

Lecture Notes for Calculus 1

Chapter 1 : Limits & Continuity

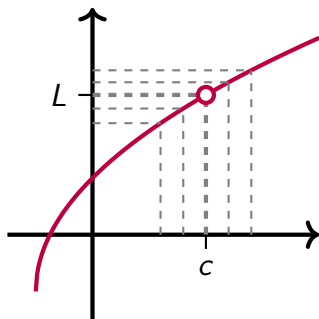
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The Idea of Limits

If $f(x)$ becomes close to a single L as x approaches c (from either sides), we say that “**the limit of $f(x)$ as x approaches to c is L .**”

$$\lim_{x \rightarrow c} f(x) = L$$



NOTE: The value of $f(c)$ has no effect on the value of $\lim_{x \rightarrow c} f(x)$.

Two Sided Limits

- * The limit of $f(x)$ as x approaches “ c ” from left is denoted by $\lim_{x \rightarrow c^-} f(x)$
- * The limit of $f(x)$ as x approaches “ c ” from right is denoted by $\lim_{x \rightarrow c^+} f(x)$

Theorem 1

Let $L \in \mathbb{R}$. We say that $\lim_{x \rightarrow c} f(x)$ **exists** if and only if

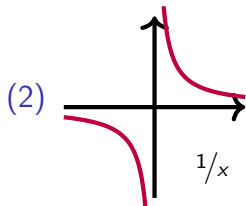
$$\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = L$$

NOTE: $x \rightarrow c^+$ means $x > c$

$x \rightarrow c^-$ means $x < c$

When the Limit Does Not Exist (d.n.e)?

$$(1) \lim_{x \rightarrow c^+} f(x) \neq \lim_{x \rightarrow c^-} f(x)$$



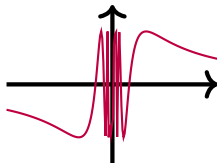
$$\lim_{x \rightarrow c^+} f(x) = \pm\infty$$

$$\lim_{x \rightarrow c^-} f(x) = \pm\infty$$

$$\lim_{x \rightarrow c} f(x) = \pm\infty$$

(3) $f(x)$ oscillating between two fixed values as x approaches to “ c ”.
For example,

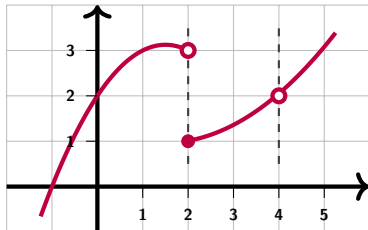
$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) \quad \text{d.n.e}$$



Limits from Function Graph

Example 1

From the given graph of $f(x)$, evaluate:



(1) $f(2) = 1$

(2) $\lim_{x \rightarrow 2^+} f(x) = 1$

(3) $\lim_{x \rightarrow 2^-} f(x) = 3$

(4) $\lim_{x \rightarrow 2} f(x)$ **d.n.e**

(5) $f(4)$ **undefined**

(6) $\lim_{x \rightarrow 4^+} f(x) = 2$

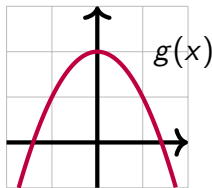
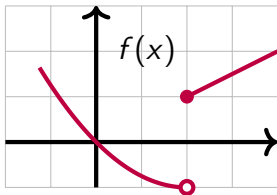
(7) $\lim_{x \rightarrow 4^-} f(x) = 2$

(8) $\lim_{x \rightarrow 4} f(x) = 2$

Think About It

Exercise 1

The figures below show the graphs of $f(x)$ and $g(x)$. Find $\lim_{x \rightarrow 0} f(g(x))$.



(A) 1

(B) -1

(C) 0

(D) Does not exist

Laws of Limits

Theorem 2

Suppose that $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ **exist**, then:

$$(1) \lim_{x \rightarrow a} \left(f \overset{\bullet}{\underset{-}{+}} g \right) = \lim_{x \rightarrow a} f \overset{\bullet}{\underset{-}{+}} \lim_{x \rightarrow a} g$$

$$(2) \lim_{x \rightarrow a} \left(\frac{f}{g} \right) = \frac{\lim_{x \rightarrow a} f}{\lim_{x \rightarrow a} g} \quad ; \quad \lim_{x \rightarrow a} g \neq 0$$

$$(3) \lim_{x \rightarrow a} k = k \quad ; \quad k \text{ is constant}$$

$$(4) \lim_{x \rightarrow a} kf(x) = k \lim_{x \rightarrow a} f(x) \quad ; \quad k \text{ is constant}$$

$$(5) \lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n \quad ; \quad n \text{ is positive integer}$$

$$(6) \lim_{x \rightarrow a} \sqrt[n]{f} = \sqrt[n]{\lim_{x \rightarrow a} f} \quad ; \quad \lim_{x \rightarrow a} f(x) \geq 0 \text{ if } n \text{ is even}$$

