Course: Mathematical Modeling

Chapter: [3]
MODELING WITH DIFFERENTIAL EQUATIONS

Section: [3.1]
Terminology

ORDINARY DIFFERENTIAL EQUATIONS

ODE

A **differential equation** is an equation involving one or more derivatives of an unknown function.

$$\frac{dy}{dx} = 3y$$ Order 1

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$ Order 2

$$y''' - ty' + (t^2 - 1)y = e^t$$ Order 3

The **order** of a differential equation is the order of the highest derivative that it contains.

ORDINARY DIFFERENTIAL EQUATIONS

Solutions of ODEs

A function $y = y(x)$ is a **solution** of a differential equation on an open interval if the equation is satisfied identically on the interval when y and its derivatives are substituted into the equation.

Example

Verify that $y = e^{2x}$ is a solution to the following differential equation on the interval $(-\infty, \infty)$.

$$\frac{dy}{dx} - y = e^{2x}$$

$$\text{LHS} = \frac{dy}{dx} - y = \frac{d}{dx}[e^{2x}] - e^{2x} = 2e^{2x} - e^{2x} = e^{2x} = \text{RHS}$$

Note

The function $y = e^{2x} + Ce^{x}$; $x \in (-\infty, \infty)$ is the **general solution** to the ODE in the previous example where C is an arbitrary constant.

INITIAL VALUE PROBLEMS

IVPs

* If the general solution of an ODE has n $-$arbitrary constants, then we need n $-$conditions to specify their values at an initial value x_0.

* An ODE with n $-$initial conditions is called an **Initial Value Problem**.

$$\frac{dy}{dx} = f(x,y) \;\; ; \;\; y(x_0) = y_0$$

1st Order IVP

$$\frac{d^2y}{dx^2} = f(x,y,y') \;\; ; \;\; y(x_0) = y_0 \text{ and } y'(x_0) = y_1$$

2nd Order IVP

INITIAL VALUE PROBLEMS

Example Find the solution of the IVP: $\dfrac{dy}{dt} = \sin t + 1 \;;\; y\left(\dfrac{\pi}{3}\right) = \dfrac{1}{2}$

$$\frac{dy}{dt} = \sin t + 1$$

$$dy = (\sin t + 1)dt$$

$$\int dy = \int (\sin t + 1)dt$$

$$y = t - \cos t + C$$

But $y = \dfrac{1}{2}$ if $t = \dfrac{\pi}{3}$

$$\frac{1}{2} = \frac{\pi}{3} - \cos\frac{\pi}{3} + C$$

$$C = 1 - \frac{\pi}{3}$$

$$\therefore y = t - \cos t + 1 - \frac{\pi}{3}$$

SEPARATION OF VARIABLES

1st Order Separable Equations

A differential equation of the form $\frac{dy}{dx} = h(x, y)$ is called **separable** if it can be written in the form $g(y)dy = f(x)dx$, and the process is called **separating variables**.

Example

$$\frac{dy}{dx} = \frac{x}{y} \longrightarrow ydy = xdx$$

$$\frac{dy}{dx} = x^2y^3 \longrightarrow \frac{1}{y^3}dy = x^2dx$$

$$\frac{dy}{dx} = y - \frac{y}{x} \longrightarrow \frac{1}{y}dy = \left(1 - \frac{1}{x}\right)dx$$

All Equations are separable

SEPARATION OF VARIABLES

Solving Separable Equations

$$g(y)dy = f(x)dx \longrightarrow \int g(y)dy = \int f(x)dx$$

$$\longrightarrow G(y) = H(x) + C$$

Example Solve the differential equation $\dfrac{dy}{dx} = -4xy^2$

$$\frac{dy}{dx} = -4xy^2 \longrightarrow y^{-2}dy = -4xdx \quad \text{(Separable)}$$

$$\longrightarrow \int y^{-2}dy = \int -4xdx \longrightarrow \frac{1}{y} = 2x^2 + C$$

$$\therefore y = \frac{1}{2x^2 + C}$$

SEPARATION OF VARIABLES

Example Solve the initial value problem $(4y - \cos y)\dfrac{dy}{dx} - 3x^2 = 0$; $y(0) = 0$

$$(4y - \cos y)\frac{dy}{dx} - 3x^2 = 0 \longrightarrow (4y - \cos y)dy = 3x^2 dx \quad \text{(Separable)}$$

$$\longrightarrow \int (4y - \cos y)dy = \int 3x^2 dx$$

$$\longrightarrow 2y^2 - \sin y = x^3 + C$$

But $y(0) = 0 \longrightarrow 2(0)^2 - \sin 0 = 0^3 + C \longrightarrow C = 0$

$$\therefore 2y^2 - \sin y = x^3$$

$$x = \sqrt[3]{2y^2 - \sin y}$$

Course: Mathematical Modeling

Chapter: [3]

MODELING WITH DIFFERENTIAL EQUATIONS

Section: [3.2]

Models with 1st Order Differential Equations

EXPONENTIAL GROWTH AND DECAY MODELS

Exponential Models

Exponential models arise in situations where a quantity **increases** or **decreases** at a **rate** that is *proportional to the amount of the quantity present*.

y > 0 in both cases

Exponential Growth (النمو) Model

$$\frac{dy}{dt} \propto y \longrightarrow \frac{dy}{dt} = ky \ ; \ y(0) = y_0$$

(K > 0 growth constant)

Exponential Decay (التحلّل) Model

$$\frac{dy}{dt} \propto y \longrightarrow \frac{dy}{dt} = -ky \ ; \ y(0) = y_0$$

(K > 0 decay constant)

EXPONENTIAL GROWTH AND DECAY MODELS

Notes

* A single differential equation can serve as a mathematical model for many different phenomena.
* The growth and decay ODEs are **separable**.

Growth Model

$$\frac{dy}{dt} = ky \longrightarrow \frac{1}{y}dy = kdt \longrightarrow \int \frac{1}{y}dy = \int kdt \longrightarrow \ln y = kt + C$$

But $y(0) = y_0 \longrightarrow \ln y_0 = k \cdot 0 + C \longrightarrow C = \ln y_0$

$$\therefore \ln y = kt + \ln y_0 \longrightarrow y = e^{kt+\ln y_0} = e^{kt} \cdot e^{\ln y_0} = y_0 e^{kt}$$

Decay Model

$$\frac{dy}{dt} = -ky \; ; \; y(0) = y_0 \longrightarrow y = y_0 e^{-kt}$$

EXPONENTIAL GROWTH AND DECAY MODELS

Example Suppose that an initial population of 10000 bacteria grows exponentially at a rate of 2% per hour and that $y = y(t)$ is the number of bacteria present t hours later.

a) **Find an initial-value problem whose solution is $\boldsymbol{y(t)}$.**

$$\frac{dy}{dt} = 0.02y \ ; \ \ y(0) = 10000$$

b) **Find a formula for $\boldsymbol{y(t)}$.**

$$y(t) = 10000 \cdot e^{0.02t}$$

c) **Find the population of bacteria after 5 hours.**

$$y(5) = 10000 \cdot e^{(0.02)(5)} \ \approx 11051.71$$

EXPONENTIAL GROWTH AND DECAY MODELS

Example Suppose that an initial population of 10000 bacteria grows exponentially at a rate of 2% per hour and that $y = y(t)$ is the number of bacteria present t hours later.

d) How long does it take for the population of bacteria to reach $\mathbf{45000}$?

$$y(t) = 10000 \cdot e^{0.02t}$$

$$45000 = 10000 \cdot e^{0.02t}$$

$$e^{0.02t} = 4.5$$

$$0.02t = \ln 4.5$$

$$t = \frac{\ln 4.5}{0.02} \approx 75.2$$ Hours

EXPONENTIAL GROWTH AND DECAY MODELS

DOUBLING TIME

* If a quantity y has an *exponential growth* model, then the time required for the original size to double is called the **doubling time**.
* Doubling time depends only on the growth rate k and not on the amount present initially y_0.

Let T denote the amount of time required for y to double in size. Then

$$y = y_0 e^{kt} \longrightarrow 2y_0 = y_0 e^{kT}$$

$$\longrightarrow e^{kT} = 2$$

$$\longrightarrow kT = \ln 2 \longrightarrow \boxed{T = \frac{\ln 2}{k}}$$

EXPONENTIAL GROWTH AND DECAY MODELS

HALF-LIFE
عُمر النِّصف

* If a quantity y has an *exponential decay* model, then the time required for the original size to reduce by half is called the **half-life**.
* Half-life depends only on the growth rate k and not on the amount present initially y_0.

