pi/3)) &= \sin^{-1}(\sin(\pi - 4\pi/3)) = \sin^{-1}(\sin(-\pi/3)) \\ &= -\pi/3 \end{aligned}$$

$$\begin{aligned} (3) \quad \cos^{-1}(\cos(17\pi/4)) &= \cos^{-1}(\cos(4\pi - 17\pi/4)) \\ &= \cos^{-1}(\cos(-\pi/4)) = \cos^{-1}(\cos(\pi/4)) = \pi/4 \end{aligned}$$

Inverse Trigonometric Functions

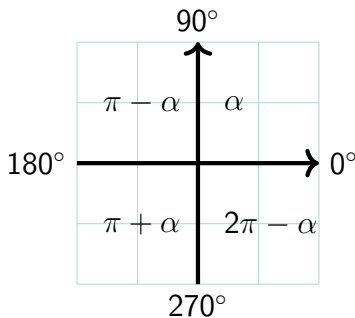
Example 45

Find the exact value of $\tan^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right)$.

$$\pi - \alpha = \frac{5\pi}{6}$$

$$\alpha = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$$

$$\begin{aligned}\tan^{-1}\left(\tan\left(\frac{5\pi}{6}\right)\right) &= -\tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right) \\ &= -\frac{\pi}{6}\end{aligned}$$



Inverse Trigonometric Functions

Example 46

Simplify the expression $\tan(\sin^{-1} x)$.

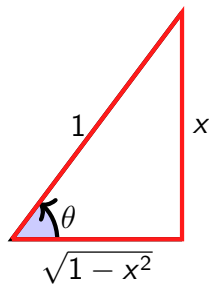
Let $\theta = \sin^{-1} x$

$$\sin \theta = \sin(\sin^{-1} x)$$

$$\sin \theta = x ; x \in [-1, 1]$$

$$\therefore \tan(\sin^{-1} x) = \tan \theta = \frac{x}{\sqrt{1-x^2}}$$

$$\text{where } x \in (-1, 1)$$



Inverse Trigonometric Functions

Example 47

Find the exact value of $\sin(2 \sec^{-1} 3)$.

Let

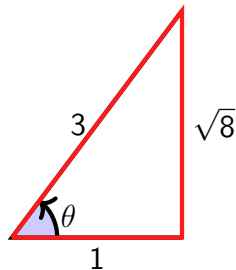
$$\theta = \sec^{-1} 3$$

$$\sec \theta = \sec(\sec^{-1} 3)$$

$$\sec \theta = 3$$

$$\therefore \sin(2 \sec^{-1} 3) = \sin(2\theta) = 2 \sin \theta \cos \theta$$

$$= 2 \times \frac{\sqrt{8}}{3} \times \frac{1}{3} = \frac{4\sqrt{2}}{9}$$



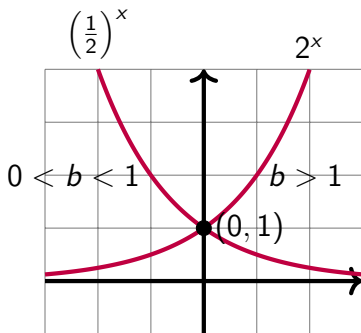
The Exponential Function

Definition 6

The exponential function to the base b is $f(x) = b^x$ where $b > 0$ and $b \neq 1$.

domain of $b^x = \mathbb{R}$

range of $b^x = (0, \infty)$



Example 48

2^x , π^x , $(1/3)^x$, $\dots$ are exponential

x^2 , x^π , $x^{1/3}$, $\dots$ are NOT exponential

The Exponential Function

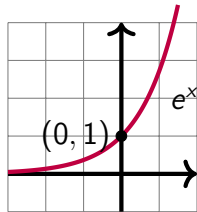
Example 49

Find the domain of the function $f(x) = 4^{\frac{x}{x^2-4}}$.

$$\text{dom}(f) = \mathbb{R} - \{x^2 - 4 = 0\} = \mathbb{R} - \{\pm 2\}$$

NOTE:

- The number $e \approx 2.7182818285 \dots$ is called the **Natural Number**.
- The exponential function $f(x) = e^x$ is called the **Natural Exponential Function**.