Laws of Limits

Example 2

Let $\lim_{x \rightarrow 1/2} f(x) = 2$ and $\lim_{x \rightarrow 1/2} g(x) = -1$, then

$$\lim_{x \rightarrow 1/2} \left(\frac{f(x) - 2g(x)}{f(x)} \right)^2 = \left(\frac{2 - 2(-1)}{2} \right)^2 = 2^2 = 4$$

Example 3

Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{-1}{x}$. Note that $\lim_{x \rightarrow 0} \frac{1}{x}$ d.n.e and $\lim_{x \rightarrow 0} \frac{-1}{x}$ d.n.e, but:

$$\lim_{x \rightarrow 0} (f + g) = \lim_{x \rightarrow 0} \left(\frac{1}{x} + \frac{-1}{x} \right) = \lim_{x \rightarrow 0} 0 = 0$$

Direct Substitution Property

If $f(x)$ is almost any function (**except piecewise**) and $a \in \text{dom}(f)$, then $\lim_{x \rightarrow a} f(x) = f(a)$.

Example 4

Evaluate the following limits.

$$(1) \lim_{x \rightarrow -2} x = -2$$

$$(2) \lim_{x \rightarrow 1} (x^2 + 3x - 1)^3 = 27$$

$$(3) \lim_{x \rightarrow 3} \frac{x+1}{x^2-4} = \frac{4}{5}$$

$$(4) \lim_{x \rightarrow -2} \frac{x+1}{x^2-4} = \frac{-1}{0} \text{ d.n.e}$$

$$(5) \lim_{x \rightarrow -9} \sqrt[3]{x+1} = \sqrt[3]{-8} = -2$$

$$(6) \lim_{x \rightarrow -1} \sqrt[3]{x+1} = \sqrt[3]{0} = 0$$

$$(7) \lim_{x \rightarrow 3} \sqrt{x+1} = 2$$

$$(8) \lim_{x \rightarrow -3} \sqrt{x+1} = \sqrt{-2} \text{ d.n.e}$$

Direct Substitution Property

Example 5

Let $f(x) = \begin{cases} x + 2 & : x \leq -2 \\ x^2 - 5 & : -2 < x < 3. \text{ Find:} \\ \sqrt{x + 13} & : x > 3 \end{cases}$

(1) $\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x^2 - 5) = -4$

(2) $\lim_{x \rightarrow -2} f(x)$ **d.n.e**

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (x + 2) = 0$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} (x^2 - 5) = -1$$

(3) $\lim_{x \rightarrow 3} f(x) = 4$

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x^2 - 5) = 4$$

$$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \sqrt{x + 13} = 4$$

Direct Substitution Property

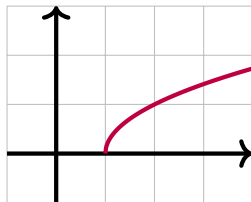
Example 6

Evaluate $\lim_{t \rightarrow 1} \sqrt{t-1} = \sqrt{0} = 0$. **Wait !!**

$$\lim_{t \rightarrow 1^+} \sqrt{t-1} = 0$$

$$\lim_{t \rightarrow 1^-} \sqrt{t-1} \text{ d.n.e}$$

$$\therefore \lim_{t \rightarrow 1} \sqrt{t-1} \text{ d.n.e}$$



$$\text{dom}(\sqrt{t-1}) = [1, \infty)$$

Direct Substitution Property

Example 7

Evaluate $\lim_{t \rightarrow 1} \sqrt{t^2 - 2t + 1} = \sqrt{0} = 0$. **Wait !!**

$$\begin{aligned} \lim_{t \rightarrow 1^+} |t - 1| &= \lim_{t \rightarrow 1} (t - 1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{t \rightarrow 1^-} |t - 1| &= \lim_{t \rightarrow 1} (1 - t) \\ &= 0 \end{aligned}$$

$$\therefore \lim_{t \rightarrow 1} |t - 1| = 0$$

$$\begin{aligned} \sqrt{t^2 - 2t + 1} &= \sqrt{(t - 1)^2} \\ &= |t - 1| \\ &= \begin{cases} t - 1 & : t \geq 1 \\ -(t - 1) & : t < 1 \end{cases} \end{aligned}$$

The Indeterminate Form $0/0$

Example 8

Evaluate $\lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} = \frac{0}{0} \left\{ \begin{array}{l} \text{Direct substitution fails.} \\ \text{The limit may exist or not.} \end{array} \right.$

$$\begin{aligned} \lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3} &= \lim_{x \rightarrow -3} \frac{(x - 3)(x + 3)}{x + 3} \quad (\text{By Factoring}) \\ &= \lim_{x \rightarrow -3} (x - 3) \\ &= -6 \end{aligned}$$

