Course: Calculus (3)

Chapter: [12]

VECTOR-VALUED FUNCTIONS

Section: [12.1]

INTRODUCTION TO VECTOR-VALUED FUNCTIONS

IN THIS CHAPTER

- ✓ We will consider *functions whose values are* vectors.

  Functions that associate vectors with real numbers.

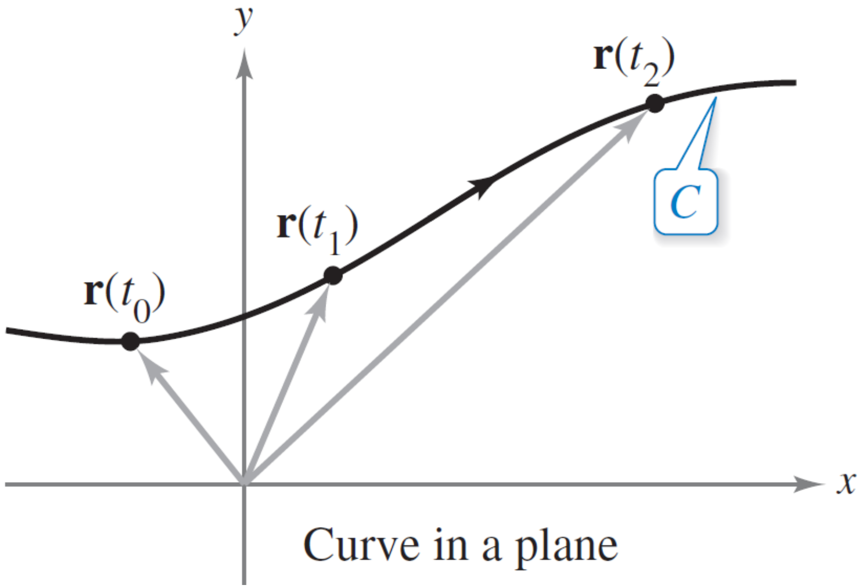
- ✓ In this section we will discuss more general parametric curves, and we will show how vector notation can be used to express parametric equations in a more compact form.

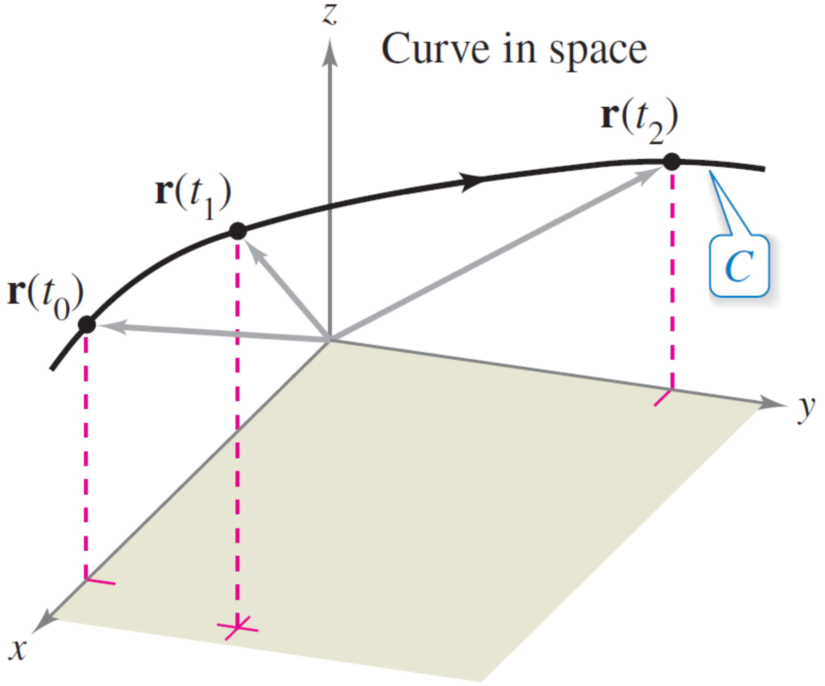
VECTOR-VALUED FUNCTIONS

A function of the form

$$\begin{aligned}\mathbf{r}(t) &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \\ &= \langle f(t), g(t), h(t)\rangle\end{aligned}$$

is a vector-valued function, where the component functions f, g and h are real-valued functions of the parameter t.





