Lecture Notes for Calculus 101

Chapter 3 : Integration

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Anti-derivative

QUESTION: What is the function whose derivative is $3x^2$?

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

$$\frac{d}{dx}\left(x^3 - 1\right) = 3x^2$$

$$\frac{d}{dx}\left(x^3 + \sqrt{5}\right) = 3x^2$$

$$\vdots$$

$$\frac{d}{dx}\left(x^3 + C\right) = 3x^2$$

The function $F(x) = x^3 + C$ is called the anti-derivative of $f(x) = 3x^2$.

Anti-derivative

The process of finding anti-derivatives is called **integration**, and can be expressed as integral notation

$$\int f(x)\, dx = F(x) + C$$

NOTE: $\int f(x)\, dx = F(x) + C \Leftrightarrow \frac{d}{dx}F(x) = f(x)$

Example 1

Verify that $\int x \sin x\, dx = \sin x - x\cos x + C$.

$$\begin{aligned} \frac{d}{dx}(\sin x - x\cos x) &= \cos x - (-x\sin x + \cos x) \\ &= \cos x + x\sin x - \cos x = x\sin x \end{aligned}$$

Basic Integration Rules

Rule (1): $\int k\ f(x)\ dx = k \int f(x)\ dx$

Rule (2): $\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

Rule (3): $\int k\ dx = kx + C$

Example 2

$$\int 1\ dx = x + C$$

$$\int \pi\ dt = \pi t + C$$

Basic Integration Rules

Rule (4): $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad ; \quad n \neq -1$$

$$\int (ax+b)^n\,dx = \frac{1}{a}\cdot\frac{(ax+b)^{n+1}}{n+1} + C \quad ; \quad n \neq -1$$

Example 3

(1) $$\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2}\right) + C = \frac{3}{2}x^2 + C$$

(2) $$\int (2x-3)^2\,dx = \frac{(2x-3)^3}{(2)\cdot(3)} + C = \frac{1}{6}(2x-3)^3 + C$$

(3) $$\int \left(x^2 + \frac{1}{x^2}\right)\,dx = \int x^2\,dx + \int x^{-2}\,dx$$

$$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + C = \frac{1}{3}x^3 - \frac{1}{x} + C$$

Basic Integration Rules

Example 4

$$\int (x-1)(x+3)\,dx = \int \left(x^2+2x-3\right)\,dx$$
$$= \frac{x^3}{3} + 2\cdot\frac{x^2}{2} - 3x + C$$
$$= \frac{x^3}{3} + x^2 - 3x + C$$

Example 5

$$\int \sqrt[3]{x^2}\,dx = \int x^{2/3}\,dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}\sqrt[3]{x^5} + C$$

Basic Integration Rules

Example 6

$$\int \frac{x^2 - 3\sqrt{x}}{x}\,dx = \int \left(\frac{x^2}{x} - \frac{3x^{1/2}}{x}\right)\,dx$$

$$= \int \left(x - 3x^{-1/2}\right)\,dx$$

$$= \frac{x^2}{2} - 3\cdot\frac{x^{1/2}}{1/2} + C = \frac{x^2}{2} - 6\sqrt{x} + C$$

Exercise 1

Evaluate $\int \frac{1}{\sqrt{3-2x}}\,dx$

Basic Integration Rules

Rule (5): $$\int \frac{1}{x} = \ln|x| + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Example 7

(1) $$\int \frac{e^x}{9+e^x}\,dx = \ln|9+e^x| + C$$

(2) $$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln\left|x^2+1\right| + C$$

(3) $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

Basic Integration Rules

Rule (6): $\int e^x \, dx = e^x + C \quad ; \quad \int e^{mx+n} \, dx = \frac{1}{m} \cdot e^{mx+n} + C$

$$\int b^x \, dx = \frac{b^x}{\ln b} + C \quad ; \quad \int b^{mx+n} \, dx = \frac{1}{m} \cdot \frac{b^{mx+n}}{\ln b} + C$$

Example 8

(1) $\int \sqrt{e^x} \, dx = \int e^{x/2} \, dx = \frac{e^{x/2}}{1/2} + C = 2\sqrt{e^x} + C$

(2) $\int 2^{8-3x} \, dx = \frac{2^{8-3x}}{-3 \ln 2} + C$

Basic Integration Rules

Rule (7):

$$\begin{aligned}
\int \sin x \; dx &= -\cos x + C \\
\int \cos x \; dx &= \quad \sin x + C \\
\int \sec^2 x \; dx &= \quad \tan x + C \\
\int \csc^2 x \; dx &= -\cot x + C \\
\int \sec x \tan x \; dx &= \quad \sec x + C \\
\int \csc x \cot x \; dx &= -\csc x + C
\end{aligned}$$

