Course: Mathematical Modeling

Chapter: [1]

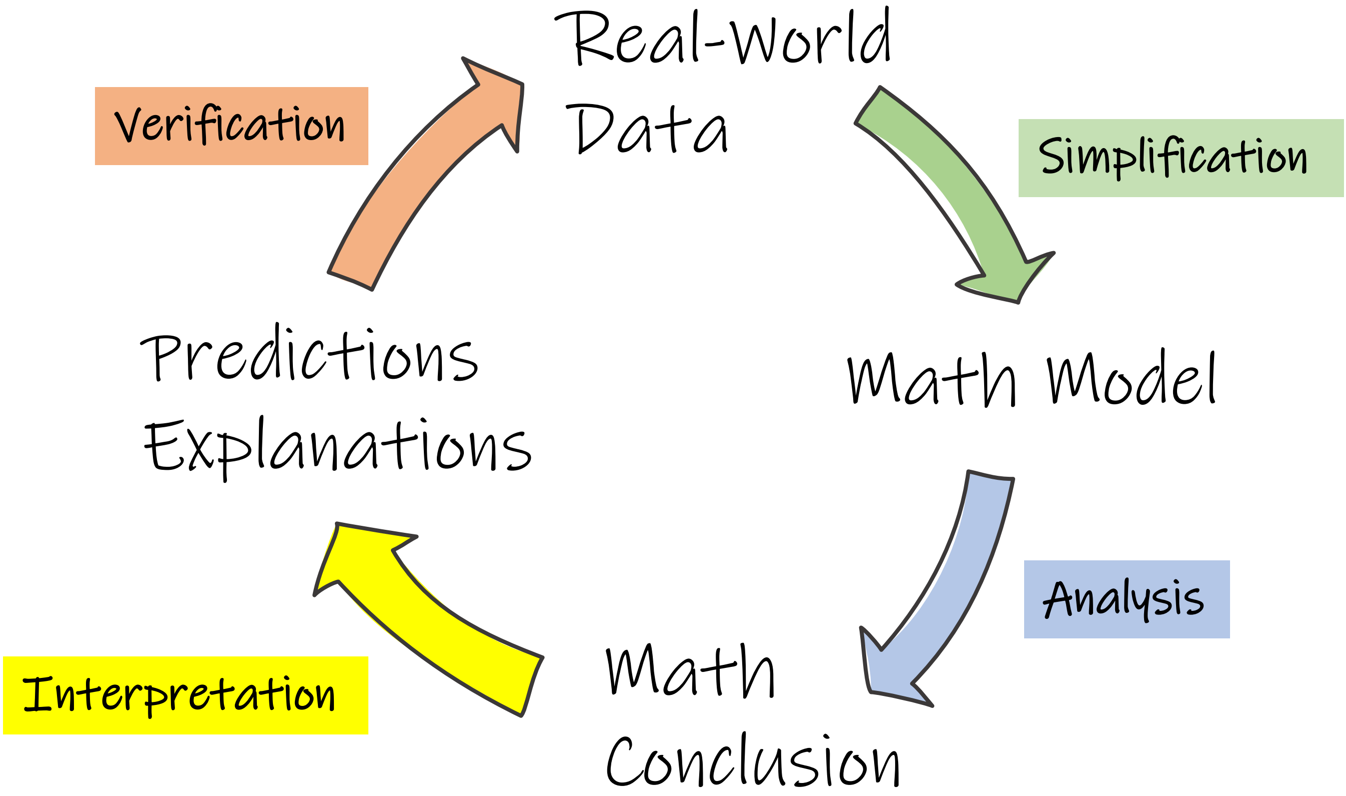
MODELING CHANGE

Section: [*]

ITRODUCTION

MATHEMATICAL MODELS

Mathematical Model A function or an equation describes a particular phenomenon mathematically to understand real-world problems.

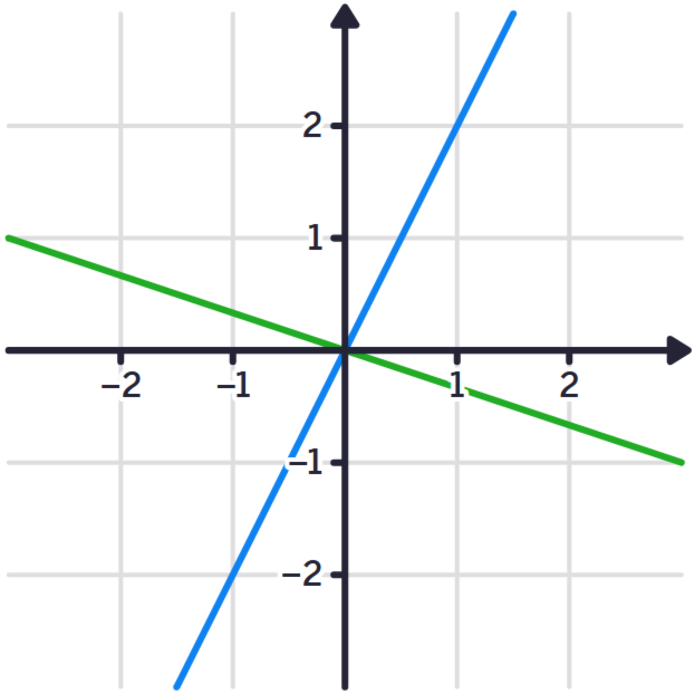


PROPORTIONALITY

Definition Two variables y and x are proportional to each other if one is always a constant multiple of the other, and we write $y \propto x$.