NOTE: $a^2 - b^2 = (a - b)(a + b)$

The Indeterminate Form $0/0$

Example 9

$$\begin{aligned}\text{Evaluate } \lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4x + 4} &= \frac{0}{0} \\ &= \lim_{x \rightarrow 2} \frac{(x + 3)(x - 2)}{(x - 2)(x - 2)} \\ &= \lim_{x \rightarrow 2} \frac{x + 3}{x - 2} \\ &= \frac{5}{0} \quad \mathbf{d.n.e}\end{aligned}$$

The Indeterminate Form $0/0$

Example 10

$$\begin{aligned}
 \text{Evaluate } \lim_{t \rightarrow 2} \frac{t^3 - 8}{t - 2} &= \frac{0}{0} \\
 &= \lim_{t \rightarrow 2} \frac{(t - 2)(t^2 + 2t + 4)}{t - 2} \\
 &= \lim_{t \rightarrow 2} (t^2 + 2t + 4) \\
 &= 12
 \end{aligned}$$

NOTE: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

The Indeterminate Form $0/0$

Example 11

Evaluate $\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} = \frac{0}{0}$ (By Conjugate)

$$= \lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2}$$

$$= \lim_{x \rightarrow 4} \frac{(x - 4)(\sqrt{x} + 2)}{x - 4}$$

$$= \lim_{x \rightarrow 4} (\sqrt{x} + 2) = 4$$

NOTE: $a^2 - b^2 = (a - b)(a + b)$

$$x - 4 = (\sqrt{x} - 2)(\sqrt{x} + 2)$$

The Indeterminate Form $0/0$

Example 12

$$\begin{aligned}
 \text{Evaluate } \lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} &= \frac{0}{0} \\
 &= \lim_{x \rightarrow 0} \frac{\sqrt{x+4} - 2}{x} \cdot \frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2} \\
 &= \lim_{x \rightarrow 0} \frac{(x+4) - 4}{x(\sqrt{x+4} + 2)} \\
 &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+4} + 2} \\
 &= \frac{1}{4}
 \end{aligned}$$

The Indeterminate Form $0/0$

Example 13

$$\begin{aligned}
 \text{Evaluate } \lim_{x \rightarrow -8} \frac{\sqrt[3]{x} + 2}{x + 8} &= \frac{0}{0} \quad \left\{ \begin{array}{l} \text{By substituting } y = \sqrt[3]{x} \Rightarrow x = y^3 \\ \text{If } x \rightarrow -8, \text{ then } y \rightarrow \sqrt[3]{-8} = -2 \end{array} \right. \\
 &= \lim_{y \rightarrow -2} \frac{y + 2}{y^3 + 8} \\
 &= \lim_{y \rightarrow -2} \frac{y + 2}{(y + 2)(y^2 - 2y + 4)} \\
 &= \lim_{y \rightarrow -2} \frac{1}{y^2 - 2y + 4} \\
 &= \frac{1}{12}
 \end{aligned}$$

The Indeterminate Form $0/0$

Example 14

Evaluate $\lim_{x \rightarrow 5} \frac{\frac{1}{x} - \frac{1}{5}}{x - 5} = \frac{0}{0}$

$$\begin{aligned}
 &= \lim_{x \rightarrow 5} \frac{\left(\frac{5-x}{5x}\right)}{x-5} \\
 &= \lim_{x \rightarrow 5} \left(\frac{5-x}{5x} \cdot \frac{1}{x-5} \right) \\
 &= \lim_{x \rightarrow 5} \frac{-1}{5x} \\
 &= \frac{-1}{25}
 \end{aligned}$$

The Indeterminate Form $0/0$

Example 15

Evaluate $\lim_{x \rightarrow 3} \frac{|x - 3|}{x^2 - 9} = \frac{0}{0}$

$$|x - 3| = \begin{cases} x - 3 & : x \geq 3 \\ -(x - 3) & : x < 3 \end{cases}$$

$$\frac{|x - 3|}{x^2 - 9} = \begin{cases} \frac{x-3}{x^2-9} & : x > 3 \\ \frac{-(x-3)}{x^2-9} & : x < 3 \end{cases}$$

$$= \begin{cases} \frac{1}{x+3} & : x > 3 \\ \frac{-1}{x+3} & : x < 3 \end{cases}$$

$$\lim_{x \rightarrow 3^+} \frac{|x - 3|}{x^2 - 9} = \lim_{x \rightarrow 3^+} \frac{1}{x + 3} = \frac{1}{6}$$

$$\lim_{x \rightarrow 3^-} \frac{|x - 3|}{x^2 - 9} = \lim_{x \rightarrow 3^-} \frac{-1}{x + 3} = \frac{-1}{6}$$

$$\therefore \lim_{x \rightarrow 3} \frac{|x - 3|}{x^2 - 9} \text{ d.n.e}$$

The Indeterminate Form $0/0$

Example 16

If $\underbrace{\lim_{x \rightarrow 1} \frac{f(x)}{x-1}} = 5$, find $\underbrace{\lim_{x \rightarrow 1} (2f(x) + x^2)} = 2 \lim_{x \rightarrow 1} f(x) + \lim_{x \rightarrow 1} x^2$