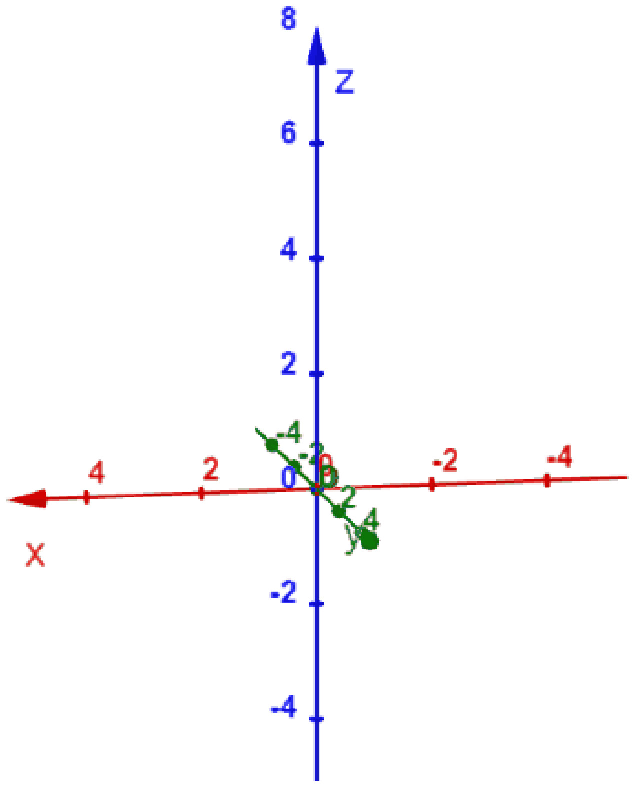
PARAMETRIC CURVES IN 3 −SPACE

Example The parametric equations

$$x = 1 - t$$
$$y = 3t$$
$$z = 2t$$

represent a line in 3 −space that passes through the point $(1,0,0)$ and is parallel to the vector $\langle -1, 3, 2\rangle$.

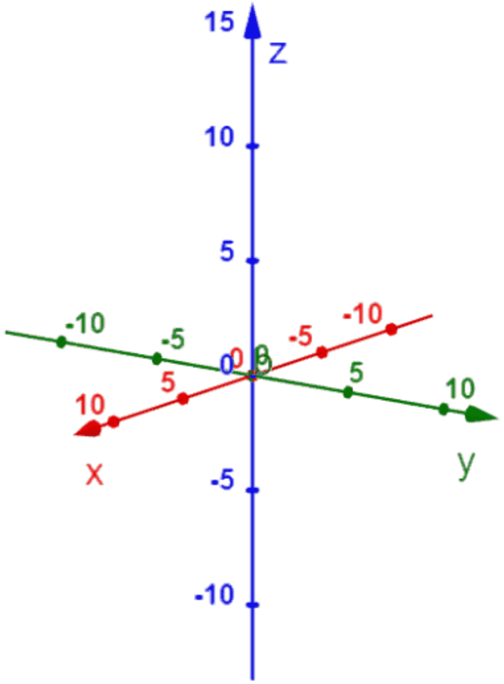
$$\mathbf{r}(t) = (1 - t)\mathbf{i} + 3t\,\mathbf{j} + 2t\,\mathbf{k}$$
$$= \langle 1 - t, 3t\,, 2t\rangle$$



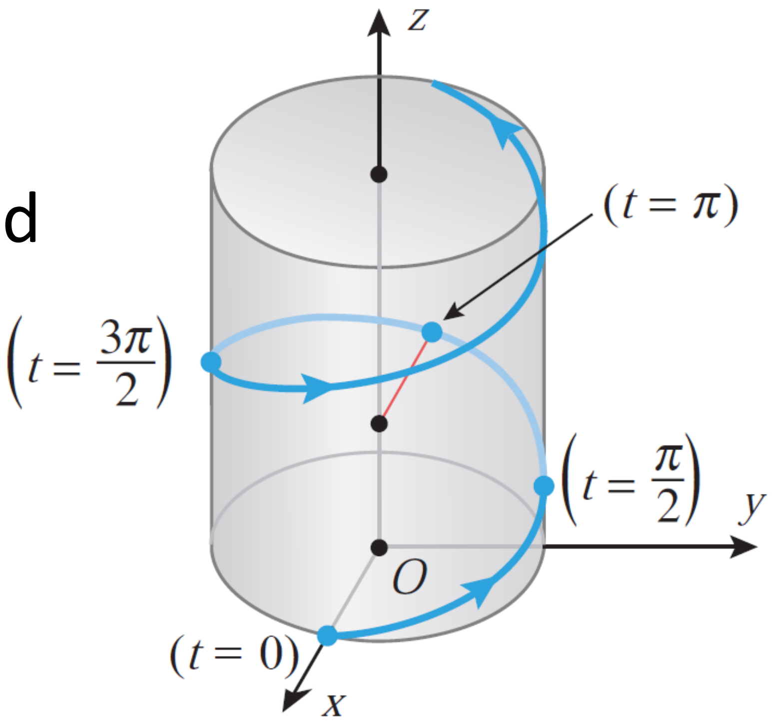
PARAMETRIC CURVES IN 3 −SPACE

Example Describe the parametric curve represented by the equations

$$x = 10\cos t$$
$$y = 10\sin t$$
$$z = t$$



$$\mathbf{r}(t) = 10\cos t\,\mathbf{i} + 10\sin t\,\mathbf{j} + t\mathbf{k}$$
$$= \langle 10\cos t, 10\sin t, t\rangle$$