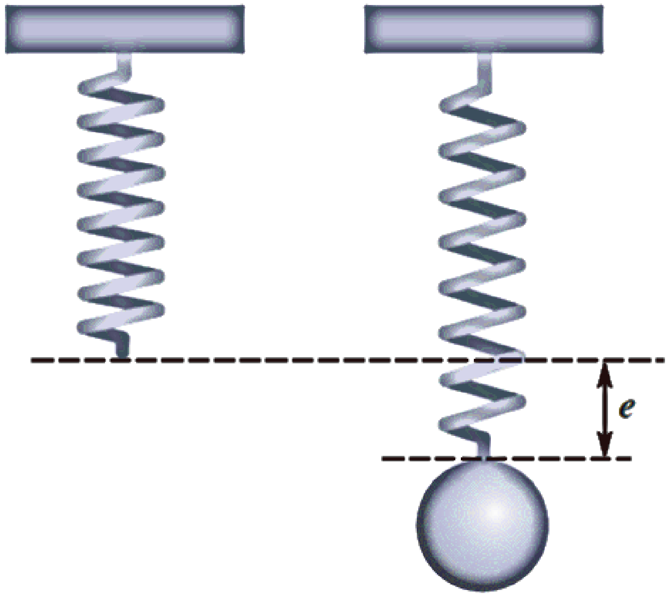
$y = kx$ for some nonzero constant k

If a proportionality is reasonable, a plot of one variable against the other should **approximate** a straight line through the origin.



PROPORTIONALITY

Example Consider a spring-mass system shown in the Figure. We conduct an experiment to measure the stretch of the spring (**Elongation** $\boldsymbol{e}$) as a function of the mass ($\boldsymbol{m}$) placed on the spring.



Mass	Elongation
50	1.000
100	1.875
150	2.750
200	3.250
250	4.375
300	4.875
350	5.675
400	6.500
450	7.250
500	8.000
550	8.750

PROPORTIONALITY

Example

Mass	Elongation
50	1.000
100	1.875
150	2.750
200	3.250
250	4.375
300	4.875
350	5.675
400	6.500
450	7.250
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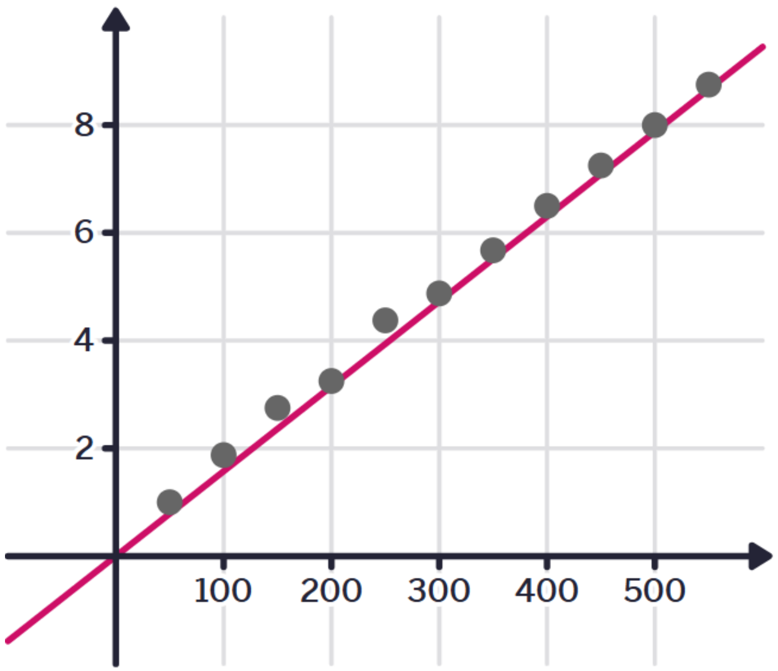

$e \propto m$

$e = km$

$e = 0.01575m$

The scatterplot reveals a straight line passing approximately through the origin.

$$k(\text{slope}) = \frac{7.250 - 5.675}{450 - 350} = 0.01575$$

MODELING CHANGE

Idea Often, we wish to predict the future based on what we know in the present.

$$\boxed{\text{future value} = \text{present value} + \text{change}}$$

We begin by studying the change itself. →

* Collecting data over **period of time.**
* Plotting the data.
* Observing any pattern.

$$\boxed{\text{Change} = \text{future value} - \text{present value}}$$

Discrete Time Period

Difference Equation

Continuous Time Period

Differential Equation

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MODELING CHANGE

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Modeling Change with Difference Equations

SEQUENCES

Definition A **sequence** a_n is a function whose domain is the set of all nonnegative integers and whose range is a subset of the real numbers.

