Lecture Notes for Calculus 101

Chapter 3 : Integration

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Anti-derivative

QUESTION: What is the function whose derivative is $3x^2$?

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

$$\frac{d}{dx}\left(x^3 - 1\right) = 3x^2$$

$$\frac{d}{dx}\left(x^3 + \sqrt{5}\right) = 3x^2$$

$$\vdots$$

$$\frac{d}{dx}\left(x^3 + C\right) = 3x^2$$

The function $F(x) = x^3 + C$ is called the anti-derivative of $f(x) = 3x^2$.

Anti-derivative

The process of finding anti-derivatives is called **integration**, and can be expressed as integral notation

$$\int f(x)\, dx = F(x) + C$$

NOTE: $\int f(x)\, dx = F(x) + C \Leftrightarrow \frac{d}{dx}F(x) = f(x)$

Example 1

Verify that $\int x \sin x \, dx = \sin x - x \cos x + C$.

$$\begin{aligned} \frac{d}{dx}(\sin x - x\cos x) &= \cos x - (-x\sin x + \cos x) \\ &= \cos x + x\sin x - \cos x = x\sin x \end{aligned}$$

Basic Integration Rules

Rule (1): $\int k\, f(x)\, dx = k \int f(x)\, dx$

Rule (2): $\int \left(f(x) \pm g(x)\right)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

Rule (3): $\int k\, dx = kx + C$

Example 2

$$\int 1\, dx = x + C$$

$$\int \pi\, dt = \pi t + C$$

Basic Integration Rules

Rule (4):

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad ; \quad n \neq -1$$

$$\int (ax+b)^n \, dx = \frac{1}{a} \cdot \frac{(ax+b)^{n+1}}{n+1} + C \quad ; \quad n \neq -1$$

Example 3

(1) $\displaystyle \int 3x \, dx = 3 \int x \, dx = 3\left(\frac{x^2}{2}\right) + C = \frac{3}{2}x^2 + C$

(2) $\displaystyle \int (2x-3)^2 \, dx = \frac{(2x-3)^3}{(2)\cdot(3)} + C = \frac{1}{6}(2x-3)^3 + C$

(3) $\displaystyle \int \left(x^2 + \frac{1}{x^2}\right) \, dx = \int x^2 \, dx + \int x^{-2} \, dx$

$$= \frac{x^3}{3} + \frac{x^{-1}}{-1} + C = \frac{1}{3}x^3 - \frac{1}{x} + C$$

Basic Integration Rules

Example 4

$$\begin{aligned}\int (x-1)(x+3)\ dx &= \int \left(x^2+2x-3\right)\ dx \\ &= \frac{x^3}{3} + 2\cdot\frac{x^2}{2} - 3x + C \\ &= \frac{x^3}{3} + x^2 - 3x + C\end{aligned}$$

Example 5

$$\int \sqrt[3]{x^2}\ dx = \int x^{2/3}\ dx = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}\sqrt[3]{x^5} + C$$

Basic Integration Rules

Example 6

$$\int \frac{x^2 - 3\sqrt{x}}{x}\, dx = \int \left(\frac{x^2}{x} - \frac{3x^{1/2}}{x}\right)\, dx$$
$$= \int \left(x - 3x^{-1/2}\right)\, dx$$
$$= \frac{x^2}{2} - 3 \cdot \frac{x^{1/2}}{1/2} + C = \frac{x^2}{2} - 6\sqrt{x} + C$$

Exercise 1

Evaluate $\int \frac{1}{\sqrt{3 - 2x}}\, dx$

Basic Integration Rules

Rule (5): $\int \frac{1}{x} = \ln|x| + C$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Example 7

(1) $\int \frac{e^x}{9+e^x}\,dx = \ln|9+e^x| + C$

(2) $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln\left|x^2+1\right| + C$

(3) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C$

$$= \ln|\sec x| + C$$

Basic Integration Rules

Rule (6): $\int e^x \, dx = e^x + C$; $\int e^{mx+n} \, dx = \frac{1}{m} \cdot e^{mx+n} + C$

$\int b^x \, dx = \frac{b^x}{\ln b} + C$; $\int b^{mx+n} \, dx = \frac{1}{m} \cdot \frac{b^{mx+n}}{\ln b} + C$

Example 8

(1) $\int \sqrt{e^x} \, dx = \int e^{x/2} \, dx = \frac{e^{x/2}}{1/2} + C = 2\sqrt{e^x} + C$

(2) $\int 2^{8-3x} \, dx = \frac{2^{8-3x}}{-3 \ln 2} + C$

Basic Integration Rules

Rule (7):

$$\begin{aligned}
\int \sin x \, dx &= -\cos x + C \\
\int \cos x \, dx &= \quad \sin x + C \\
\int \sec^2 x \, dx &= \quad \tan x + C \\
\int \csc^2 x \, dx &= -\cot x + C \\
\int \sec x \tan x \, dx &= \quad \sec x + C \\
\int \csc x \cot x \, dx &= -\csc x + C
\end{aligned}$$