Since $5 \in \mathbb{R}$

then the limit exists

Since $\lim_{x \rightarrow 1} (x-1) = 0$

then $\lim_{x \rightarrow 1} f(x) = 0$

$= 2(0) + 1$

$= 1$

The Indeterminate Form $0/0$

Example 17

Evaluate $\lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{\sqrt[3]{x} - 1} = \frac{0}{0}$ **(By substitution: $x = y^6$)**

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{\sqrt[3]{x} - 1} &= \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{\sqrt[3]{x} - 1} \cdot \frac{\sqrt{x} + 1}{\sqrt{x} + 1} \cdot \frac{(\sqrt[3]{x})^2 + \sqrt[3]{x} + 1}{(\sqrt[3]{x})^2 + \sqrt[3]{x} + 1} \\ &= \lim_{x \rightarrow 1} \frac{x - 1}{x - 1} \cdot \frac{(\sqrt[3]{x})^2 + \sqrt[3]{x} + 1}{\sqrt{x} + 1} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt[3]{x})^2 + \sqrt[3]{x} + 1}{\sqrt{x} + 1} = \frac{3}{2} \end{aligned}$$

NOTE: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$$x - 1 = (\sqrt[3]{x} - 1) \left((\sqrt[3]{x})^2 + \sqrt[3]{x} + 1 \right)$$

The Indeterminate Form $0/0$

Example 18

Let $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & : x \neq 2 \\ 5 & : x = 2 \end{cases}$. Find:

(1) $f(2) = 5$

(2) $\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$

Infinite Limits and Vertical Asymptotes

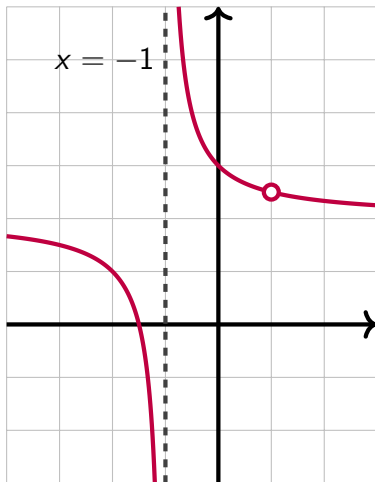
The figure shows the graph of

$$f(x) = \frac{2x^2 + x - 3}{x^2 - 1}.$$

$$\lim_{x \rightarrow -1^-} f(x) = -\infty$$

$$\lim_{x \rightarrow -1^+} f(x) = \infty$$

$\therefore x = -1$ is **Vertical Asymptote**



Infinite Limits and Vertical Asymptotes

Definition 1

We say that $x = a$ is vertical asymptote of $f(x)$ if one of the following holds:

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty$$

or
$$\lim_{x \rightarrow a^-} f(x) = \pm\infty$$

or
$$\lim_{x \rightarrow a} f(x) = \pm\infty$$

Example 19

Find the vertical asymptotes for $f(x) = \frac{x-1}{x^2-1}$.

Since $f(x) = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$, then $x = -1$ is **V.A.**

Infinite Limits and Vertical Asymptotes

Example 20

Find the vertical asymptotes for $f(x) = \frac{2x^3 - 5x + 7}{x^2 + 4x - 5}$.

$$\begin{aligned}\text{denominator} = 0 &\Rightarrow x^2 + 4x - 5 = 0 \\ &\Rightarrow (x - 1)(x + 5) = 0 \\ &\Rightarrow x = 1, x = -5\end{aligned}$$

Now, substitute $x = 1$ in the numerator $\Rightarrow 4 \neq 0$

$x = -5$ in the numerator $\Rightarrow -218 \neq 0$

$\therefore x = 1$ and $x = -5$ are **V.As**

Limits at Infinity and Horizontal Asymptote

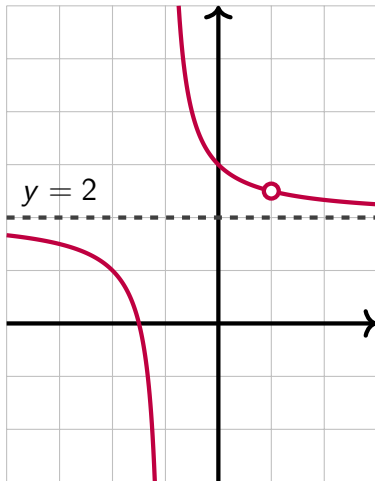
The figure shows the graph of

$$f(x) = \frac{2x^2 + x - 3}{x^2 - 1}.$$

$$\lim_{x \rightarrow \infty} f(x) = 2$$

$$\lim_{x \rightarrow -\infty} f(x) = 2$$

$\therefore y = 2$ is **Horizontal Asymptote**



Limits at Infinity and Horizontal Asymptote

Definition 2

We say that $y = b$ is a horizontal asymptote of $f(x)$ if either $\lim_{x \rightarrow \infty} f(x) = b$ or $\lim_{x \rightarrow -\infty} f(x) = b$.