General Term

Example Write out the first five terms of the sequence $a_n = 4^n$.

$$a_0 = 4^0 = 1$$

$$a_1 = 4^1 = 4$$

$$a_2 = 4^2 = 16$$

$$a_3 = 4^3 = 64$$

$$a_4 = 4^4 = 256$$

We can write a_n using the recursion formula:

$$\boxed{\begin{aligned} a_{n+1} &= 4a_n \quad ; \quad \mathrm{n} = 0,1,2,\cdots \\ a_0 &= 1 \end{aligned}}$$

SEQUENCES

Example Write out the first five terms of the sequence

$$\boxed{\begin{aligned} a_{n+1} &= a_n^2 - 1 \\ a_0 &= 2 \end{aligned}}$$

$$a_0 = 2$$

$$a_1 = a_0^2 - 1 = 2^2 - 1 = 3$$

$$a_2 = a_1^2 - 1 = 3^2 - 1 = 8$$

$$a_3 = a_2^2 - 1 = 8^2 - 1 = 63$$

$$a_4 = a_3^2 - 1 = 63^2 - 1 = 3968$$

Example Find a formula for the nth term of the sequence $1, 3, 7, 15, 31, \cdots$

$\boxed{2}$ $\boxed{4}$ $\boxed{8}$ $\boxed{16}$ $\boxed{32}$

$\boxed{2^1}$ $\boxed{2^2}$ $\boxed{2^3}$ $\boxed{2^4}$ $\boxed{2^5}$

$\boxed{2^1-1}$ $\boxed{2^2-1}$ $\boxed{2^3-1}$ $\boxed{2^4-1}$ $\boxed{2^5-1}$

$$a_n = 2^n - 1 \quad ; \quad n = 1,2,3, \cdots$$

MODELING OBSERVED CHANGES

Definition For a sequence of numbers $A = \{a_0, a_1, a_2, a_3, \cdots\}$ the **first differences** are

$$\begin{aligned}\Delta a_0 &= a_1 - a_0\\ \Delta a_1 &= a_2 - a_1\\ &\vdots\\ \Delta a_n &= a_{n+1} - a_n\end{aligned}$$

Example By examining the sequence $\{1,2,5,12,27,\cdots\}$, write a difference equation to represent the change during the nth interval as a function of the previous term in the sequence.

$$\begin{aligned}\Delta a_0 &= a_1 - a_0 = 2 - 1 = 1\\ \Delta a_1 &= a_2 - a_1 = 5 - 2 = 3\\ \Delta a_2 &= a_3 - a_2 = 12 - 5 = 7\\ \Delta a_3 &= a_4 - a_3 = 27 - 12 = 15\end{aligned}$$

$$\Delta a_n = a_{n+1} - a_n = a_n + n$$

$$\boxed{\begin{aligned}a_{n+1} &= 2a_n + n\\ a_0 &= 1\end{aligned}}$$

MODELING OBSERVED CHANGES

Example Write out the first five terms of the sequence satisfying the difference equations

$$\begin{aligned} \Delta p_n &= 0.001(500 - p_n) \\ p_0 &= 10 \end{aligned}$$

$$\Delta p_n = p_{n+1} - p_n$$

$$0.001(500 - p_n) = p_{n+1} - p_n$$

$$p_{n+1} = 0.001(500 - p_n) + p_n$$

$$\boxed{\begin{aligned} p_{n+1} &= 0.999 p_n + 0.5 \\ p_0 &= 10 \end{aligned}}$$

Dynamical System

$$p_0 = 10$$

$$p_1 = (0.999)(10) + 0.5 = 10.49$$

$$p_2 = (0.999)(10.49) + 0.5 = 10.97951$$

$$p_3 = (0.999)(10.97951) + 0.5 = 11.46853$$

$$p_4 = (0.999)(11.46853) + 0.5 = 11.95706$$

MODELING OBSERVED CHANGES

Definition A **dynamical system** is a relationship among terms in a sequence.

Example **SAVINGS CERTIFICATE (شهادات الادخار)**

A savings certificate is initially valued at $1000 and earns compound interest (فائدة مركبة) at a monthly rate of 1%. Formulate a dynamical system that exactly models the value of the certificate over time.

$$\Delta a_n = 0.01a_n = a_{n+1} - a_n$$

$$a_{n+1} = a_n + 0.01a_n$$

$$\boxed{\begin{aligned} a_{n+1} &= 1.01a_n \\ a_0 &= 1000 \end{aligned}}$$

The value of the certificate for the first few months:

1000

1010

1020.1

1030.301

1040.60401

MODELING OBSERVED CHANGES

Example **GRANDPARENTS ANNUITY**

Your grandparents (الجدّين) have an annuity (راتب تقاعد). The value of the annuity increases each month by an automatic deposit (إيداع) of 1% interest on the previous month's balance (الرصيد). Your grandparents withdraw (يسحب) \$1000 at the beginning of each month for living expenses (نفقات المعيشة). Currently, they have \$60,000 in the annuity. Model the annuity with a dynamical system.