Basic Integration Rules

Example 9

(1) $\displaystyle\int \sin(4x-10)\,dx = -\frac{\cos(4x-10)}{4} + C$

(2) $\displaystyle\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)\,dx = \tan x - x + C$

(3) $$\begin{aligned}\int \sec x(\tan x + \cos x)\,dx &= \int (\sec x \tan x + \sec x \cos x)\,dx \\ &= \int \sec x \tan x\,dx + \int 1\,dx \\ &= \sec x + x + C\end{aligned}$$

Basic Integration Rules

Example 10

$$\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) \, dx$$
$$= \frac{1}{2}x - \frac{1}{2} \cdot \frac{\sin(2x)}{2} + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

NOTE: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) \quad ; \quad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Exercise 2

Evaluate the following integrals:

(1) $\displaystyle\int \frac{\sin(2x)}{\cos^2 x} \, dx$

(2) $\displaystyle\int e^{2\ln x} \, dx$

(3) $\displaystyle\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

Basic Integration Rules

Rule (8): (1) $$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{a+bx^2}\,dx = \frac{1}{\sqrt{ab}}\tan^{-1}\left(\sqrt{\frac{b}{a}}x\right) + C$$

(2) $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \frac{1}{\sqrt{a-bx^2}}\,dx = \frac{1}{\sqrt{b}}\sin^{-1}\left(\sqrt{\frac{b}{a}}x\right) + C$$

Basic Integration Rules

Example 11

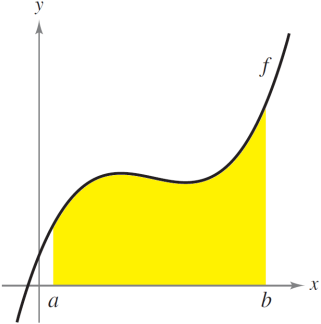
$$(1)\quad \int \frac{1}{25+4x^2}\,dx = \frac{1}{\sqrt{(25)(4)}}\tan^{-1}\left(\sqrt{\frac{4}{25}}x\right)+C$$
$$= \frac{1}{10}\tan^{-1}\left(\frac{2}{5}x\right)+C$$

$$(2)\quad \int \frac{1}{\sqrt{16-9x^2}}\,dx = \frac{1}{\sqrt{9}}\sin^{-1}\left(\sqrt{\frac{9}{16}}x\right)+C$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{3}{4}x\right)+C$$

Area and the Definite Integral

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the $x-$axis, and the vertical lines $x = a$ and $x = b$ is

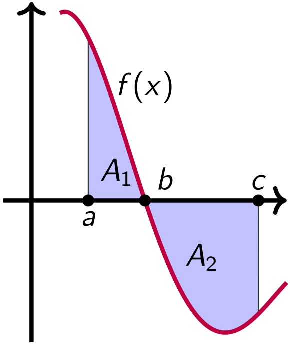
$$\text{Area} = \int_a^b f(x)\,dx$$



Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\,dx$ exists.

Area and the Definite Integral



* $\displaystyle\int_a^b f(x)\,dx = A_1$

* $\displaystyle\int_b^c f(x)\,dx = -A_2$

* $\displaystyle\int_a^c f(x)\,dx = A_1 - A_2$

  $= \textbf{net signed area}$

* **Total area** $= A_1 + A_2$

  $\displaystyle = \int_a^c \left|f(x)\right|\,dx$

Area and the Definite Integral

Example 12

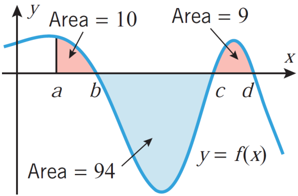
Use the areas shown in the figure to find:

(1) $\int_a^b f(x)\,dx = 10$

(2) $\int_b^c f(x)\,dx = -94$

(3) $\int_a^c f(x)\,dx = 10 - 94 = -84$

(4) $\int_a^d f(x)\,dx = 10 - 94 + 9 = -75$



The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and $F(x)$ is an antiderivative of on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b$$

Example 13

(1) $\displaystyle\int_1^2 x\,dx = \frac{x^2}{2}\Big]_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

(2) $\displaystyle\int_0^{\pi/3} \sec^2 x\,dx = \tan x\Big]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$

The Fundamental Theorem of Calculus

Example 14

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x\Big]_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$$
$$= \sin^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

Exercise 3

(1) *Evaluate* $\displaystyle\int_0^{\ln 3} 5e^x\, dx$

(2) *Evaluate* $\displaystyle\int_{-e}^{-1} \frac{1}{x}\, dx$

Properties of the Definite Integral

* If $a \in \mathrm{dom}(f)$, we define $\int_a^a f(x)\,dx = 0$
* If f is integrable on $[a, b]$, we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
* If f is integrable on a closed interval containing the three points a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

* $f(x) \geq 0\ ;\ \forall x \in \left[a, b\right] \Rightarrow \int_a^b f(x)\,dx \geq 0$

  $f(x) \leq 0\ ;\ \forall x \in \left[a, b\right] \Rightarrow \int_a^b f(x)\,dx \leq 0$

Properties of the Definite Integral

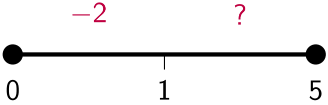
Example 15

Given that $\int_1^0 f(x)\,dx = 2$ and $\int_0^5 f(x)\,dx = 1$, find $\int_1^5 f(x)\,dx$.