Let T denote the amount of time required for y to reduce by half. Then

$$y = y_0 e^{-kt} \longrightarrow \frac{1}{2} y_0 = y_0 e^{-kT}$$

$$\longrightarrow e^{-kT} = \frac{1}{2}$$

$$\longrightarrow -kT = -\ln 2 \longrightarrow \boxed{T = \frac{\ln 2}{k}}$$

EXPONENTIAL GROWTH AND DECAY MODELS

Example **RADIOACTIVE DECAY (الاضمحلال الإشعاعي)**

a) **If the half-life of radioactive carbon-14 is about 5730 years. Find the decay constant for this element.**

$$\boxed{T = \frac{\ln 2}{k}}$$

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5730} \approx 0.000121$$

b) **If 100 grams of radioactive carbon-14 are stored in a cave for 1000 years, how many grams will be left at that time?**

$$y(t) = y_0 e^{-kt}$$

$$y(1000) = 100 \cdot e^{-(0.000121)(1000)} \approx 88.6 \quad \text{grams}$$

EXPONENTIAL GROWTH AND DECAY MODELS

Exercise A cell of the bacterium *E. coli* divides into two cells every 20 minutes when placed in a nutrient culture. Let $y = y(t)$ be the number of cells that are present t minutes after a single cell is placed in the culture. Assume that the growth of the bacteria is approximated by an exponential growth model.

a) **Find an initial-value problem whose solution is $\boldsymbol{y(t)}$.**

b) **Find a formula for $\boldsymbol{y(t)}$.**

c) **How many cells are present after 2 hours?**

EXPONENTIAL GROWTH AND DECAY MODELS

Exercise A scientist wants to determine the half-life of a certain radioactive substance. She determines that in exactly 5 days a $10.0-$milligram sample of the substance decays to 3.5 milligrams. Based on these data, what is the half-life?

Exercise Suppose that 30% of a certain radioactive substance decays in 5 years. What is the half-life of the substance in years?

NEWTON'S LAW OF COOLING AND WARMING

The Law

* The rate at which the temperature of a body changes is proportional to the *difference* between the temperature of the body and the temperature of the surrounding medium (**ambient temperature حرارة الأجواء المحيطة**).
* If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and $\frac{dT}{dt}$ the rate at which the temperature of the body changes, then

$$\boxed{\frac{dT}{dt} \propto T - T_m} \qquad \boxed{\frac{dT}{dt} = k(T - T_m)}$$

$k < 0$ in either cooling or warming

NEWTON'S LAW OF COOLING AND WARMING

Example Let y represent the temperature (in F°) of an object in a room whose temperature is kept at a constant 60°. The object cools from 100° to 90° in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°.

Separable

$$\frac{dT}{dt} = k(T - 60) \qquad T(0) = 100 \qquad (k < 0)$$

$$\frac{1}{T-60}\,dT = k\,dt \longrightarrow \int \frac{1}{T-60}\,dT = \int k\,dt$$

$$\longrightarrow \ln(T - 60) = kt + C \qquad (T > 60)$$

But $T(0) = 100 \longrightarrow \ln(100 - 60) = k \cdot 0 + C \longrightarrow C = \ln 40$

NEWTON'S LAW OF COOLING AND WARMING

Example Let y represent the temperature (in F°) of an object in a room whose temperature is kept at a constant 60°. The object cools from 100° to 90° in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°.

$$\ln(T-60) = kt + \ln 40 \longrightarrow T - 60 = e^{kt+\ln 40}$$

$$\longrightarrow \boxed{T(t) = 60 + 40e^{kt}}$$

But $T(10) = 90 \longrightarrow 90 = 60 + 40e^{10k}$

$$\longrightarrow \frac{3}{4} = e^{10k}$$

$$\longrightarrow k = \frac{1}{10}\ln\frac{3}{4} \approx -0.02877$$

NEWTON'S LAW OF COOLING AND WARMING

Example Let y represent the temperature (in $\mathrm{F}°$) of an object in a room whose temperature is kept at a constant $60°$. The object cools from $100°$ to $90°$ in 10 minutes. How much longer will it take for the temperature of the object to decrease to $80°$.

$$T(t) = 60 + 40e^{-0.02877t}$$

If $T(t) = 80 \longrightarrow 60 + 40e^{-0.02877t} = 80$

$$\longrightarrow e^{-0.02877t} = 0.5$$

$$\longrightarrow -0.02877t = \ln 0.5$$

$$\longrightarrow t = \frac{-0.69315}{-0.02877} \approx 24.09$$ minutes

Therefore, we need to extra 14.09 minutes to reach 80°

NEWTON'S LAW OF COOLING AND WARMING

General Solution $$\frac{dT}{dt} = k(T - T_m) \qquad T(0) = t_0$$

$$\frac{dT}{T - T_m} = k\,dt \longrightarrow \int \frac{1}{T - T_m}\,dT = \int k dt$$

$$\longrightarrow \ln|T - T_m| = kt + C$$

But $T(0) = 100 \longrightarrow \ln|t_0 - T_m| = k \cdot 0 + C \longrightarrow C = \ln|t_0 - T_m|$

$$\longrightarrow \ln|T - T_m| = kt + \ln|t_0 - T_m|$$

$$\longrightarrow T - T_m = \pm e^{kt} \cdot e^{\ln|t_0 - T_m|}$$

$$\longrightarrow T = T_m \pm |t_0 - T_m| e^{kt}$$

$$\longrightarrow T = T_m + (t_0 - T_m)e^{kt} \qquad (k < 0)$$

NEWTON'S LAW OF COOLING AND WARMING

Example A glass of lemonade with a temperature of 40°F is placed in a room with a constant temperature of 70°F. How many minutes will it take for the lemonade to reach a temperature of 65°F if it heats to 52°F in 1 hour?

$t_0 = 40 \qquad T_m = 70 \qquad T(60) = 52$

$$\begin{aligned} T(t) &= T_m + (t_0 - T_m)e^{kt} \\ &= 70 + (40 - 70)e^{kt} \\ &= 70 - 30e^{kt} \end{aligned}$$

But $T(60) = 52$

$$52 = 70 - 30e^{60k}$$

$$30e^{60k} = 18$$

$$k = \frac{1}{60}\ln\frac{18}{30} \approx -0.0085$$

$$T(t) = 70 - 30e^{-0.0085t}$$

NEWTON'S LAW OF COOLING AND WARMING

Example A glass of lemonade with a temperature of 40°F is placed in a room with a constant temperature of 70°F. How many minutes will it take for the lemonade to reach a temperature of 65°F if it heats to 52°F in 1 hour?

$$T(t) = 70 - 30e^{-0.0085t}$$

$$T(t) = 65 \longrightarrow 65 = 70 - 30e^{-0.0085t}$$

$$\longrightarrow e^{-0.0085t} = \frac{5}{30} \longrightarrow -0.0085t = \ln\frac{1}{6}$$