NOTE:

$$(1) \quad \lim_{x \rightarrow \pm\infty} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) = \lim_{x \rightarrow \pm\infty} a_n x^n$$

$$(2) \quad \lim_{x \rightarrow \pm\infty} k = k \text{ where } k \text{ is constant}$$

$$(3) \quad \lim_{x \rightarrow \pm\infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0} = \begin{cases} \pm\infty & : n > m \\ 0 & : n < m \\ a_n/b_m & : n = m \end{cases}$$

Limits at Infinity and Horizontal Asymptote

Example 21

Find the horizontal asymptotes for $f(x) = \frac{x-1}{x^2-1}$.

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x-1}{x^2-1} &= \lim_{x \rightarrow \infty} \frac{x}{x^2} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} \\ &= \frac{1}{\infty} \\ &= 0\end{aligned}$$

$\therefore y = 0$ is **H.A**

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x-1}{x^2-1} &= \lim_{x \rightarrow -\infty} \frac{x}{x^2} \\ &= \lim_{x \rightarrow -\infty} \frac{1}{x} \\ &= \frac{1}{-\infty} \\ &= 0\end{aligned}$$

$\therefore y = 0$ is **H.A**

Limits at Infinity and Horizontal Asymptote

Example 22

Find the horizontal asymptotes for $f(x) = \frac{2x^3 - 5x + 7}{4x - 3x^2}$.

$$\lim_{x \rightarrow \infty} \frac{2x^3 - 5x + 7}{4x - 3x^2} = \lim_{x \rightarrow \infty} \frac{2x^3}{-3x^2} = \lim_{x \rightarrow \infty} \frac{-2x}{3} = -\infty$$

$\therefore$ There is **NO H.A**

$$\lim_{x \rightarrow -\infty} \frac{2x^3 - 5x + 7}{4x - 3x^2} = \lim_{x \rightarrow -\infty} \frac{2x^3}{-3x^2} = \lim_{x \rightarrow -\infty} \frac{-2x}{3} = \infty$$

$\therefore$ There is **NO H.A**

Limits at Infinity and Horizontal Asymptote

Example 23

Find the horizontal asymptotes for $f(x) = \frac{(2x^2 - 1)(x + 1)}{-4x^3 + 2x^2 - x + 1}$.

$$\lim_{x \rightarrow \infty} \frac{(2x^2 - 1)(x + 1)}{-4x^3 + 2x^2 - x + 1} = \lim_{x \rightarrow \infty} \frac{(2x^2)(x)}{-4x^3} = \lim_{x \rightarrow \infty} \frac{2x^3}{-4x^3} = -\frac{1}{2}$$

$\therefore y = -1/2$ is **H.A**

$$\lim_{x \rightarrow -\infty} \frac{(2x^2 - 1)(x + 1)}{-4x^3 + 2x^2 - x + 1} = \lim_{x \rightarrow -\infty} \frac{(2x^2)(x)}{-4x^3} = \lim_{x \rightarrow -\infty} \frac{2x^3}{-4x^3} = -\frac{1}{2}$$

$\therefore y = -1/2$ is **H.A**

Limits at Infinity and Horizontal Asymptote

Example 24

Find the horizontal asymptotes for $f(x) = \frac{\sqrt{4x^2 + 1}}{3x - 1}$.

$$\lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 + 1}}{3x - 1} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2}}{3x} = \lim_{x \rightarrow \infty} \frac{2|x|}{3x} = \lim_{x \rightarrow \infty} \frac{2x}{3x} = \frac{2}{3}$$

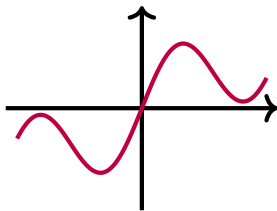
$\therefore y = 2/3$ is **H.A**

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 + 1}}{3x - 1} = \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2}}{3x} = \lim_{x \rightarrow -\infty} \frac{2|x|}{3x} = \lim_{x \rightarrow -\infty} \frac{-2x}{3x} = \frac{-2}{3}$$

$\therefore y = -2/3$ is **H.A**

Continuous Functions

Simply, a function is continuous if it does not have any “breaks” or “holes” in its graph.