The Exponential Function: Properties

$$(1) \quad b^0 = 1$$

$$\text{Ex: } 2^0 = 1$$

$$(2) \quad b^x \cdot b^y = b^{x+y}$$

$$\text{Ex: } 2^4 \cdot 2^3 = 2^7$$

$$(3) \quad b^x \div b^y = b^{x-y}$$

$$\text{Ex: } 3^5 / 3^2 = 3^3$$

$$(4) \quad b^{-x} = 1/b^x$$

$$\text{Ex: } e^{-2} = 1/e^2$$

$$(5) \quad (b^x)^y = b^{xy}$$

$$\text{Ex: } (3^2)^3 = 3^6$$

$$(6) \quad \sqrt[n]{b^m} = b^{m/n}$$

$$\text{Ex: } \sqrt[3]{2^6} = 2^{6/3} = 2^2$$

$$(7) \quad (a \cdot b)^x = a^x \cdot b^x$$

$$\text{Ex: } (2e)^3 = 2^3 \cdot e^3 = 8e^3$$

$$(8) \quad (a \div b)^x = a^x \div b^x$$

$$\text{Ex: } (5/2)^2 = 5^2/2^2$$

$$(9) \quad b^x = b^y \Leftrightarrow x = y$$

One-to-One Property

The Exponential Function: Properties

Example 50

Find the exact value:

$$(1) (-8)^{\frac{2}{3}} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$$

$$(2) 9^{-1/2} = \frac{1}{9^{1/2}} = \frac{1}{\sqrt{9}} = \frac{1}{3}$$

Example 51

Solve the equation $2^x = 64$.

$$2^x = 64 \Rightarrow 2^x = 2^6 \Rightarrow x = 6$$

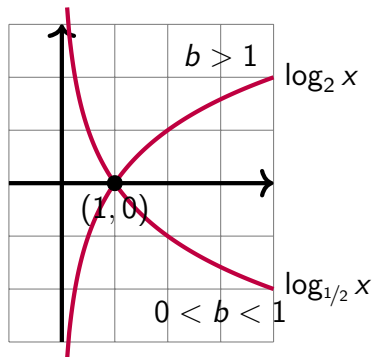
The Logarithmic Function

Definition 7

The logarithmic function to the base b is $f(x) = \log_b x$ where $b > 0$ and $b \neq 1$.

domain of $\log_b x = (0, \infty)$

range of $\log_b x = \mathbb{R}$



NOTE: The function $\log_e x$ is called the **Natural Logarithmic Function** and denoted by $\ln x$.

The Logarithmic Function

NOTE: $\text{dom}(\log_b g(x)) =$ All real numbers such that $g(x) > 0$ and defined.

Example 52

Find the domain of $f(x) = \ln(9 - x^2)$.

$$\text{dom } \ln(9 - x^2) = \text{All } x \in \mathbb{R} \text{ such that } 9 - x^2 > 0$$

$$\Rightarrow x^2 < 9$$

$$\Rightarrow \sqrt{x^2} < \sqrt{9}$$

$$\Rightarrow |x| < 3$$

$$\Rightarrow -3 < x < 3$$

$$\Rightarrow x \in (-3, 3)$$

The Logarithmic Function: Properties

$$(1) \log_b 1 = 0$$

$$\textbf{Ex: } \log_{10} 1 = 0$$

$$(2) \log_b b = 1$$

$$\textbf{Ex: } \ln e = \log_e e = 1$$

$$(3) \log_b (x^n) = n \log_b x$$

$$\textbf{Ex: } \log_3 9 = \log_3 (3^2) = 2$$

$$(4) \log_b a = \frac{\ln a}{\ln b}$$

$$\textbf{Ex: } \log_4 3 = \ln 3 / \ln 4$$

$$(5) \log_b (xy) = \log_b x + \log_b y$$

$$(6) \log_b (x/y) = \log_b x - \log_b y$$

$$(7) \log_b x = \log_b y \Leftrightarrow x = y$$

One-to-One Property

$$(8) y = b^x \Leftrightarrow \log_b y = x$$

The Logarithmic Function: Properties

Example 53

Find the exact value.

$$(1) \log_2 32 = \log_2 (2^5) = 5 \log_2 2 = 5$$

$$(2) \log_4 2 = \log_4 \sqrt{4} = \log_4 (4^{1/2}) = 1/2$$

Example 54

Simplify the expression $\log_6 9 - \log_6 5 + \log_6 20$.

$$\begin{aligned} (\log_6 9 - \log_6 5) + \log_6 20 &= \log_6 \left(\frac{9}{5} \right) + \log_6 20 \\ &= \log_6 \left(\frac{9}{5} \times 20 \right) \\ &= \log_6 36 = \log_6 (6^2) = 2 \end{aligned}$$

Think About It

Exercise 3

1. If $a = \log_{10} 2$ and $b = \log_{10} 3$, find the value of each of the following in terms of a and b :
 - (a) $\log_{10} 6$
 - (b) $\log_{10} 5$
 - (c) $\log_{10} 30$
2. Find the exact value of the following product:

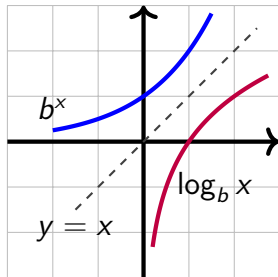
$$\log_2 3 \times \log_3 4 \times \log_4 5 \times \cdots \times \log_{15} 16$$

Logarithmic & Exponential are Inverses

Function	Domain	Range
b^x	$\mathbb{R}$	$(0, \infty)$
$\log_b x$	$(0, \infty)$	$\mathbb{R}$

$$\log_b(b^x) = x ; x \in \mathbb{R}$$

$$b^{\log_b x} = x ; x \in (0, \infty)$$



Example 55

Find the exact value of $5^{2 \log_5 4}$.

$$5^{2 \log_5 4} = 5^{\log_5(4^2)} = 4^2 = 16$$

Logarithmic & Exponential are Inverses

Example 56

Solve the following equations.

$$(1) \quad e^x = 7 \Rightarrow \ln(e^x) = \ln 7 \Rightarrow x = \ln 7$$

$$(2) \quad \log_{10} x = -2 \Rightarrow 10^{\log_{10} x} = 10^{-2} \Rightarrow x = \frac{1}{10^2} = 0.01$$

$$(3) \quad \ln x = 3 \Rightarrow e^{\ln x} = e^3 \Rightarrow x = e^3$$

Logarithmic & Exponential are Inverses

Example 57

Solve the equation $e^{2x} + e^x - 6 = 0$.

$$\begin{aligned}e^{2x} + e^x - 6 = 0 &\Rightarrow (e^x)^2 + e^x - 6 = 0 && (\text{Let } y = e^x) \\&\Rightarrow y^2 + y - 6 = 0 \\&\Rightarrow (y + 3)(y - 2) = 0\end{aligned}$$

$$\begin{aligned}\text{Solution (1)} &\Rightarrow y = -3 \\&\Rightarrow e^x = -3 \quad \times\end{aligned}$$

$$\begin{aligned}\text{Solution (2)} &\Rightarrow y = 2 \\&\Rightarrow e^x = 2 \quad \checkmark \\&\Rightarrow x = \ln 2\end{aligned}$$

Logarithmic & Exponential are Inverses

Example 58

Solve the equation $\ln(x - 2) + \ln(2x - 3) = 2 \ln x$.

$$\begin{aligned}\ln((x - 2)(2x - 3)) &= \ln(x^2) \Rightarrow (x - 2)(2x - 3) = x^2 \\ &\Rightarrow 2x^2 - 7x + 6 = x^2 \\ &\Rightarrow x^2 - 7x + 6 = 0 \\ &\Rightarrow (x - 6)(x - 1) = 0 \\ &\Rightarrow x = 6 \quad \checkmark \quad \text{or} \quad x = 1 \quad \times\end{aligned}$$

Logarithmic & Exponential are Inverses

Example 59

True or False?

“The functions $\ln(x^2)$ and $2\ln x$ have the same domain !!”

$$\text{domain of } \ln(x^2) = \mathbb{R} - \{0\}$$

$$\text{domain of } 2\ln(x) = (0, \infty) \quad \therefore \text{ **FALSE**}$$

Example 60

Find the inverse function of $f(x) = \ln(x + 3)$.

$$y = \ln(x + 3) \Rightarrow e^y = e^{\ln(x+3)}$$

$$\Rightarrow x + 3 = e^y$$

$$\Rightarrow x = e^y - 3 \quad \therefore f^{-1}(x) = e^x - 3$$

Logarithmic & Exponential are Inverses

Example 61

Find the inverse function of $g(x) = e^{2x-1}$.

$$\begin{aligned}y &= e^{2x-1} \Rightarrow \ln y = \ln(e^{2x-1}) \\&\Rightarrow 2x - 1 = \ln y \\&\Rightarrow x = \frac{1 + \ln y}{2} \quad \therefore g^{-1}(x) = \frac{1 + \ln x}{2}\end{aligned}$$

Think About It

Exercise 4

1. Which of the following functions are even and which are odd:

$$(a) f(x) = \frac{a^x - 1}{a^x + 1}$$

$$(b) g(x) = a^x + \frac{1}{a^x}$$

$$(c) h(x) = a^x - \frac{1}{a^x}$$

2. Which elementary function possesses the following properties:

$$f(1) = 0; \quad f(a) = 1; \quad f(xy) = f(x) + f(y)$$

3. Which elementary function possesses the following properties:

$$f(0) = 1; \quad f(1) = a; \quad f(x + y) = f(x)f(y)$$