$$\longrightarrow t = 210.765$$ Minutes

LOGISTIC MODELS

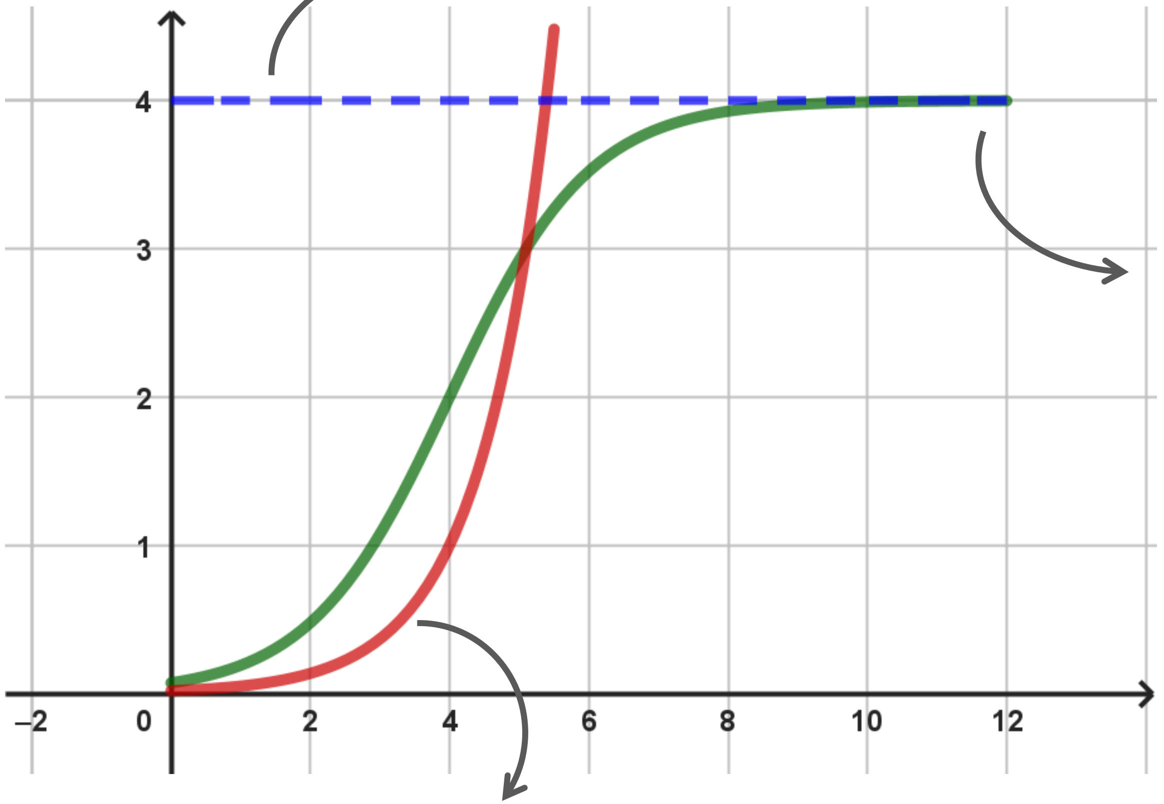
Notes

* The uninhibited population growth (النموّ السكاني غير المقيّد) model was predicated on the assumption that the population $y = y(t)$ was **not constrained by the environment**.
* In general, populations grow within ecological systems (الأنظمة البيئية) that can only support a certain number of individuals L.
* The number L is called the **carrying capacity** (القدرة الاستيعابية) of the system.

LOGISTIC MODELS

Notes

Carrying Capacity L



Exponential Growth

$$y(t) = y_0 e^{kt}$$

Logistic Growth $\quad \dfrac{dy}{dt} = ?$

$$\frac{y}{L} > 1 \longrightarrow \frac{dy}{dt} < 0$$

$$\frac{y}{L} < 1 \longrightarrow \frac{dy}{dt} > 0$$

$$\frac{y}{L} = 1 \longrightarrow \frac{dy}{dt} = 0$$

LOGISTIC MODELS

The Model

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \quad ; \quad y(0) = y_0 \quad \text{and} \quad k > 0$$

Separable

Solution

$$\frac{dy}{dt} = ky\left(\frac{L-y}{L}\right) \longrightarrow \frac{L}{y(L-y)}dy = kdt$$

$$\longrightarrow \int \frac{L}{y(L-y)}dy = \int kdt$$

By Partial Fractions

$$\frac{L}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y} = \frac{A(L-y) + By}{y(L-y)}$$

LOGISTIC MODELS

Solution $$\int \frac{L}{y(L-y)}\,dy = \int k\,dt$$

$$L = A(L-y) + By$$

$$y = L \longrightarrow \boxed{B = 1}$$

$$y = 0 \longrightarrow \boxed{A = 1}$$

$$\int \left(\frac{1}{y} + \frac{1}{L-y}\right) dy = \int k\,dt$$

$$\ln y - \ln(L-y) = kt + C \longrightarrow \ln\left(\frac{y}{L-y}\right) = kt + C$$

$$\longrightarrow \frac{y}{L-y} = e^{kt} \cdot e^{C}$$

$$\longrightarrow \frac{L-y}{y} = e^{-kt} \cdot e^{-C}$$

$$\longrightarrow \frac{L}{y} - 1 = e^{-kt} \cdot e^{-C} \longrightarrow y = \frac{L}{1 + e^{-kt} \cdot e^{-C}}$$

LOGISTIC MODELS

Solution

$$y = \frac{L}{1 + e^{-kt} \cdot \boxed{e^{-C}}} \longrightarrow y = \frac{L}{1 + b \cdot e^{-kt}}$$

Constant

But $y(0) = y_0 \longrightarrow y_0 = \dfrac{L}{1 + b}$

$$\longrightarrow b = \frac{L}{y_0} - 1 = \frac{L - y_0}{y_0}$$

$$y = \frac{L}{1 + \frac{L - y_0}{y_0} \cdot e^{-kt}} = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-kt}}$$

LOGISTIC MODELS

Example Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{60}{5 + 7e^{-t}}$$

$$y = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-kt}}$$

a) What is the population at time $t = 0$?

$$y(0) = \frac{60}{5 + 7 \cdot e^0} = 5 \qquad y_0 = 5$$

b) What is the carrying capacity L?

$$y_0 L = 60 \longrightarrow 5L = 60 \longrightarrow L = 12$$

LOGISTIC MODELS

Example Suppose that the growth of a population $y = y(t)$ is given by the logistic equation

$$y = \frac{60}{5 + 7e^{-t}}$$

$$y = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-kt}}$$

$y_0 = 5$ $L = 12$

c) What is the constant k?

$k = 1$

d) When does the population reach half of the carrying capacity?

$$y(t) = \frac{L}{2} = 6 \longrightarrow \frac{60}{5 + 7e^{-t}} = 6 \longrightarrow 30 + 42e^{-t} = 60$$

$$\longrightarrow e^{-t} = \frac{30}{42} \longrightarrow t \approx 0.3365$$

LOGISTIC MODELS

Example Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10000. If the population grows to 2500 after one year, what will the population be after another three years?

$$y = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-kt}}$$

$y_0 = 1000$ $L = 10000$

$$y(t) = \frac{(1000)(10000)}{1000 + (10000 - 1000) \cdot e^{-kt}}$$

$$= \frac{10000}{1 + 9 \cdot e^{-kt}}$$

But $y(1) = 2500 \longrightarrow \dfrac{10000}{1 + 9 \cdot e^{-k(1)}} = 2500 \longrightarrow 1 + 9 \cdot e^{-k} = 4$

$$\longrightarrow e^{-k} = \frac{1}{3} \longrightarrow k = \ln 3$$

LOGISTIC MODELS

Example Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10000. If the population grows to 2500 after one year, what will the population be after another three years?

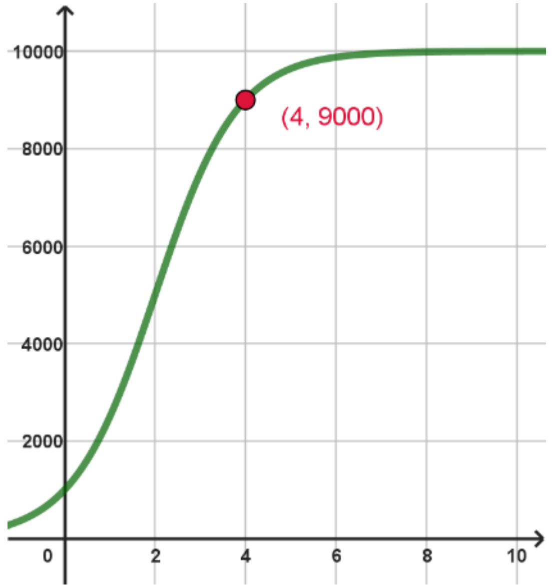
$$y = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-kt}}$$