$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^5 f(x)\,dx$$

$$1 = -2 + \int_1^5 f(x)\,dx$$

$$\therefore \int_1^5 f(x)\,dx = 3$$



The Substitution Technique

$$\int \overset{\text{Outside function}}{f}\big(\underset{\text{Inside function}}{g(x)}\big)\underbrace{g'(x)}_{\text{Derivative of inside function}}\,dx = F(g(x)) + C$$

By letting $u = g(x)$ gives $du = g'(x)dx$ and $\int f(u)\,du = F(u) + C$

The Substitution Technique

Example 16

Evaluate $\int 2x\left(x^2+1\right)^{50}\,dx$

$$\begin{aligned}\int 2x\left(x^2+1\right)^{50}\,dx &= \int 2x\,(u)^{50}\,\frac{du}{2x}\\ &= \int u^{50}\,du\\ &= \frac{u^{51}}{51}+C\\ &= \frac{1}{51}\left(x^2+1\right)^{51}+C\end{aligned}$$

Let $u = x^2+1$

$$du = 2x\,dx$$

$$dx = \frac{du}{2x}$$

The Substitution Technique

Example 17

Evaluate $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$$\begin{aligned}\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx &= \int \frac{e^{u}}{\sqrt{x}}\,2\sqrt{x}du\\ &= \int 2e^{u}\,du = 2e^{u}+C\\ &= 2e^{\sqrt{x}}+C\end{aligned}$$

$$\begin{aligned}\text{Let } u &= \sqrt{x}\\ du &= \frac{1}{2\sqrt{x}}\,dx\\ dx &= 2\sqrt{x}\,du\end{aligned}$$

Exercise 4

Evaluate $\displaystyle\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\,dx$

The Substitution Technique

Example 18

Evaluate $\displaystyle\int \frac{\left(\tan^{-1} x\right)^3}{1+x^2}\,dx$

$$\begin{aligned}\int \frac{\left(\tan^{-1} x\right)^3}{1+x^2}\,dx &= \int \frac{(u)^3}{1+x^2}\,\left(1+x^2\right)\,du \\ &= \int u^3\,du = \frac{1}{4}u^4 + C \\ &= \frac{1}{4}\left(\tan^{-1} x\right)^4 + C\end{aligned}$$

Let $u = \tan^{-1} x$

$$du = \frac{1}{1+x^2}\,dx$$

$$dx = \left(1+x^2\right)\,du$$

The Substitution Technique

Example 19

Evaluate $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

$$\begin{aligned}
\int \frac{x}{\sqrt{1-x^2}} &= \int \frac{x}{\sqrt{u}}\,\frac{-du}{2x} \\
&= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\
&= -\frac{1}{2}\int u^{-1/2}\,du \\
&= -\frac{1}{2}\frac{u^{1/2}}{1/2} = -\sqrt{u} = -\sqrt{1-x^2}
\end{aligned}$$

$$\begin{aligned}
\text{Let } u &= 1-x^2 \\
du &= -2x\,dx \\
dx &= \frac{-du}{2x}
\end{aligned}$$

The Substitution Technique

Example 20

Evaluate $\displaystyle\int_1^e \frac{(\ln x)^2}{x}\, dx$

$$\begin{aligned}
\int_1^e \frac{(\ln x)^2}{x}\, dx &= \int_0^1 \frac{u^2}{x}\, x du \\
&= \int_0^1 u^2\, du \\
&= \frac{u^3}{3}\bigg]_0^1 \\
&= \frac{1}{3} - 0 = \frac{1}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Let } u &= \ln x \\
du &= \frac{1}{x}\, dx \\
dx &= x\, du \\
\text{If } x = 1 &\Rightarrow u = 0 \\
\text{If } x = e &\Rightarrow u = 1
\end{aligned}$$