Basic Integration Rules

Example 9

(1) $\displaystyle\int \sin(4x-10)\,dx = -\frac{\cos(4x-10)}{4} + C$

(2) $\displaystyle\int \tan^2 x\,dx = \int \left(\sec^2 x - 1\right)\,dx = \tan x - x + C$

(3)
$$\begin{aligned}\int \sec x(\tan x + \cos x)\,dx &= \int (\sec x \tan x + \sec x \cos x)\,dx \\ &= \int \sec x \tan x\,dx + \int 1\,dx \\ &= \sec x + x + C\end{aligned}$$

Basic Integration Rules

Example 10

$$\int \sin^2 x \, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) \, dx$$
$$= \frac{1}{2}x - \frac{1}{2} \cdot \frac{\sin(2x)}{2} + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

NOTE: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$; $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Exercise 2

Evaluate the following integrals:

(1) $\int \frac{\sin(2x)}{\cos^2 x} \, dx$

(2) $\int e^{2 \ln x} \, dx$

(3) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} \, dx$

Basic Integration Rules

Rule (8): (1) $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$

$$\int \frac{1}{a+bx^2}\,dx = \frac{1}{\sqrt{ab}}\tan^{-1}\left(\sqrt{\frac{b}{a}}x\right) + C$$

(2) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

$$\int \frac{1}{\sqrt{a-bx^2}}\,dx = \frac{1}{\sqrt{b}}\sin^{-1}\left(\sqrt{\frac{b}{a}}x\right) + C$$

Basic Integration Rules

Example 11

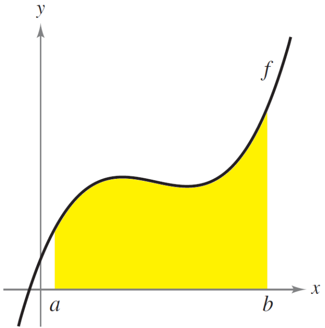
(1) $$\int \frac{1}{25+4x^2}\,dx = \frac{1}{\sqrt{(25)(4)}}\tan^{-1}\left(\sqrt{\frac{4}{25}}x\right)+C$$
$$= \frac{1}{10}\tan^{-1}\left(\frac{2}{5}x\right)+C$$

(2) $$\int \frac{1}{\sqrt{16-9x^2}}\,dx = \frac{1}{\sqrt{9}}\sin^{-1}\left(\sqrt{\frac{9}{16}}x\right)+C$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{3}{4}x\right)+C$$

The Definite Integral

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the $x-$axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\, dx$$



Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is, $\int_a^b f(x)\, dx$ exists.

The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and $F(x)$ is an antiderivative of on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b$$

Example 12

(1) $\int_1^2 x\,dx = \frac{x^2}{2}\Big]_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

(2) $\int_0^{\pi/3} \sec^2 x\,dx = \tan x\Big]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$

The Fundamental Theorem of Calculus

Example 13

$$\int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x\Big]_{-1/2}^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

Exercise 3

(1) *Evaluate* $\int_{0}^{\ln 3} 5e^{x}\, dx$

(2) *Evaluate* $\int_{-e}^{-1} \frac{1}{x}\, dx$

Properties of the Definite Integral

* If $a \in \text{dom}(f)$, we define $\int_a^a f(x)\,dx = 0$
* If f is integrable on $[a, b]$, we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
* If f is integrable on a closed interval containing the three points a, b, and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

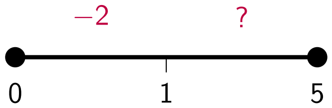
Properties of the Definite Integral

Example 14

Given that $\int_1^0 f(x)\,dx = 2$ and $\int_0^5 f(x)\,dx = 1$, find $\int_1^5 f(x)\,dx$.

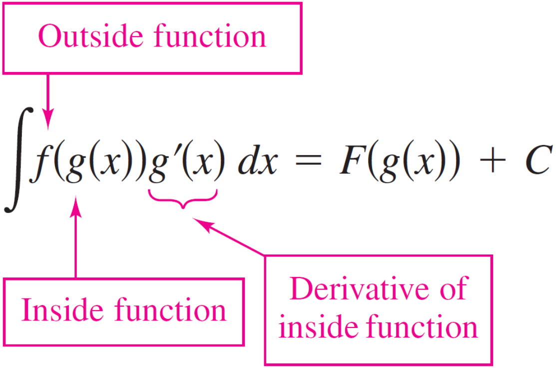
$$\int_0^5 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^5 f(x)\,dx$$

$$1 = -2 + \int_1^5 f(x)\,dx$$



$$\therefore \int_1^5 f(x)\,dx = 3$$

The Substitution Technique



$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

By letting $u = g(x)$ gives $du = g'(x)dx$ and $\int f(u)\,du = F(u) + C$

The Substitution Technique

Example 15

Evaluate $\int 2x\left(x^2+1\right)^{50}\ dx$

$$\begin{aligned}\int 2x\left(x^2+1\right)^{50}\ dx &= \int 2x\,(u)^{50}\ \frac{du}{2x}\\ &= \int u^{50}\ du\\ &= \frac{u^{51}}{51}+C\\ &= \frac{1}{51}\left(x^2+1\right)^{51}+C\end{aligned}$$