$y_0 = 1000$ $L = 10000$

$$y(t) = \frac{10000}{1 + 9 \cdot e^{-(\ln 3)t}} = \frac{10000}{1 + 3^{2-t}}$$

After another 3-years: $$y(4) = \frac{10000}{1 + 3^{-2}}$$

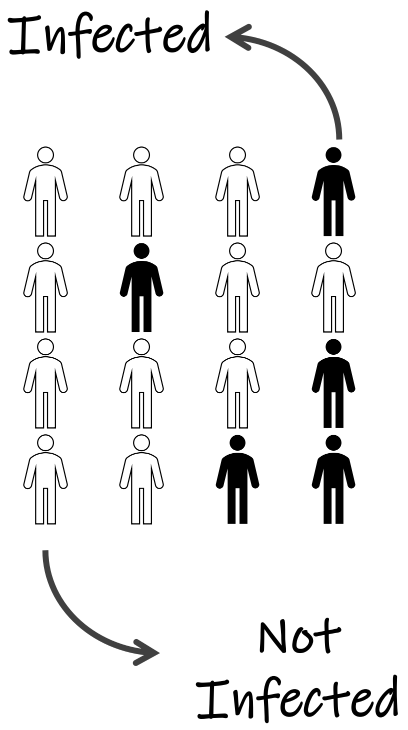
$$= 9000$$



SPREAD OF DISEASE

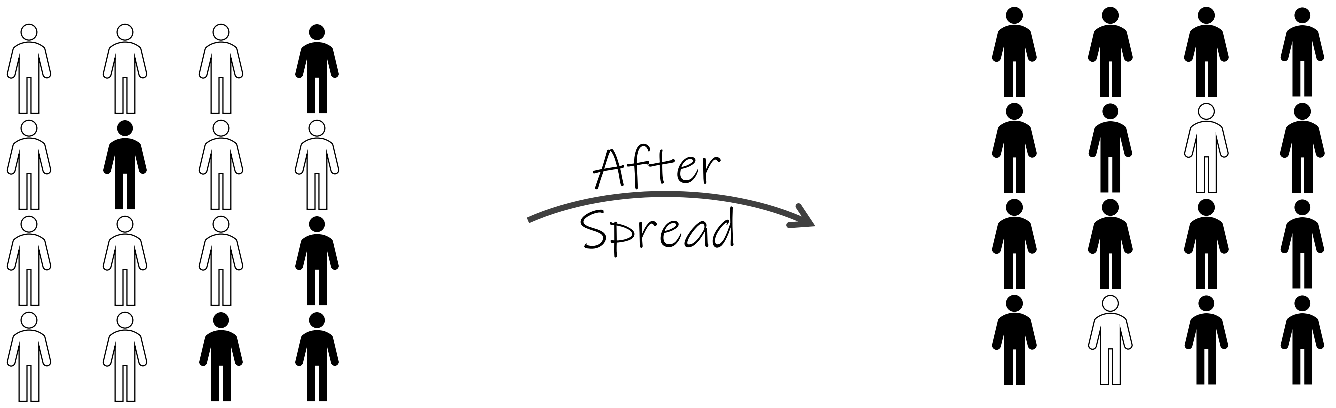
The Model

* Suppose that a disease begins to spread in a population of L individuals.
* The spread of a disease is determined by the contact between **infected individuals** and those who are **not infected**.



SPREAD OF DISEASE

The Model



Since # uninfected people is larger than who are infected, then the spread increases

Since # infected people is larger than who are uninfected, then the spread decreases

$$\begin{pmatrix}\text{Rate of}\\ \text{Spread}\end{pmatrix} \propto \begin{pmatrix}\text{Number of}\\ \text{Infected}\end{pmatrix} \cdot \begin{pmatrix}\text{Number of}\\ \text{Uninfected}\end{pmatrix}$$

SPREAD OF DISEASE

The Model

Let $y(t)$ is the number of individuals who have the disease at time t, then

$$\frac{dy}{dt} = ky(L-y) \qquad\qquad y(0) = y_0$$

Separable

k > 0 depends on the nature of the disease

The number of affected individuals at some point in time

Solution

$$\frac{dy}{dt} = ky(L-y) \longrightarrow \frac{1}{y(L-y)}dy = kdt$$

$$\longrightarrow \int \frac{1}{y(L-y)}dy = \int kdt$$

By Partial Fractions

$$\frac{1}{y(L-y)} = \frac{A}{y} + \frac{B}{L-y} = \frac{A(L-y)+By}{y(L-y)}$$

SPREAD OF DISEASE

Solution

$$\int \frac{1}{y(L-y)}dy = \int kdt$$

$$1 = A(L-y) + By$$

$$y = L \longrightarrow \boxed{B = 1/L}$$

$$y = 0 \longrightarrow \boxed{A = 1/L}$$

$$\frac{1}{L}\int \left(\frac{1}{y} + \frac{1}{L-y}\right)dy = \int kdt$$

$$\ln y - \ln(L-y) = Lkt + C \longrightarrow \ln\left(\frac{y}{L-y}\right) = Lkt + C$$

$$\longrightarrow \frac{y}{L-y} = e^{Lkt} \cdot e^{C}$$

$$\longrightarrow \frac{L-y}{y} = e^{-Lkt} \cdot e^{-C}$$

$$\longrightarrow \frac{L}{y} - 1 = e^{-Lkt} \cdot e^{-C} \longrightarrow y = \frac{L}{1 + e^{-Lkt} \cdot e^{-C}}$$

SPREAD OF DISEASE

Solution $$y = \frac{L}{1 + e^{-Lkt} \cdot e^{-C}} \longrightarrow y = \frac{L}{1 + b \cdot e^{-Lkt}}$$

Constant

But $y(0) = y_0 \longrightarrow y_0 = \dfrac{L}{1+b}$

$$\longrightarrow b = \frac{L}{y_0} - 1 = \frac{L - y_0}{y_0}$$

$$y = \frac{L}{1 + \frac{L - y_0}{y_0} \cdot e^{-Lkt}} = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-Lkt}}$$

SPREAD OF DISEASE

Example Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine the number of infected students after 6 days if it is further observed that after 4 days there were 50 infected students.

$$y(t) = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-Lkt}}$$

$L = 1000$ $y_0 = 1$

$$y(t) = \frac{1000}{1 + 999 \cdot e^{-1000kt}}$$

But $y(4) = 50 \longrightarrow 50 = \dfrac{1000}{1 + 999 \cdot e^{-4000k}}$

$$\longrightarrow 1 + 999 \cdot e^{-4000k} = 20$$

$$\longrightarrow k = -\frac{1}{4000} \ln \frac{19}{999} \approx 0.00099$$

SPREAD OF DISEASE

Example Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine the number of infected students after 6 days if it is further observed that after 4 days there were 50 infected students.

$$y(t) = \frac{y_0 L}{y_0 + (L - y_0) \cdot e^{-Lkt}}$$

$L = 1000$ $\quad y_0 = 1$

$$y(t) = \frac{1000}{1 + 999 \cdot e^{-0.99t}}$$

$$y(6) = \frac{1000}{1 + 999 \cdot e^{-(0.99)(6)}}$$

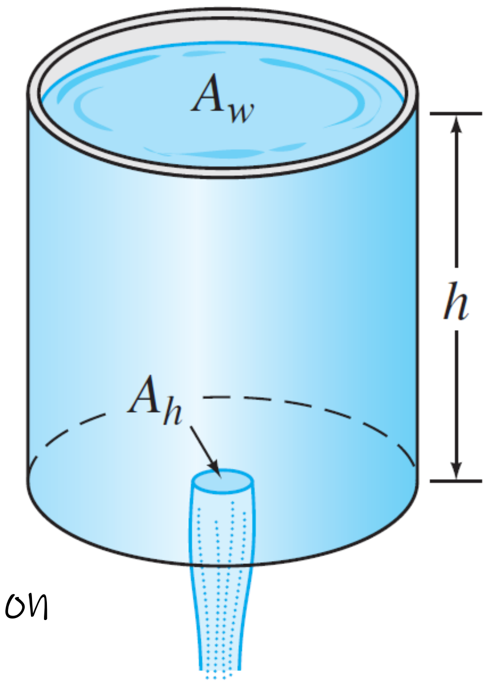
$$= \frac{1000}{1 + 999 \cdot e^{-5.94}} \approx 275.53$$

Students

DRAINING A TANK AND TORRICELLI'S LAW

Torricelli's Law

The law states that the speed v of efflux of water though a sharp-edged at the bottom of a tank filled to a depth h is the same as the speed that a body (water) would acquire in falling freely from a height h.



KINETIC ENERGY = POTENTIAL ENERGY

Law of Conservation of Energy

(طاقة الحركة) (طاقة الوضع)

$$\frac{1}{2}mv^2 = mgh$$

Body mass

$g = 32\ \text{ft/s}^2$

Acceleration due to gravity

$$v = \sqrt{2gh}$$

DRAINING A TANK AND TORRICELLI'S LAW

The Model

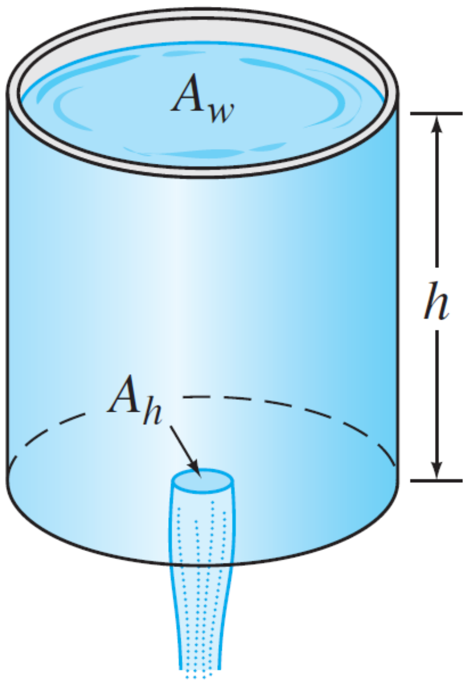
Let $h(t)$ the depth of water remaining in the tank at time t (in seconds)

$V(t)$ the volume of water in the tank at time t (in seconds)

A_h the area of the hole (in ft^2)

A_w the area of the upper surface (in ft^2)