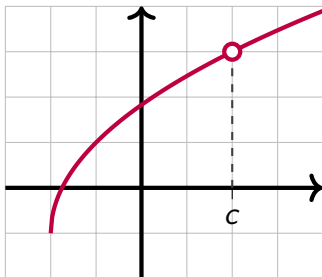


Definition 3

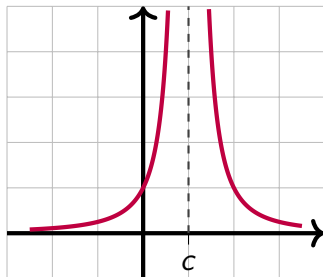
A function $f(x)$ is continuous at $x = c$ if all the following three conditions are hold:

- (1) $f(c)$ is defined
- (2) $\lim_{x \rightarrow c} f(x)$ exists
- (3) $f(c) = \lim_{x \rightarrow c} f(x)$

When $f(x)$ is discontinuous at $x = c$?



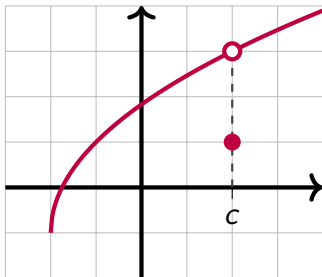
$f(c)$ is not defined



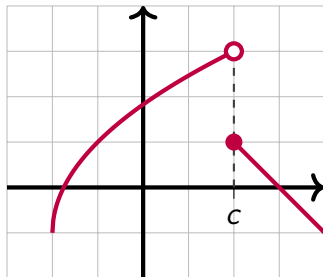
$f(c)$ is not defined

$\lim_{x \rightarrow c} f(x)$ d.n.e

When $f(x)$ is discontinuous at $x = c$?



$$f(c) \neq \lim_{x \rightarrow c} f(x)$$



$$\lim_{x \rightarrow c} f(x) \text{ d.n.e}$$

When $f(x)$ is discontinuous at $x = c$?

Example 25

Determine whether $f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & : x \neq 2 \\ 4 & : x = 2 \end{cases}$ is continuous at $x = 2$?

$$✓ \quad f(2) = 4$$

$$✓ \quad \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = 4$$

$$✓ \quad f(2) = \lim_{x \rightarrow 2} f(x)$$

$\therefore f(x)$ is continuous at $x = 2$

Continuity on Intervals

- * $f(x)$ is continuous on (a, b) or $(-\infty, \infty)$ if f is continuous at each $x = c$ in the interval.
- * $f(x)$ is continuous on $[a, b]$ if
 - ▶ f is continuous on (a, b)
 - ▶ f is continuous at $x = a$ from **right**.

$$\lim_{x \rightarrow a^+} f(x) = f(a)$$
 - ▶ f is continuous at $x = b$ from **left**.

$$\lim_{x \rightarrow b^-} f(x) = f(b)$$

NOTE: In general, every function (**except piecewise**) is continuous on its domain.

Continuity on Intervals

Example 26

Let $f(x) = \frac{x^2 - 9}{x^2 - 5x + 6}$. Where f is discontinuous?

$$f \text{ is discontinuous} \Leftrightarrow x^2 - 5x + 6 = 0$$

$$\Leftrightarrow (x - 2)(x - 3) = 0 \Leftrightarrow x = 2, x = 3$$

Example 27

Let $g(x) = \sqrt{4 - x^2}$. Where g is continuous?

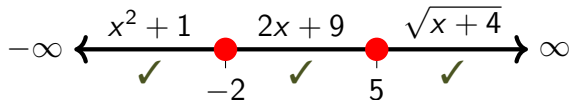
$$g \text{ is continuous} \Leftrightarrow 4 - x^2 \geq 0$$

$$\Leftrightarrow x^2 \leq 4 \Leftrightarrow |x| \leq 2 \Leftrightarrow -2 \leq x \leq 2$$

Continuity on Intervals

Example 28

Let $g(x) = \begin{cases} x^2 + 1 & : x \leq -2 \\ 2x + 9 & : -2 < x < 5 \\ \sqrt{x+4} & : x \geq 5 \end{cases}$. Where g is discontinuous?



$$x = -2 \quad g(-2) = 5 \quad \lim_{x \rightarrow -2^-} g(x) = 5 \quad \lim_{x \rightarrow -2^+} g(x) = 5 \quad \checkmark$$

$$x = 5 \quad g(5) = 3 \quad \lim_{x \rightarrow 5^-} g(x) = 19 \quad \lim_{x \rightarrow 5^+} g(x) = 3 \quad \times$$

$\therefore g(x)$ is discontinuous at $x = 5$.