$$a_{n+1} = a_n + 0.01a_n - 1000$$

$$\begin{aligned} a_{n+1} &= 1.01a_n - 1000 \\ a_0 &= 60000 \end{aligned}$$

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MODELING CHANGE

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Solutions to Dynamical Systems

LINEAR DYNAMICAL SYSTEMS $a_{n+1} = ra_n$, FOR r CONSTANT

Note Given a dynamical system $a_{n+1} = ra_n$ where a_0 is an initial value, then

$$\begin{aligned} a_{n+1} &= r(ra_{n-1}) = r^2 a_{n-1} \\ &= r^2(ra_{n-2}) = r^3 a_{n-2} = \cdots = r^{n+1} a_0 \end{aligned}$$

Theorem The solution of the linear dynamical system $a_{n+1} = ra_n$ for any nonzero constant r is

$$a_n = r^n a_0$$

where a_0 is a given initial value.

Example Find the solution to the difference equations $a_{n+1} = 5a_n$ where $a_0 = 10$.

$$a_n = 10 \cdot 5^n \qquad n = 0, 1, 2, \cdots$$

LINEAR DYNAMICAL SYSTEMS $a_{n+1} = ra_n$, FOR r CONSTANT

Example **SAVINGS CERTIFICATE**

1) Determine the value of the certificate after **one year**.
2) In how many months will the certificate's value **double**?

$$a_{n+1} = 1.01a_n$$
$$a_0 = 1000$$

$$a_n = 1000 \cdot (1.01)^n$$

Part 1

$$a_{12} = 1000 \cdot (1.01)^{12}$$
$$= 1126.83$$

Part 2

$$a_n = 2000$$
$$1000 \cdot (1.01)^n = 2000$$
$$(1.01)^n = 2$$
$$n \ln 1.01 = \ln 2$$
$$n \approx 69.6607$$
$$\therefore n = 70$$ Months

LINEAR DYNAMICAL SYSTEMS $a_{n+1} = ra_n$, FOR r CONSTANT

Example **Sewage Treatment (تنقية مياه الصرف الصحّي)**

A sewage treatment plant (محطة تنقية) processes raw sewage to produce usable fertilizer and clean water by removing all other contaminants (الملوثات و الشوائب). The process is such that each hour 12% of remaining contaminants in a processing tank are removed.

A. What percentage of the sewage would remain after 1 day?

Let the initial amount of sewage contaminants be a_0 and let a_n denote the amount after n hours.

$$a_{n+1} = a_n - 0.12a_n$$
$$= 0.88a_n$$

$$a_n = (0.88)^n a_0$$

$$a_{24} = (0.88)^{24}\, a_0 = 0.0465\, a_0$$

The amount of sewage after the end if the 1st day is about 4.65% of its initial amount

LINEAR DYNAMICAL SYSTEMS $a_{n+1} = ra_n$, FOR r CONSTANT

Example **Sewage Treatment (تنقية مياه الصرف الصحّي)**

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B. How long would it take to lower the amount of sewage by half?

$$a_n = \frac{1}{2}a_0 = (0.88)^n a_0$$

$$(0.88)^n = 0.5$$

$$n = \frac{\ln(0.5)}{\ln(0.88)} \approx 5.42$$ hours

LINEAR DYNAMICAL SYSTEMS $a_{n+1} = ra_n$, FOR r CONSTANT

Example **Sewage Treatment (تنقية مياه الصرف الصحّي)**

A sewage treatment plant (محطة تنقية) processes raw sewage to produce usable fertilizer and clean water by removing all other contaminants (الملوثات والشوائب). The process is such that each hour 12% of remaining contaminants in a processing tank are removed.

C. How long until the level of sewage is down to 10% of the original level?

$$a_n = 0.10a_0 = (0.88)^n a_0$$

$$(0.88)^n = 0.1$$

$$n = \frac{\ln(0.1)}{\ln(0.88)} \approx 18.01$$ hours

LONG-TERM BEHAVIOR OF $a_{n+1} = ra_n$, FOR r CONSTANT

Definition Values for which a dynamical system remains constant at those values, once reached, are called **equilibrium values** of the system.

Case [1] $r = 0$

Constant solution and equilibrium value at 0

All the values of the sequence (except possibly a_0) are zero.

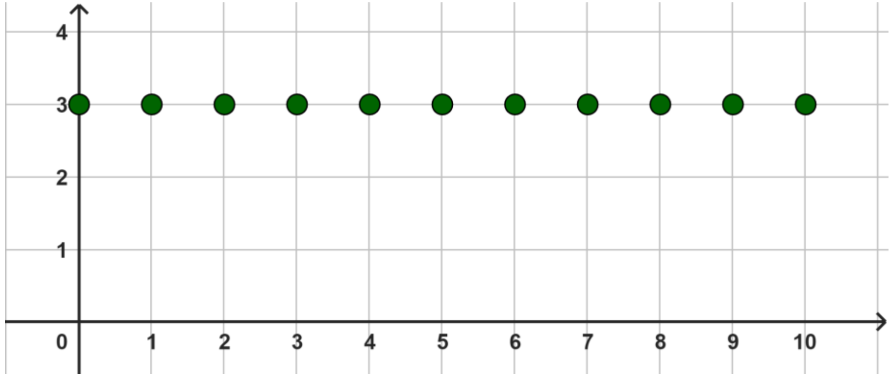
Case [2] $r = 1$

All initial values are constant solutions and equilibrium value at a_0

Then $a_{n+1} = a_n$. No matter where the sequence starts, it stays there forever.