FIXED



$v = \sqrt{2gh}$

$g = 32 \text{ ft/s}^2$

Then Flow Rate = Hole Area × Velocity of Flow

$$-\frac{dV}{dt} = A_h \sqrt{2gh}$$

$$\boxed{\frac{dV}{dt} = -8A_h \sqrt{h}}$$

DRAINING A TANK AND TORRICELLI'S LAW

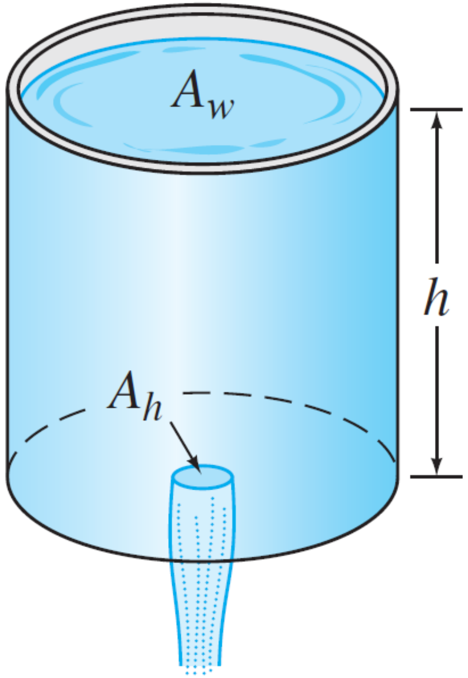
The Model

$$\boxed{\frac{dV}{dt} = -8A_h\sqrt{h}}$$

But $V(t) = A_w h(t) \longrightarrow \dfrac{dV}{dt} = A_w \dfrac{dh}{dt}$

$$\longrightarrow A_w \frac{dh}{dt} = -8A_h\sqrt{h}$$

$$\longrightarrow \boxed{\frac{dh}{dt} = -8\frac{A_h}{A_w}\sqrt{h}}$$



$v = \sqrt{2gh}$

DRAINING A TANK AND TORRICELLI'S LAW

Solution $$\frac{dh}{dt} = -8\frac{A_h}{A_w}\sqrt{h}$$

$$\longrightarrow \frac{dh}{dt} = \underbrace{-8\frac{A_h}{A_w}}_{\text{Constant } b}\sqrt{h}$$

$$\longrightarrow \frac{dh}{\sqrt{h}} = b\,dt$$ Separable

$$\longrightarrow \int \frac{1}{\sqrt{h}}dh = \int b\,dt$$

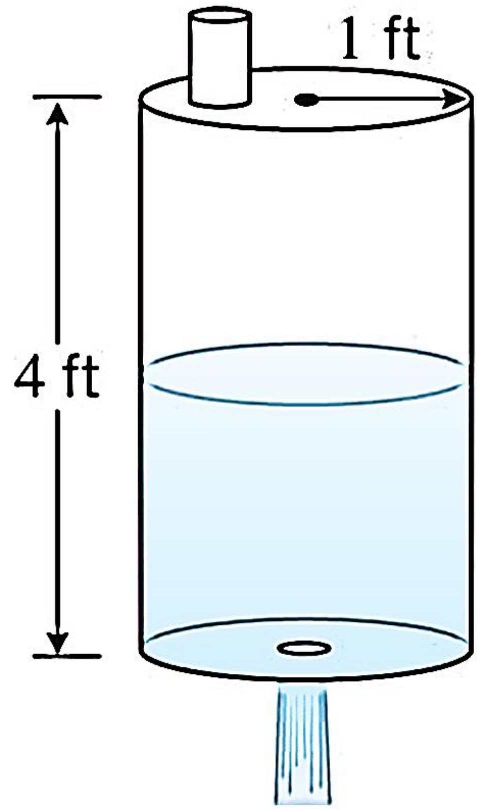
$$\longrightarrow 2\sqrt{h} = bt + C$$

$$\longrightarrow 2\sqrt{h} = -8\frac{A_h}{A_w}t + C$$

$$\longrightarrow h(t) = \left(-4\frac{A_h}{A_w}t + C\right)^2 \qquad C > 0$$

DRAINING A TANK AND TORRICELLI'S LAW

Example Suppose that the cylindrical tank in the figure is filled to a depth of 4 feet at time $t = 0$ and the radius of the circular hole is 0.045 feet.



a) **Find $\boldsymbol{h(t)}$**

$$h(t) = \left(-4\,\frac{A_h}{A_w}\,t + C\right)^2$$

$$A_h = r_h^2\pi \;=(0.045)^2\pi \;=0.002025\,\pi$$

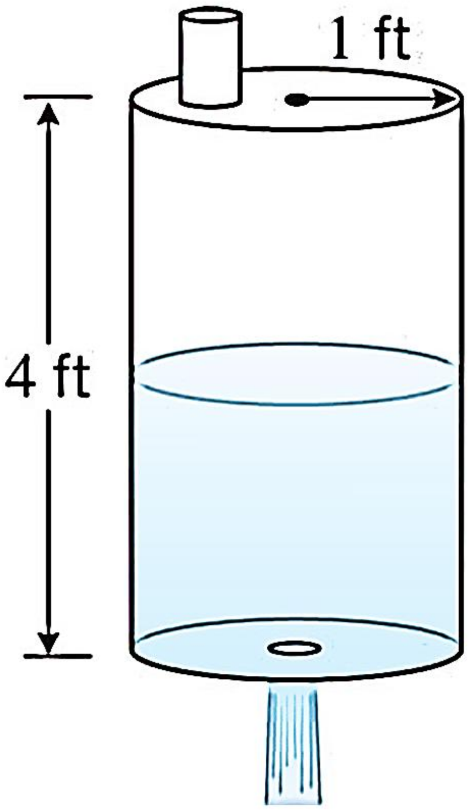
$$A_w = r_h^2\pi \;=(1)^2\pi \;=\pi$$

But $h(0) = 4 \longrightarrow 4 = (0 + C)^2 \longrightarrow C = 2$

$$\therefore h(t) = \left(-4\,\frac{0.002025\,\pi}{\pi}\,t + 2\right)^2 \;= (2 - 0.0081\,t)^2$$

DRAINING A TANK AND TORRICELLI'S LAW

Example Suppose that the cylindrical tank in the figure is filled to a depth of 4 feet at time $t = 0$ and the radius of the circular hole is 0.045 foot.



b) How many minutes will it take for the tank to drain completely?

$$h(t) = (2 - 0.0081\, t)^2 \quad = 0$$

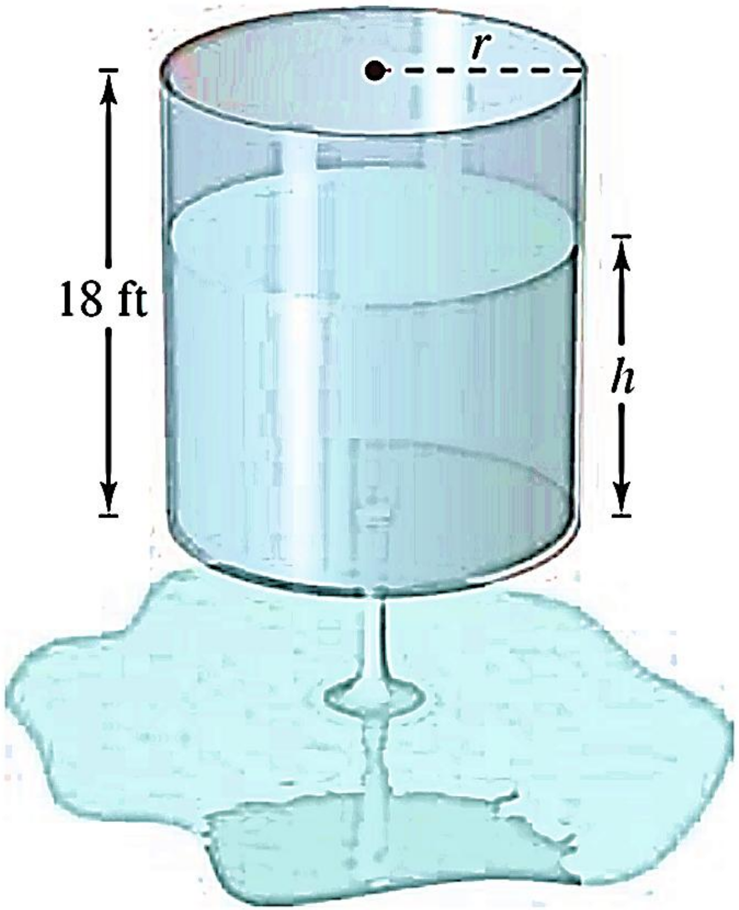
$$2 - 0.0081\, t = 0$$

$$t = \frac{2}{0.0081} = 246.9 \quad \text{seconds}$$

$$= 4.12 \quad \text{minutes}$$

DRAINING A TANK AND TORRICELLI’S LAW

Exercise The cylindrical water tank shown in the figure has a height of 18 feet. When the tank is full, a circular valve (صمّام) is opened at the bottom of the tank. After 30 minutes, the depth of the water is 12 feet.