Continuity on Intervals

Example 29

Find c such that $f(x) = \begin{cases} \frac{x^2 - 9}{x + 3} & : x \neq -3 \\ c^3 & : x = -3 \end{cases}$ is continuous every where.

f continuous every where $\Rightarrow f$ continuous at $x = -3$

$$\Rightarrow f(-3) = \lim_{x \rightarrow -3} f(x)$$

$$\Rightarrow c^3 = \lim_{x \rightarrow -3} \frac{x^2 - 9}{x + 3}$$

$$\Rightarrow c^3 = -6 \Rightarrow c = \sqrt[3]{-6} = -\sqrt[3]{6}$$

Some Theorems

Theorem 3

Suppose f and g are continuous at $x = c$, then:

- (1) $f \pm g$ is continuous at $x = c$.*
- (2) $\frac{f}{g}$ is continuous at $x = c$ if $g(c) \neq 0$.*

Theorem 4

If g is continuous at $x = c$, and f is continuous at $g(c)$, then $f \circ g$ is continuous at $x = c$.

Some Theorems

Theorem 5

The absolute value of continuous function is continuous.

For example, $g(x) = |3x^2 - 4x + 1|$ is continuous on $\mathbb{R}$.

Theorem 6

If $\lim_{x \rightarrow c} g(x) = L$ (exists) and $f(x)$ is continuous at $x = L$, then

$$\lim_{x \rightarrow c} f(g(x)) = f\left(\lim_{x \rightarrow c} g(x)\right)$$

Example 30

$$\lim_{x \rightarrow 2} \log_2 \left(\frac{x^2 - 4}{x - 2} \right) = \log_2 \left(\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \right) = \log_2 4 = 2$$

Think About It

Exercise 2

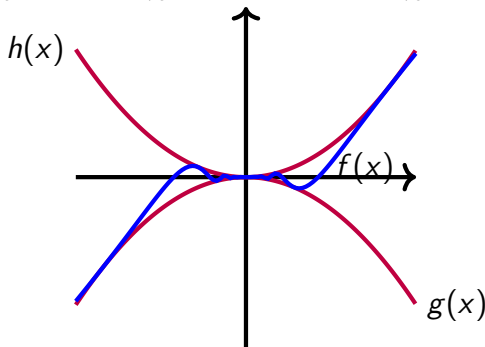
- (1) Find k such that $f(x) = \begin{cases} 7x - 1 & : x \leq 1 \\ kx^2 & : x > 1 \end{cases}$ is continuous on $\mathbb{R}$.
- (2) Find the interval of continuity of $g(x) = \frac{\ln(4 - x^2)}{x}$
- (3) Find the vertical and horizontal asymptotes of $f(x) = \frac{5x^3 + 6x}{x^3 - 4x}$
- (4) For what values of b , the function $f(x) = \frac{1}{x^2 + bx + 4}$ is continuous everywhere?

Squeeze Theorem

Let f , g and h be functions such that

$$g(x) \leq f(x) \leq h(x) ; \forall x \in \text{some interval}$$

If $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, then $\lim_{x \rightarrow c} f(x) = L$.



Squeeze Theorem

Example 31

Given that $f(x)$ satisfies the inequality

$$1 - x^2 \leq f(x) \leq \cos x ; \text{ for all } x \in (-\pi/2, \pi/2)$$

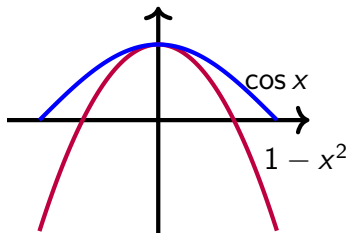
Find $\lim_{x \rightarrow 0} f(x)$

$$\text{Since } \lim_{x \rightarrow 0} (1 - x^2) = 1$$

$$\lim_{x \rightarrow 0} \cos x = 1$$

then by Squeezing Theorem

$$\lim_{x \rightarrow 0} f(x) = 1$$



Squeeze Theorem

Example 32

Use Squeeze Theorem to evaluate $\lim_{x \rightarrow 0} x^2 \sin \left(\frac{1}{x} \right)$.

$$\begin{aligned} -1 \leq \sin x \leq 1 &\Rightarrow -1 \leq \sin \left(\frac{1}{x} \right) \leq 1 \\ &\Rightarrow -x^2 \leq x^2 \sin \left(\frac{1}{x} \right) \leq x^2 \end{aligned}$$

Since $\lim_{x \rightarrow 0} (-x^2) = \lim_{x \rightarrow 0} (x^2) = 0$, then $\lim_{x \rightarrow 0} x^2 \sin \left(\frac{1}{x} \right) = 0$

The Limit of $\sin x/x$

Theorem 7

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sin(ax)}{bx} = \frac{a}{b}$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{x}{\tan x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\tan(ax)}{bx} = \frac{a}{b}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{\tan x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{\sin x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sin(ax)}{\tan(bx)} = \frac{a}{b}$$