Circular HELIX

VECTOR-VALUED FUNCTIONS

The domain of a vector-valued function $\mathbf{r}(t)$ is the set of allowable values for t.

NOTE Usual reasons to restrict a domain:

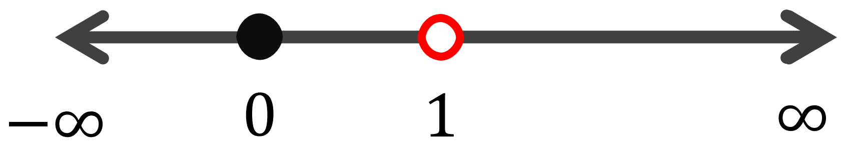
1. Avoid division by 0.
2. Avoid even roots of negative numbers.
3. Avoid logarithms of negative numbers or 0.

VECTOR-VALUED FUNCTIONS

Example Find the natural domain of $\mathbf{r}(t) = \ln|t-1|\,\mathbf{i} + e^{t}\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$

$x(t) = \ln|t-1|$ ☑ Domain $= \mathbb{R} - \{1\}$

$y(t) = e^{t}$ ☑ Domain $= \mathbb{R}$



$z(t) = \sqrt{t}$ ☑ Domain $= [0, \infty)$

$\therefore$ The domain of $\mathbf{r}(t)$ is the *intersection of these sets*. $[0,1) \cup (1, \infty)$

Course: Calculus (3)

Chapter: [12]

VECTOR-VALUED FUNCTIONS

Section: [12.2]

CALCULUS OF VECTOR-VALUED FUNCTIONS

LIMITS AND CONTINUITY

- Many techniques and definitions used in the calculus of real-valued functions can be applied to vector-valued functions.
- For instance, you can add and subtract vector-valued functions, multiply a vector-valued function by a scalar, take the limit of a vector-valued function, differentiate a vector-valued function, and so on.

LIMITS AND CONTINUITY

$$\begin{aligned}\mathbf{r}_1(t) + \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j}.\end{aligned}$$

$$\begin{aligned}\mathbf{r}_1(t) - \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] - [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) - f_2(t)]\mathbf{i} + [g_1(t) - g_2(t)]\mathbf{j}.\end{aligned}$$

$$\begin{aligned}c\mathbf{r}(t) &= c[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] \\ &= cf_1(t)\mathbf{i} + cg_1(t)\mathbf{j}.\end{aligned}$$

$$\begin{aligned}\frac{\mathbf{r}(t)}{c} &= \frac{[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}]}{c}, \quad c \neq 0 \\ &= \frac{f_1(t)}{c}\mathbf{i} + \frac{g_1(t)}{c}\mathbf{j}.\end{aligned}$$

LIMITS AND CONTINUITY

If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}$$

provided f, g, and h have limits as $t \to a$.

Example If $\mathbf{r}(t) = \frac{3}{t^2}\mathbf{i} + \frac{\ln t}{t^2-1}\mathbf{j} + \cos(\pi t)\,\mathbf{k}$, find $\lim_{t\to 1} \mathbf{r}(t)$.

$$\lim_{t\to 1} \mathbf{r}(t) = \left\langle 3, \frac{1}{2}, -1 \right\rangle$$

$$= 3\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$\lim_{t\to 1} \frac{3}{t^2} = 3$$

$$\lim_{t\to 1} \frac{\ln t}{t^2 - 1} = \lim_{t\to 1} \frac{1/t}{2t} = \frac{1}{2}$$

$$\lim_{t\to 1} \cos(\pi t) = -1$$

LIMITS AND CONTINUITY

Example If $\mathbf{r}(t) = \frac{2t^2-1}{t^2+t}\mathbf{i} + \sin\left(\frac{1}{t}\right)\mathbf{j} + te^{-t}\mathbf{k}$, find $\lim_{t\to\infty} \mathbf{r}(t)$.

$$\lim_{t\to\infty} \mathbf{r}(t) = \langle 2,0,0 \rangle = 2\mathbf{i}$$

$$\lim_{t\to\infty} \frac{2t^2 - 1}{t^2 + t} = 2$$

$$\lim_{t\to\infty} \sin\left(\frac{1}{t}\right) = 0$$

$$\lim_{t\to\infty} te^{-t} = 0 \cdot \infty$$

$$= \lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0$$

LIMITS AND CONTINUITY

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ when the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** I when it is continuous at every point in the interval.