The Substitution Technique

Example 21

Evaluate $\int x\sqrt{x-1}\,dx$

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int x\cdot u\cdot 2udu \\ &= \int 2u^2\left(u^2+1\right)\,du \\ &= \int \left(2u^4+2u^2\right)\,du \\ &= 2\cdot\frac{u^5}{5}+2\cdot\frac{u^3}{3}+C \\ &= \frac{2}{5}\left(\sqrt{x-1}\right)^5+\frac{2}{3}\left(\sqrt{x-1}\right)^3+C\end{aligned}$$

$$\begin{aligned}\text{Let } u &= \sqrt{x-1} \\ u^2 &= x-1 \\ x &= u^2+1 \\ 2u\,du &= dx\end{aligned}$$

The Substitution Technique

Example 22

Evaluate $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ **Note:** $e^{2x} = (e^x)^2$

$$\begin{aligned}
\int \frac{e^x}{1+e^{2x}}\,dx &= \int \frac{e^x}{1+(e^x)^2}\,dx \\
&= \int \frac{e^x}{1+u^2}\,\frac{du}{e^x} \\
&= \int \frac{1}{1+u^2}\,du \\
&= \tan^{-1} u + C \\
&= \tan^{-1}(e^x) + C
\end{aligned}$$

$$\begin{aligned}
\text{Let } u &= e^x \\
du &= e^x\,dx \\
dx &= \frac{du}{e^x}
\end{aligned}$$

The Substitution Technique

Exercise 5

Evaluate

(1) $\displaystyle \int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

(2) $\displaystyle \int \frac{1}{x\left(1+(\ln x)^2\right)}\,dx$

(3) $\displaystyle \int \frac{\ln\left(\sqrt[3]{x^2}\right)}{x}\,dx$

(4) $\displaystyle \int x^4\,\sqrt[3]{3-5x^5}\,dx$

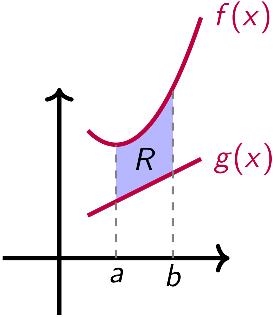
Exercise 6

Given that $\displaystyle \int_0^4 f(x)\,dx = 1$, *find* $\displaystyle \int_{-2}^0 x\,f\left(x^2\right)\,dx$

Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of R is given by

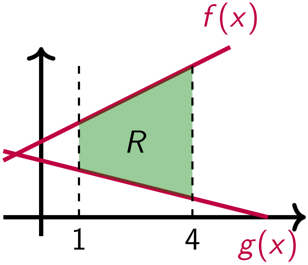
$$\int_a^b \left[f(x) - g(x) \right] \, dx$$



Finding the Area between Two Curves

Example 23

Find the area of the region R bounded above by $f(x) = x + 4$ and below by $g(x) = 3 - \dfrac{x}{2}$ over the interval $\left[1, 4\right]$.



$$\begin{aligned}
\text{Area} &= \int_1^4 \Big[f(x) - g(x)\Big]\, dx \\
&= \int_1^4 \left[\,(x+4) - \left(3 - \frac{x}{2}\right)\,\right] dx \\
&= \int_1^4 \left(\frac{3}{2}x + 1\right)\, dx \\
&= \frac{57}{4}
\end{aligned}$$

Areas of Compound Regions

Question: What if we want to look at regions bounded by the graphs of functions that cross one another?

Answer: In this case, we break up the interval $[a, b]$ at the crossing points into sub-intervals

$$[a, c_1], [c_1, c_2], \cdots, [c_n, b]$$

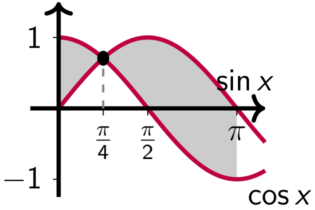
and evaluate several integrals as follows to obtain the area.

$$\text{Area} = \left| \int_a^{c_1} f(x) - g(x)\, dx \right| + \cdots + \left| \int_{c_n}^{b} f(x) - g(x)\, dx \right|$$

Areas of Compound Regions

Example 24

Find the area of the region bounded by the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $\left[0, \pi\right]$.



* **Find the point of intersection:**

$$\sin x = \cos x$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

Areas of Compound Regions

Example 24 (Continued)

Find the area of the region bounded by the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $\left[0, \pi\right]$.

* **Calculate the area:**

$$
\begin{aligned}
\text{Area} &= \left| \int_0^{\pi/4} \sin x - \cos x \ dx \right| + \left| \int_{\pi/4}^{\pi} \sin x - \cos x \ dx \right| \\
&= \left| \Big[-\cos x - \sin x \Big]_0^{\pi/4} \right| + \left| \Big[-\cos x - \sin x \Big]_{\pi/4}^{\pi} \right| \\
&= (\sqrt{2} - 1) + (\sqrt{2} + 1) \\
&= 2\sqrt{2}
\end{aligned}
$$

Think About It

Exercise 7

Find the area enclosed by the functions $f(x) = x^3 - x$ *and* $g(x) = 3x$.