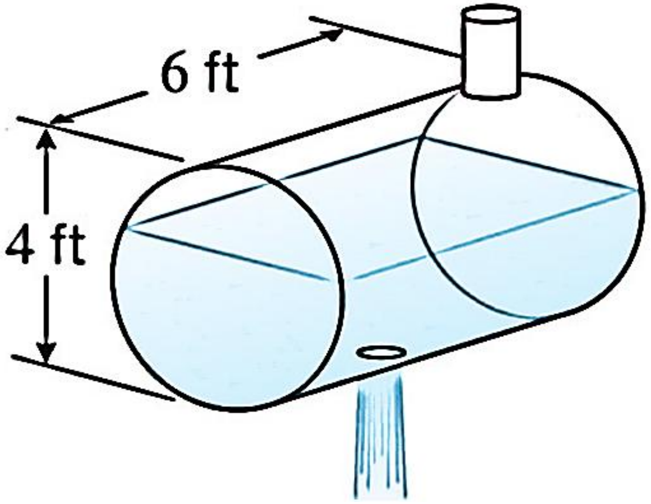


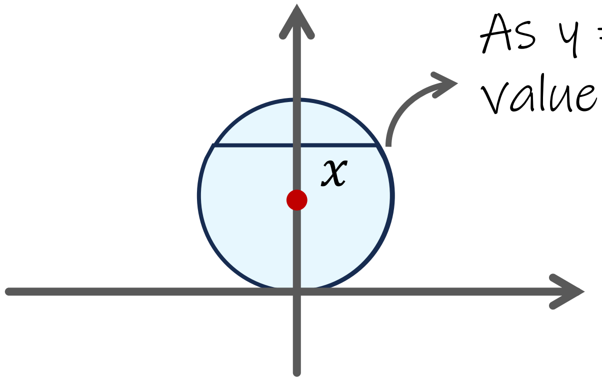
a) how long will it take for the tank to drain completely?

b) What is the depth of the water in the tank after 1 hour?

DRAINING A TANK AND TORRICELLI'S LAW

Example Suppose that the tank in the figure is filled to a depth of 4 feet at time $t = 0$ and the radius of the circular hole is 0.045 feet. Find $h(t)$





As y = h(t) changes, the value of x also changes

Center (0,2)

radius r = 2

$x^2 + (y-2)^2 = 4$

$$x^2 = 4 - (h-2)^2$$

$$x = \sqrt{4-(h-2)^2}$$

$$x = \sqrt{4h - h^2}$$

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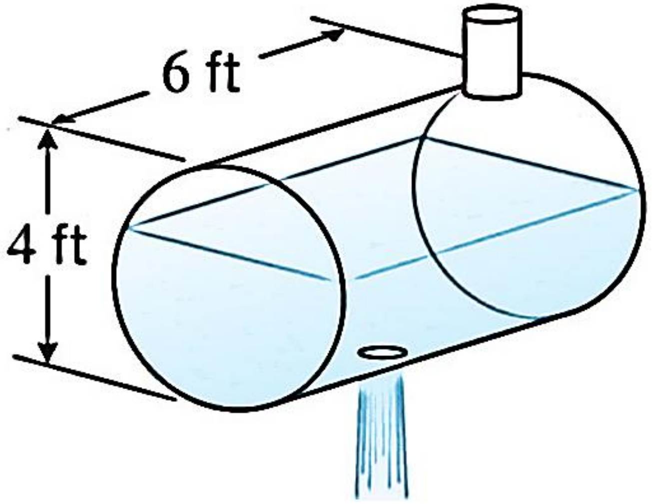
$$2x = \sqrt{4h - h^2}$$

DRAINING A TANK AND TORRICELLI'S LAW

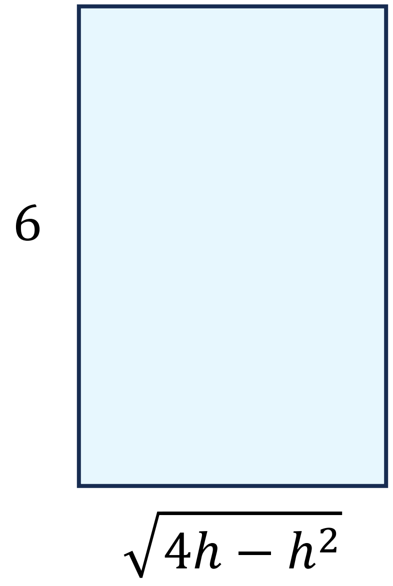
Example Suppose that the tank in the figure is filled to a depth of 4 feet at time $t = 0$ and the radius of the circular hole is 0.045 feet. Find $h(t)$



$$\frac{dh}{dt} = -8\frac{A_h}{A_w}\sqrt{h}$$

$$A_h = r_h^2\pi = (0.045)^2\pi \;= 0.002025\,\pi$$

$$A_w = (\text{heigh}) \cdot (\text{width}) \;= 12\sqrt{4h - h^2}$$

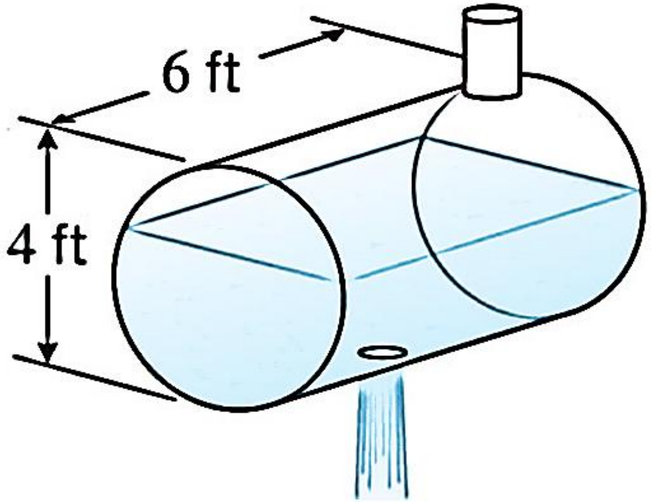


$$\frac{dh}{dt} = -8\frac{0.002025\,\pi}{12\sqrt{4h - h^2}}\sqrt{h} = -0.00424\sqrt{\frac{h}{4h - h^2}}$$

$$= -0.00424\frac{1}{\sqrt{4 - h}}$$

DRAINING A TANK AND TORRICELLI'S LAW

Example Suppose that the tank in the figure is filled to a depth of 4 feet at time $t = 0$ and the radius of the circular hole is 0.045 feet. Find $h(t)$



$$\frac{dh}{dt} = -0.00424\frac{1}{\sqrt{4-h}}$$

Separable

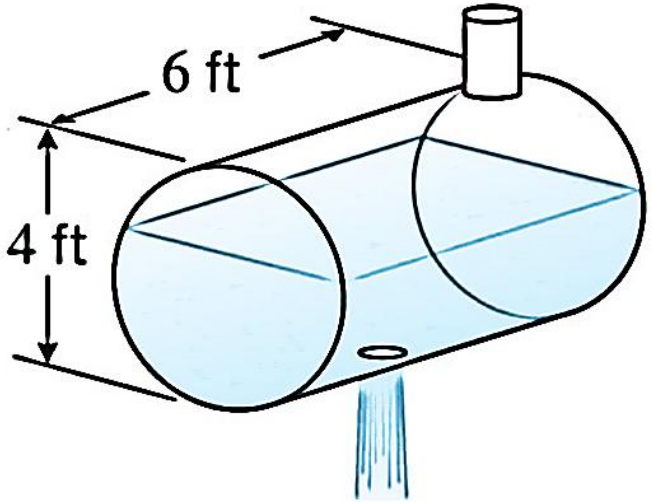
$$\sqrt{4-h}\,dh = -0.00424\,dt \longrightarrow \int \sqrt{4-h}\,dh = -\int 0.00424\,dt$$

$$\longrightarrow \frac{2}{3}(4-h)^{3/2} = -0.00424\,t + C$$

But $h(0) = 4 \longrightarrow \frac{2}{3}(4-4)^{3/2} = -0.00424 \cdot 0 + C \longrightarrow C = 0$

DRAINING A TANK AND TORRICELLI'S LAW

Example Suppose that the tank in the figure is filled to a depth of 4 feet at time $t = 0$ and the radius of the circular hole is 0.045 feet. Find $h(t)$



$$\frac{2}{3}(4-h)^{3/2} = -0.00424\,t$$

$$(4-h)^{3/2} = -0.00636\,t$$

$$(4-h)^{3} = 0.00000405\,t^{2}$$

$$4-h = 0.0343\,t^{2/3} \longrightarrow \boxed{h(t) = 4 - 0.0343\sqrt[3]{t^2}}$$

Course: Mathematical Modeling

Chapter: [3]

MODELING WITH DIFFERENTIAL EQUATIONS

Section: [3.3]

Models with 2^{nd} Order Differential Equations

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

FORM

$$g_n(x)y^{(n)} + g_{n-1}(x)y^{(n-1)} + \cdots + g_1(x)y' + g_0(x)y = f(x)$$

Higher Order Differential Equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

Higher Order Differential Equation with Constant Coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Homogeneous Higher Order Differential Equation with Constant Coefficients

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Auxiliary Equation

* Considering the special case of the homogeneous linear 2nd order differential equation

$$ay'' + by' + cy = 0 \qquad \text{------} \ (\mathbf{1})$$

where a, b, and c are constants.