Example 33

$$\lim_{x \rightarrow 0} \frac{\tan(2x)}{\sin(3x)} = \frac{2}{3}$$

The Limit of $\sin x/x$

Example 34

$$\lim_{x \rightarrow 0} \frac{\sin^2(2x)}{3x^2} = \frac{1}{3} \lim_{x \rightarrow 0} \frac{\sin^2(2x)}{x^2} = \frac{1}{3} \lim_{x \rightarrow 0} \left(\frac{\sin(2x)}{x} \right)^2 = \frac{2^2}{3} = \frac{4}{3}.$$

Example 35

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan(2x^2)}{3x} &= \lim_{x \rightarrow 0} \frac{\tan(2x^2)}{3x} \cdot \frac{x}{x} = \lim_{x \rightarrow 0} x \cdot \frac{\tan(2x^2)}{3x^2} \\ &= \left(\lim_{x \rightarrow 0} x \right) \left(\lim_{x \rightarrow 0} \frac{\tan(2x^2)}{3x^2} \right) \left\{ \begin{array}{l} \text{Let } y = x^2 \\ \text{If } x \rightarrow 0 \text{ then } y \rightarrow 0 \end{array} \right. \\ &= \left(\lim_{x \rightarrow 0} x \right) \left(\lim_{y \rightarrow 0} \frac{\tan(2y)}{3y} \right) = 0 \times \frac{2}{3} = 0 \end{aligned}$$

The Limit of $\sin x/x$

Example 36

$$\begin{aligned}
 \lim_{x \rightarrow 5} \frac{x-5}{\sin(x^2-25)} &= \lim_{x \rightarrow 5} \frac{x-5}{\sin(x^2-25)} \cdot \frac{x+5}{x+5} \\
 &= \lim_{x \rightarrow 5} \frac{1}{x+5} \cdot \frac{x^2-25}{\sin(x^2-25)} \\
 &= \left(\lim_{x \rightarrow 5} \frac{1}{x+5} \right) \underbrace{\left(\lim_{x \rightarrow 5} \frac{x^2-25}{\sin(x^2-25)} \right)}_{\substack{\text{Let } y=x^2-25 \\ \text{If } x \rightarrow 5, \text{ then } y \rightarrow 0}} \\
 &= \left(\lim_{x \rightarrow 5} \frac{1}{x+5} \right) \left(\lim_{y \rightarrow 0} \frac{y}{\sin(y)} \right) = \frac{1}{10} \times 1 = \frac{1}{10}
 \end{aligned}$$

The Limit of $\sin x/x$

Example 37

$$\lim_{x \rightarrow 0} \frac{\sin(3x) + \tan(4x)}{x^2 + 2x} = \lim_{x \rightarrow 0} \frac{\frac{\sin(3x)}{x} + \frac{\tan(4x)}{x}}{x + 2} = \frac{3 + 4}{0 + 2} = \frac{7}{2}$$

Example 38

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} \\ &= \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right) \left(\lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x} \right) = 1 \times \frac{0}{1 + 1} = 0 \end{aligned}$$

The Limit of $\sin x/x$

Example 39

Find a nonzero constant k that makes the function

$$f(x) = \begin{cases} \frac{\sin(kx)}{x} & : x < 0 \\ 3x + 2k^2 & : x \geq 0 \end{cases} \text{ continuous at } x = 0.$$

$$f \text{ continuous at } x = 0 \Leftrightarrow \lim_{x \rightarrow 0} f(x) \text{ exists}$$

$$\Leftrightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$$

$$\Leftrightarrow \lim_{x \rightarrow 0^+} (3x + 2k^2) = \lim_{x \rightarrow 0^-} \frac{\sin(kx)}{x}$$

$$\Leftrightarrow 2k^2 = k \Leftrightarrow 2k^2 - k = 0$$

$$\Leftrightarrow k(2k - 1) = 0 \Leftrightarrow k = 0 \text{ ✗ or } k = 1/2 \text{ ✓}$$

Think About It

Exercise 3

(1) Suppose that $4 - x^2 \leq \frac{f(x)}{x - 2} \leq e^x + e^{-x} - 6$. Use Squeeze Theorem to evaluate $\lim_{x \rightarrow 0} f(x)$.

(2) Suppose that $x^2 - \frac{x^4}{3} \leq f(x) \leq x^2$ for all $x \in \mathbb{R}$. Use Squeeze Theorem to evaluate $\lim_{x \rightarrow 0} \frac{f(x)}{x^2}$.

Think About It

Exercise 4

Evaluate the following limits

$$(1) \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + 2x - 3}$$

$$(2) \lim_{\theta \rightarrow 0} \frac{2 \sin^2 \theta}{\theta \cos \theta}$$

$$(3) \lim_{x \rightarrow 0} \frac{2 - \cos(3x) - \cos(4x)}{x}$$