For example:

$$a_{n+1} = a_n$$
$$a_0 = 3$$



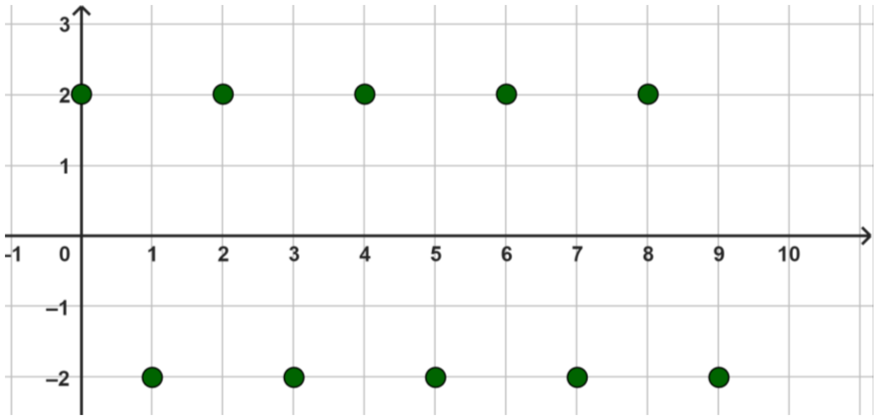
LONG-TERM BEHAVIOR OF $a_{n+1} = ra_n$, FOR r CONSTANT

Case [3] $r = -1$ No equilibrium value

The values oscillate between $-a_0$ and a_0.

For example:

$$a_{n+1} = -a_n$$
$$a_0 = 2$$

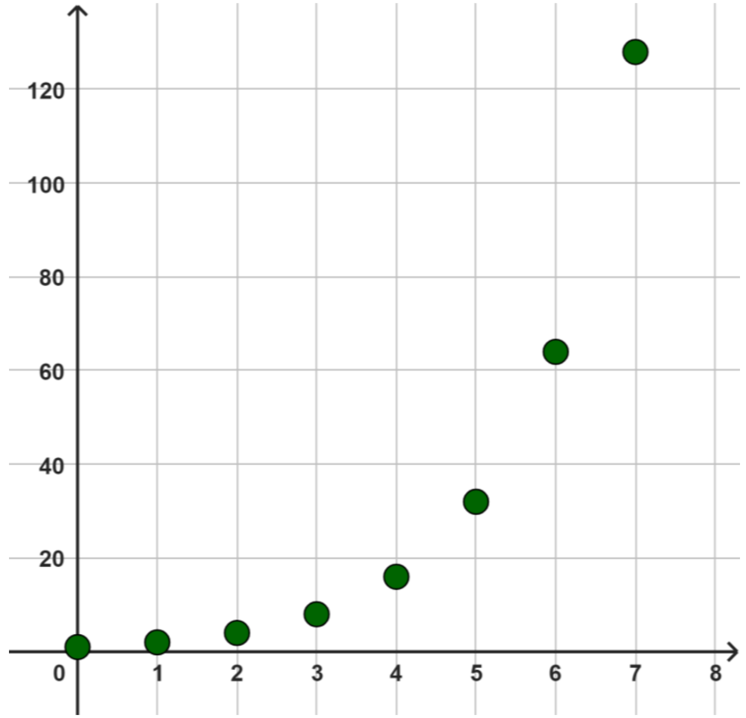


Case [4] $r > 1$ $a_0 > 0$

The sequence grows large without bound.

For example:

$$a_{n+1} = 2a_n$$
$$a_0 = 1$$



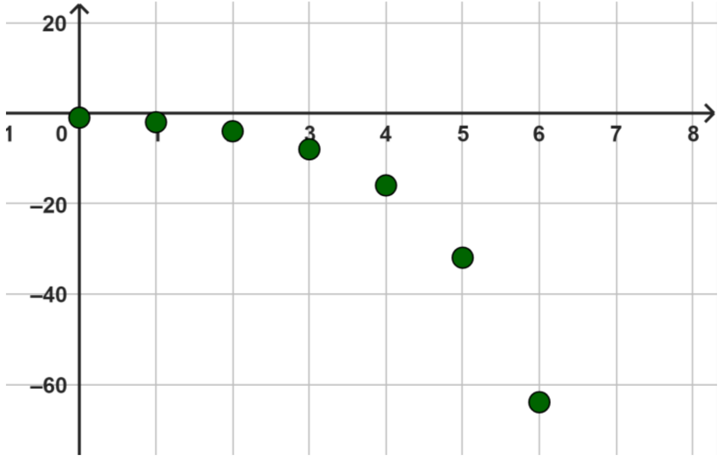
LONG-TERM BEHAVIOR OF $a_{n+1} = ra_n$, FOR r CONSTANT

Case [5] $r > 1$ $a_0 < 0$

The sequence grows negative without bound.