* If we try to find a solution of the form $y = e^{mx}$, then after substituting $y' = me^{mx}$ and $y'' = m^2e^{mx}$, equation **(1)** becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$e^{mx}\left(am^2 + bm + c\right) = 0$$

$$am^2 + bm + c = 0$$

Auxiliary Equation

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Auxiliary Equation

$$am^2 + bm + c = 0$$

The two roots are:

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Three cases of the general solution:

- Distinct real roots $(m_1 \neq m_2)$ if $b^2 - 4ac > 0$
- Equal real roots $(m_1 = m_2)$ if $b^2 - 4ac = 0$
- Conjugate complex roots $(\alpha \pm \beta i)$ if $b^2 - 4ac < 0$

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Distinct Real Roots	$m_1, m_2 \in \mathbb{R}$	and	$m_1 \neq m_2$	then	$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$
Repeated Real Roots	$m_1, m_2 \in \mathbb{R}$	and	$m_1 = m_2$	then	$\boxed{y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}}$
Complex Roots	$m_1, m_2 \in \mathbb{C}$	and	$m_1 = \alpha + i\beta$ $m_2 = \alpha - i\beta$ $\alpha, \beta > 0$	then	$\boxed{y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)}$

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Example Solve the following differential equations.

A) $2y'' - 5y' - 3y = 0$

Auxiliary Equation: $2m^2 - 5m - 3 = 0$

$$m = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)} = \frac{5 \pm \sqrt{49}}{4}$$

$$m_1 = 3 \qquad m_2 = -\frac{1}{2}$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{-x/2}}$$

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Example Solve the following differential equations.

B) $y'' - 10y' + 25y = 0$

Auxiliary Equation: $m^2 - 10m + 25 = 0$

$$m = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(25)}}{2(1)} = \frac{10 \pm \sqrt{0}}{2}$$

$m_1 = 5 \qquad m_2 = 5$

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Example Solve the following differential equations.

C) $y'' + 4y' + 7y = 0$

Auxiliary Equation: $m^2 + 4m + 7 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2(1)} = \frac{-4 \pm \sqrt{-12}}{2}$$

$$m_1 = -2 + i\sqrt{3} \qquad \alpha = -2$$

$$m_2 = -2 - i\sqrt{3} \qquad \beta = \sqrt{3}$$

$$\boxed{y = e^{-2x}\left(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right)}$$

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Example Solve the initial value problem

$$4y'' + 4y' + 17y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

Auxiliary Equation: $4m^2 + 4m + 17 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4(4)(17)}}{2(4)} = \frac{-4 \pm \sqrt{-256}}{8}$$

$$m_1 = -\frac{1}{2} + 2i \qquad \alpha = -\frac{1}{2}$$

$$m_2 = -\frac{1}{2} - 2i \qquad \beta = 2$$

$$\boxed{y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)}$$

HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Example Solve the initial value problem

$$4y'' + 4y' + 17y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

$$\boxed{y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)}$$

$$y(0) = 0 \longrightarrow 0 = e^0(c_1 \cos 0 + c_2 \sin 0) \longrightarrow 0 = c_1$$

$$\therefore y(x) = c_2 e^{-x/2} \sin 2x \longrightarrow y'(x) = \frac{1}{2} c_2 e^{-x/2}(4 \cos 2x - \sin 2x)$$

$$y'(0) = 1 \longrightarrow 1 = \frac{1}{2} c_2 e^0 (4 \cos 0 - \sin 0) \longrightarrow \frac{1}{2} = c_2$$

$$\therefore y(x) = \frac{1}{2} e^{-x/2} \sin 2x$$

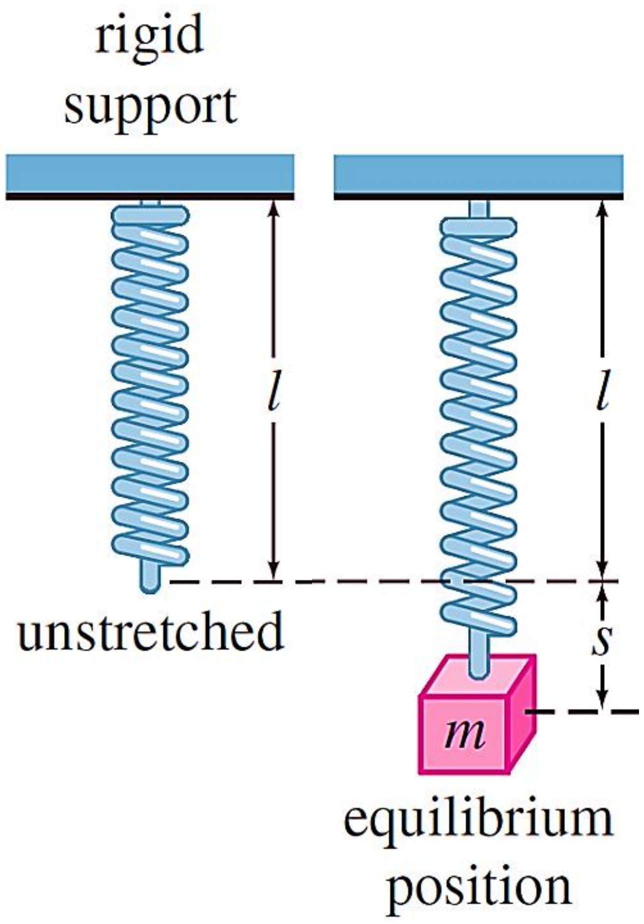
SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Hooke's Law

* Suppose a spring/mass system consists of a flexible spring suspended vertically from a rigid support (دعامة صلبة) with a mass m is attached to its free end.

* The amount of stretch s, or elongation, of the spring will depend on the mass; masses with different weights stretch the spring by differing amounts.

* Hooke's Law states that the spring applies a restoring force (قوة شد مسترجِعة) F opposite to the direction of elongation and proportional to the amount of elongation s.

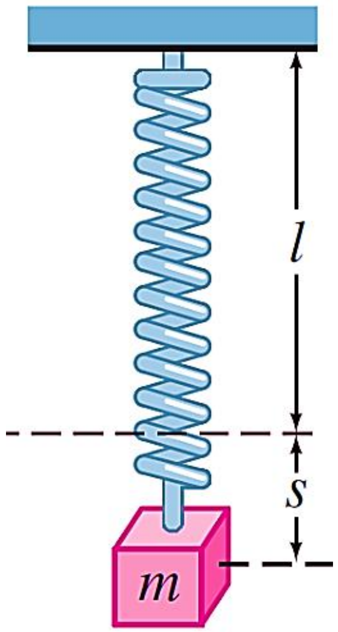


$$F = -ks \qquad (k > 0)$$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Newton's Second Law

* When a mass m is attached to the lower end of a spring it stretches the spring by an amount s and attains an equilibrium (rest) position at which its weight W is balanced by the restoring force ks of the spring.