Let $u = x^2+1$

$$du = 2x\ dx$$

$$dx = \frac{du}{2x}$$

The Substitution Technique

Example 16

Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx = \int \frac{e^u}{\sqrt{x}}\, 2\sqrt{x} du = \int 2e^u\, du = 2e^u + C = 2e^{\sqrt{x}} + C$$

Let $u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\, dx$$

$$dx = 2\sqrt{x}\, du$$

Exercise 4

Evaluate $\int \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\, dx$

The Substitution Technique

Example 17

Evaluate $\int \frac{\left(\tan^{-1} x\right)^3}{1+x^2}\,dx$

$$\int \frac{\left(\tan^{-1} x\right)^3}{1+x^2}\,dx = \int \frac{(u)^3}{1+x^2}\,\left(1+x^2\right)\,du$$

$$= \int u^3\,du = \frac{1}{4}u^4 + C$$

$$= \frac{1}{4}\left(\tan^{-1} x\right)^4 + C$$

Let $u = \tan^{-1} x$

$$du = \frac{1}{1+x^2}\,dx$$

$$dx = \left(1+x^2\right)\,du$$

The Substitution Technique

Example 18

Evaluate $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

$$\int \frac{x}{\sqrt{1-x^2}} = \int \frac{x}{\sqrt{u}}\,\frac{-du}{2x}$$
$$= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$
$$= -\frac{1}{2}\int u^{-1/2}\,du$$
$$= -\frac{1}{2}\frac{u^{1/2}}{1/2} = -\sqrt{u} = -\sqrt{1-x^2}$$

Let $u = 1 - x^2$

$$du = -2x\,dx$$
$$dx = \frac{-du}{2x}$$

The Substitution Technique

Example 19

Evaluate $\displaystyle\int_1^e \frac{(\ln x)^2}{x}\, dx$

$$\int_1^e \frac{(\ln x)^2}{x}\, dx = \int_0^1 \frac{u^2}{x}\, xdu$$
$$= \int_0^1 u^2\, du$$
$$= \frac{u^3}{3}\Big]_0^1$$
$$= \frac{1}{3} - 0 = \frac{1}{3}$$

Let $u = \ln x$

$$du = \frac{1}{x}\, dx$$
$$dx = x\, du$$

If $x = 1 \Rightarrow u = 0$

If $x = e \Rightarrow u = 1$

The Substitution Technique

Example 20

Evaluate $\int x\sqrt{x-1}\,dx$

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int x\cdot u\cdot 2udu \\ &= \int 2u^2\left(u^2+1\right)\,du \\ &= \int\left(2u^4+2u^2\right)\,du \\ &= 2\cdot\frac{u^5}{5}+2\cdot\frac{u^3}{3}+C \\ &= \frac{2}{5}\left(\sqrt{x-1}\right)^5+\frac{2}{3}\left(\sqrt{x-1}\right)^3+C\end{aligned}$$

$$\begin{aligned}\text{Let } u &= \sqrt{x-1} \\ u^2 &= x-1 \\ x &= u^2+1 \\ 2u\,du &= dx\end{aligned}$$

The Substitution Technique

Example 21

Evaluate $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ **Note:** $e^{2x} = (e^x)^2$

$$\begin{aligned}
\int \frac{e^x}{1+e^{2x}}\,dx &= \int \frac{e^x}{1+(e^x)^2}\,dx \\
&= \int \frac{e^x}{1+u^2}\,\frac{du}{e^x} \\
&= \int \frac{1}{1+u^2}\,du \\
&= \tan^{-1} u + C \\
&= \tan^{-1}(e^x) + C
\end{aligned}$$

Let $u = e^x$

$$du = e^x\,dx$$

$$dx = \frac{du}{e^x}$$

The Substitution Technique

Exercise 5

Evaluate

(1) $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

(2) $\displaystyle\int \frac{1}{x\left(1+(\ln x)^2\right)}\,dx$

(3) $\displaystyle\int \frac{\ln\left(\sqrt[3]{x^2}\right)}{x}\,dx$

(4) $\displaystyle\int x^4\,\sqrt[3]{3-5x^5}\,dx$

Exercise 6

Given that $\displaystyle\int_0^4 f(x)\,dx = 1$, *find* $\displaystyle\int_{-2}^0 x\,f\left(x^2\right)\,dx$