For example:

$$a_{n+1} = 2a_n$$
$$a_0 = -1$$

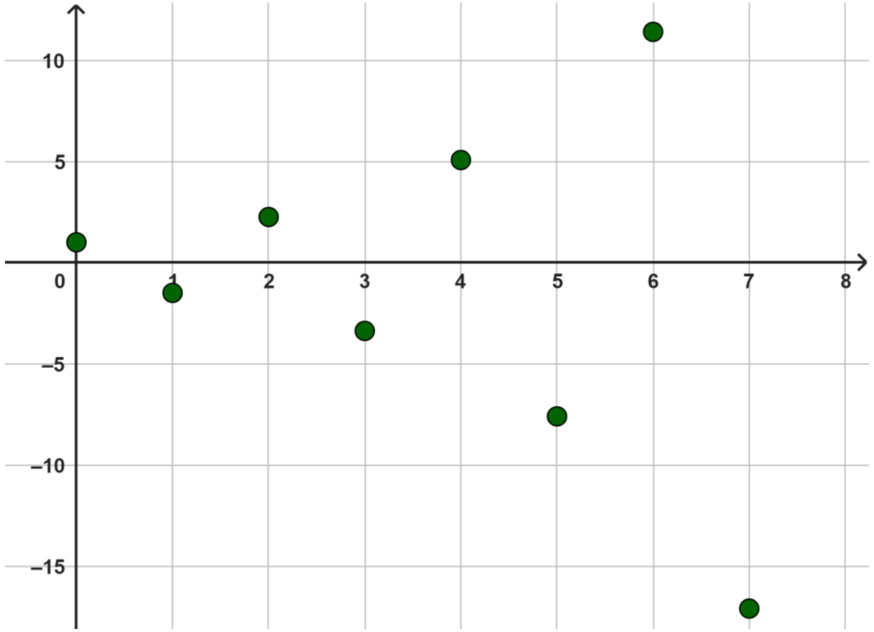


Case [6] $r < -1$ No equilibrium value

The sequence oscillates and grows.

For example:

$$a_{n+1} = -1.5a_n$$
$$a_0 = 1$$



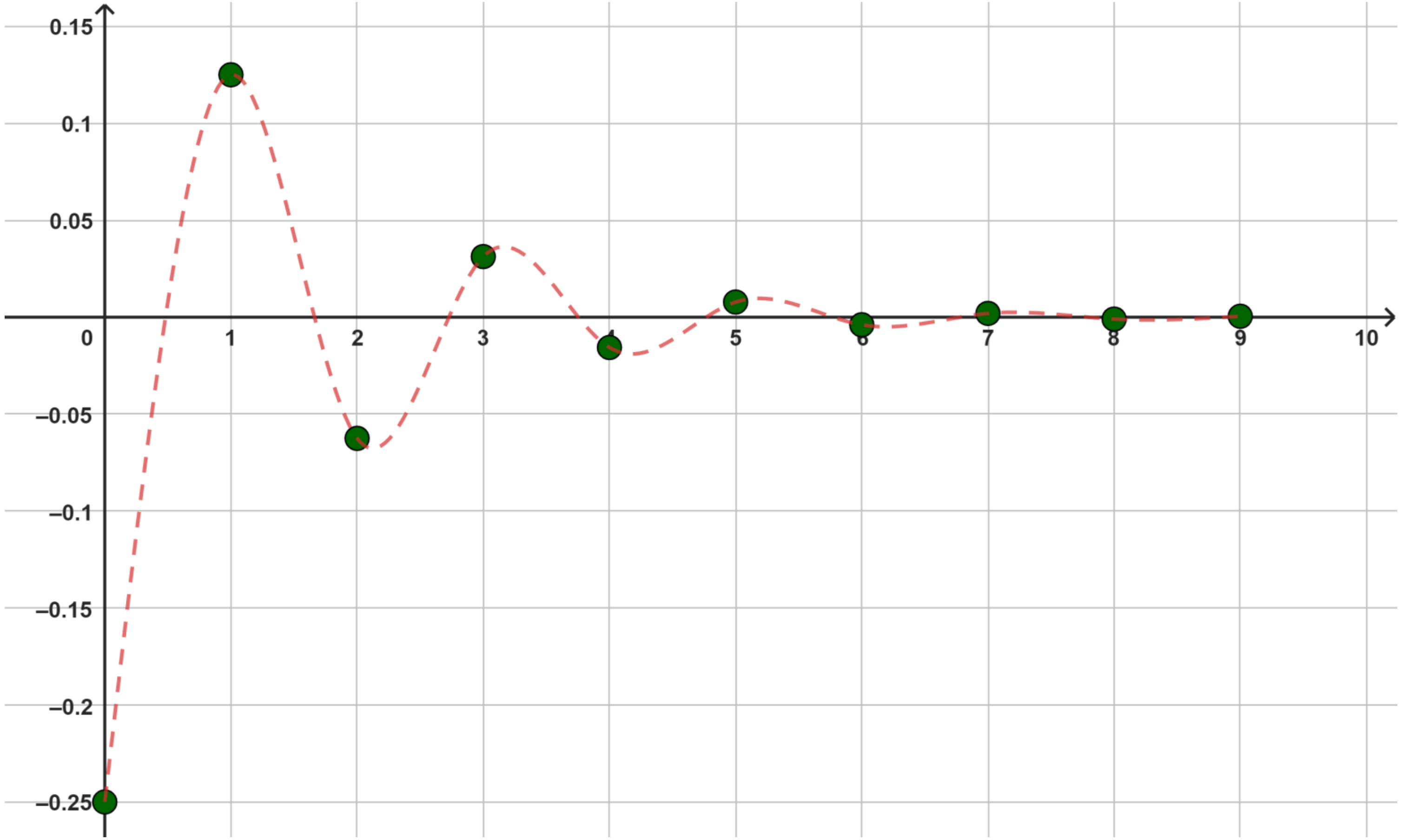
LONG-TERM BEHAVIOR OF $a_{n+1} = ra_n$, FOR r CONSTANT