equilibrium position

$$W = mg$$

g is the acceleration due to gravity
$g = 32\ ft/s^2$
$g = 9.8\ m/s^2$

m is measured in kilograms or grams

$$F = -ks$$

$$(k > 0)$$

* The condition of equilibrium is

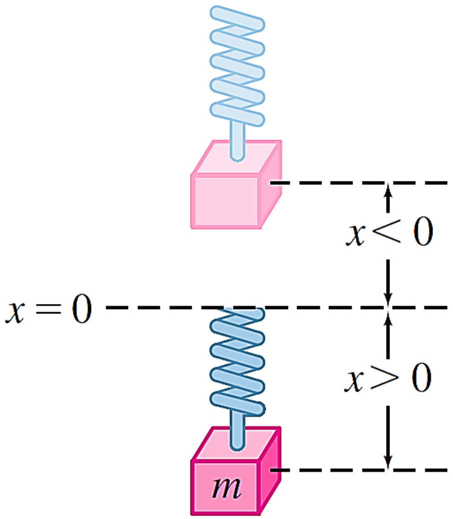
$$mg = ks$$

$$mg - ks = 0$$

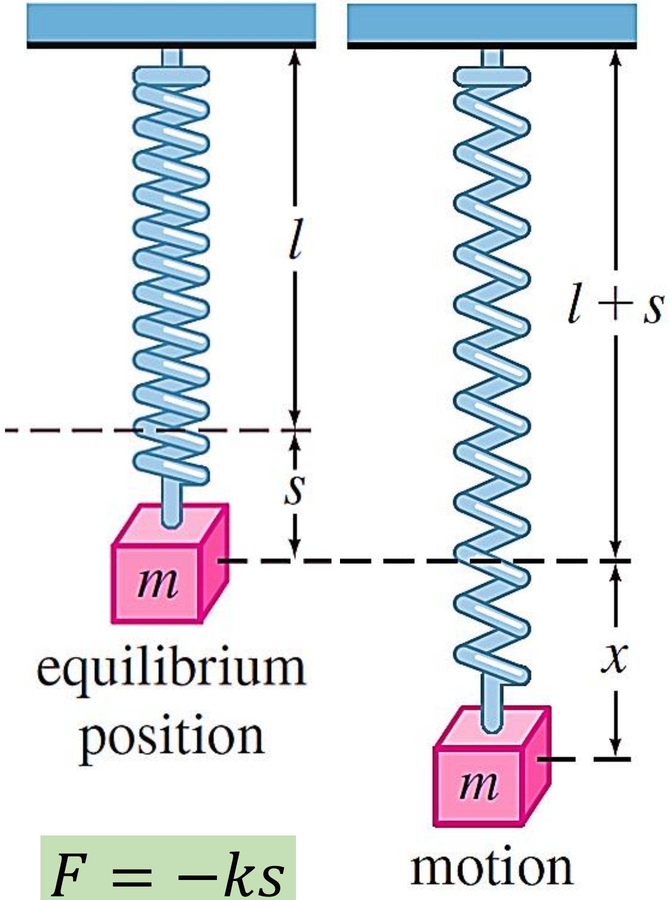
SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Newton's Second Law

* Now suppose the mass on the spring is set in motion by giving it an initial displacement (an elongation or a compression) and an initial velocity.



* Let us assume that the motion takes place in a vertical line, that the displacements $x(t)$ of the mass are measured along this line such that $x = 0$ corresponds to the equilibrium position, and that displacements measured below the equilibrium position are positive.



$F = -ks$

$mg = ks$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

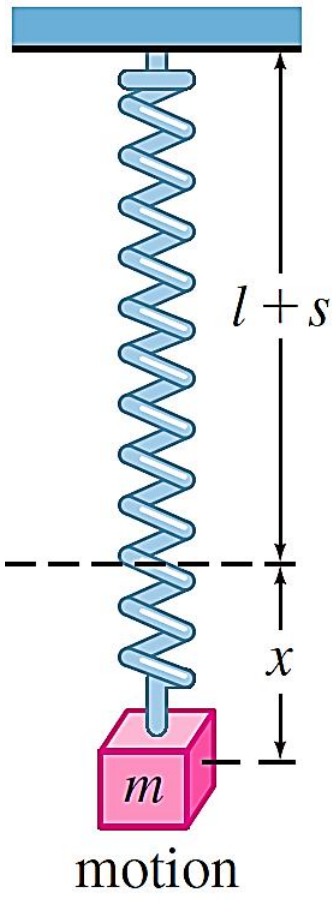
Newton's Second Law

* Newton's second law of motion: the net force on a moving body of mass m is given by $\sum F = ma$ where $a = \frac{d^2x}{dt^2}$ is its acceleration (التسارع).

* If we assume that the mass vibrates free of all other external forces, then

$$m\frac{d^2x}{dt^2} = -k(x+s) + mg$$
$$= -kx - ks + mg \ = -kx$$

$$\boxed{\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x} \qquad k, m > 0$$



$F = -ks$

$mg = ks$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Solution

$$\boxed{\therefore \frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

$$x(0) = x_0 \qquad x'(0) = x_1$$

Initial Displacement

Initial Velocity

* If $x_0 > 0$, then mass starts from a point **below** the equilibrium position

* If $x_0 < 0$, then mass starts from a point **above** the equilibrium position

* If $x_1 > 0$, then mass starts with an imparted **downward** velocity

* If $x_1 < 0$, then mass starts with an imparted **upward** velocity

* If $x_1 = 0$, then the mass is released from rest

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Solution

$$\boxed{\therefore \frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \qquad x(0) = x_0 \qquad x'(0) = x_1$$

Let $\omega^2 = \dfrac{k}{m}$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \longrightarrow p^2 + \omega^2 = 0 \longrightarrow p^2 = -\omega^2$$

$$\longrightarrow p = 0 \pm \omega i \longrightarrow \alpha = 0 \qquad \beta = \omega$$

$$\therefore x(t) = e^0(c_1 \cos \omega t + c_2 \sin \omega t)$$

$$= c_1 \cos \omega t + c_2 \sin \omega t \quad = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Example A mass weighing 9.8 N stretches a spring 0.2 m. At $t = 0$ the mass is released from a point 0.25 m below the equilibrium position with an upward velocity of 0.4 m/s. Determine the equation of motion.

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$x(0) = 0.25$$
$$x'(0) = -0.4$$

$$W = mg \longrightarrow 9.8 = 9.8m \longrightarrow m = 1$$

$$F = ks \longrightarrow 9.8 = 0.2k \longrightarrow k = 49$$

Hooke's Law

$$x(t) = c_1 \cos(7t) + c_2 \sin(7t)$$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Example A mass weighing 9.8 N stretches a spring 0.2 m. At $t = 0$ the mass is released from a point 0.25 m below the equilibrium position with an upward velocity of 0.4 m/s. Determine the equation of motion.

$$x(t) = c_1 \cos(7t) + c_2 \sin(7t) \qquad x(0) = 0.25 \qquad x'(0) = -0.4$$

$$x'(t) = -7c_1 \sin(7t) + 7c_2 \cos(7t)$$

$$x(0) = 0.25 \longrightarrow c_1 = \frac{1}{4}$$

$$x'(0) = -0.4 \longrightarrow c_2 = -\frac{2}{35}$$

$$\boxed{\therefore\ x(t) = \frac{1}{4}\cos(7t) - \frac{2}{35}\sin(7t)}$$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Example A mass weighing 7.35 newtons, attached to the end of a spring, with stiffness (صلابة) of 72 N/m. Initially, the mass is released from rest from a point 0.25 m above the equilibrium position. Find the equation of motion.

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$x(0) = -0.25$$

$$x'(0) = 0$$

$$k = 72$$

$$W = mg \longrightarrow 7.35 = 9.8m \longrightarrow m = \frac{3}{4}$$

$$x(t) = c_1 \cos\left(\sqrt{96}\,t\right) + c_2 \sin\left(\sqrt{96}\,t\right)$$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Example A mass weighing 7.35 newtons, attached to the end of a spring, with stiffness (صلابة) of 72 N/m. Initially, the mass is released from rest from a point 0.25 m above the equilibrium position. Find the equation of motion.

$$x(t) = c_1 \cos\left(4\sqrt{6}\, t\right) + c_2 \sin\left(4\sqrt{6}\, t\right)$$

$$x(0) = -0.25$$

$$x'(0) = 0$$

$$x'(t) = -4\sqrt{6}\, c_1 \sin\left(4\sqrt{6}\, t\right) + 4\sqrt{6}\, c_2 \cos\left(4\sqrt{6}\, t\right)$$

$$x(0) = -0.25 \longrightarrow c_1 = -\frac{1}{4}$$

$$x'(0) = 0 \longrightarrow c_2 = 0$$

$$\boxed{\therefore\ x(t) = -\frac{1}{4}\cos\left(4\sqrt{6}\, t\right)}$$

SPRING/MASS SYSTEMS: FREE UNDAMPED MOTION

Exercise A mass weighing 7.35 newtons, attached to the end of a spring, with stiffness (صلابة) of 72 N/m. Initially, the mass is released from the equilibrium position with a downward velocity of 2 m/s. Find the equation of motion.