Case [7] $-1 < r < 0$ The equilibrium value is 0.

The sequence oscillates and decays to 0.

For example:

$$a_{n+1} = -0.5a_n$$
$$a_0 = -0.25$$

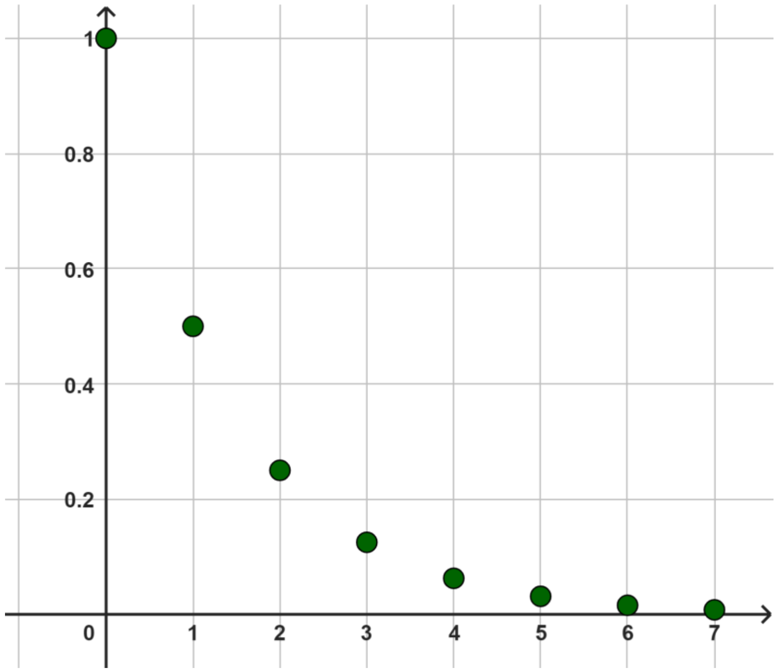


LONG-TERM BEHAVIOR OF $a_{n+1} = ra_n$, FOR r CONSTANT

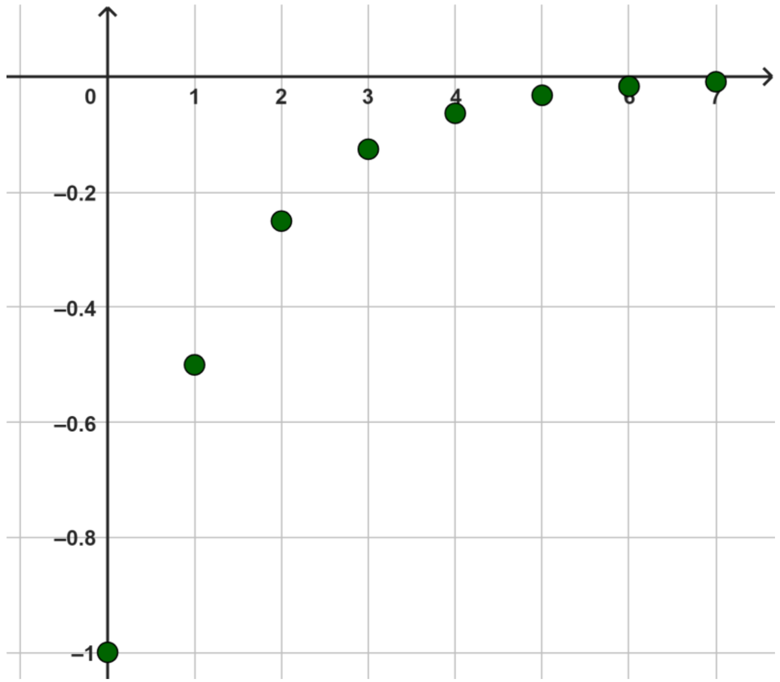
Case [8] $0 < r < 1$ The equilibrium value is 0.

The sequence decays to 0.

$a_0 > 0$ **For example:** $a_{n+1} = 0.5a_n$, $a_0 = 1$



$a_0 < 0$ **For example:** $a_{n+1} = 0.5a_n$, $a_0 = -1$



DYNAMICAL SYSTEMS $a_{n+1} = ra_n + b$, FOR r, b CONSTANTS

Theorem Given a dynamical system $a_{n+1} = ra_n + b$ where a_0 is an initial value. If $r \neq 1$, then

$$a_1 = ra_0 + b$$

$$a_2 = ra_1 + b = r\,(ra_0 + b) + b = r^2 a_0 + (r+1)b$$

$$a_3 = ra_2 + b = r\,(r^2 a_0 + (r+1)b) + b = r^3 a_0 + (r^2 + r + 1)b$$

$$\vdots$$

$$a_n = r^n a_0 + (r^{n-1} + r^{n-2} + \cdots + r^2 + r + 1)b$$

$$\therefore a_n = r^n a_0 + \frac{1 - r^n}{1 - r} \cdot b$$

DYNAMICAL SYSTEMS $a_{n+1} = ra_n + b$, FOR r, b CONSTANTS

Example Find the solution to the difference equations $a_{n+1} = -a_n + 2$ where $a_0 = -1$.

$$
\begin{aligned}
a_n &= r^n a_0 + \frac{1 - r^n}{1 - r} \cdot b \\
&= (-1)^n \cdot (-1) + \frac{1 - (-1)^n}{1 - (-1)} \cdot (2) \\
&= (-1)^n \cdot (-1) + 1 - (-1)^n \\
&= 2 \cdot (-1)^{n+1} + 1
\end{aligned}
$$

DYNAMICAL SYSTEMS $a_{n+1} = ra_n + b$, FOR r, b CONSTANTS

Example **GRANDPARENTS ANNUITY**

Will the annuity run out of money? When?

$$a_{n+1} = \underbrace{1.01}_{r}a_n - \underbrace{1000}_{b}$$
$$a_0 = 60000$$

$$a_n = 60000 \cdot (1.01)^n + \frac{1-(1.01)^n}{1-1.01} \cdot (-1000)$$
$$= 60000 \cdot (1.01)^n + (100000) \cdot (1-(1.01)^n)$$
$$= 100000 - 40000 \cdot (1.01)^n$$

We find n such that $a_n = 0$

$$100000 - 40000 \cdot (1.01)^n = 0$$

$$(1.01)^n = \frac{100000}{40000} = 2.5$$

$$n = \frac{\ln 2.5}{\ln 1.01} \approx 92.0865$$

$\therefore n = 93$ Months

FINDING AND CLASSIFYING EQUILIBRIUM VALUES

Definition A point p is a **stable equilibrium** of a dynamical system if, when the system starts close to p, it stays close to p for all future time.

Theorem The equilibrium value for the dynamical system $a_{n+1} = r\, a_n + b$ where $r \neq 1$, is

$$p = \frac{b}{1-r}$$

If $r = 1$ and $b = 0$, every number is an equilibrium value. If $r = 1$ and $b \neq 0$, no equilibrium value exists.

Stability **For the dynamical system $\boldsymbol{a_{n+1} = ra_n + b}$ where $\boldsymbol{b \neq 0}$:**

- If $|r| < 1$: Stable equilibrium
- If $|r| > 1$: Unstable equilibrium
- If $r = 1$: Graph is a line with no equilibrium

FINDING AND CLASSIFYING EQUILIBRIUM VALUES

Example For the following problems, find an equilibrium value if one exists. Classify the equilibrium value as stable or unstable.

[1] $$a_{n+1} = \frac{1}{2}a_n + 1$$

$$\text{Equilibrium value} = \frac{b}{1-r}$$

$$= \frac{1}{1-0.5} = 2$$

Since $r = 0.5$ and $|r| < 1$, then the equilibrium is stable

n	a_n	a_n	a_n
0	3	1	2
1	2.5	1.5	2
2	2.25	1.75	2
3	2.125	1.875	2
4	2.0625	1.9375	2
5	2.03125	1.96875	2
6	2.01563	1.98438	2
7	2.00781	1.99219	2

FINDING AND CLASSIFYING EQUILIBRIUM VALUES

Example For the following problems, find an equilibrium value if one exists. Classify the equilibrium value as stable or unstable.

[2] $a_{n+1} = -2a_n + 3$

Equilibrium value $= \dfrac{b}{1-r}$

$$= \frac{3}{1-(-2)} = 1$$

Since r = -2 and |r| > 1, then the equilibrium is unstable

n	a_n	a_n	a_n
0	0.5	1.5	1
1	2	0	1
2	-1	3	1
3	5	-3	1
4	-7	9	1
5	17	-15	1
6	-31	33	1
7	65	-63	1