Example The vector-valued function $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{t^2-1}\mathbf{j} + t\mathbf{k}$, is discontinuous at $t = \pm 1$.

It is continuous for all $t \in \mathbb{R} - \{-1,1\}$

DERIVATIVES

- The derivative of a vector-valued function is *defined by a limit* like that for the derivative of a real-valued function.

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

- The derivative of $\mathbf{r}(t)$ can be *expressed as*

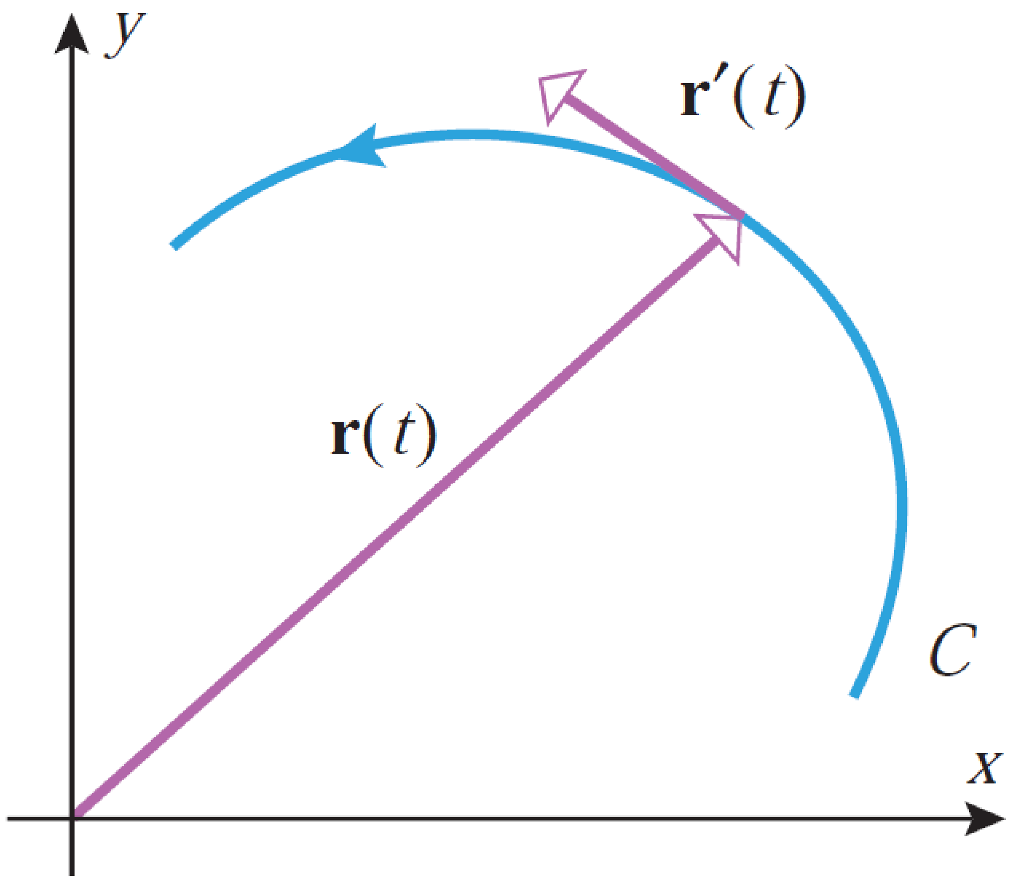
$$\frac{d}{dt}[\mathbf{r}(t)], \qquad \frac{d\mathbf{r}}{dt}, \qquad \mathbf{r}'(t), \qquad \mathbf{r}'$$

- Keep in mind that $\mathbf{r}(t)$ is a *vector*, not a number, and hence *has a magnitude and a direction* for each value of t, except if $\mathbf{r}(t) = \mathbf{0}$.

DERIVATIVES

Suppose that C is the graph of a vector-valued function $\mathbf{r}(t)$ and that $\mathbf{r}'(t)$ exists and is nonzero for a given value of t.

If the vector $\mathbf{r}'(t)$ is positioned with its initial point at the terminal point of the radius vector $\mathbf{r}(t)$, then $\mathbf{r}'(t)$ is tangent to C and points in the direction of increasing parameter.

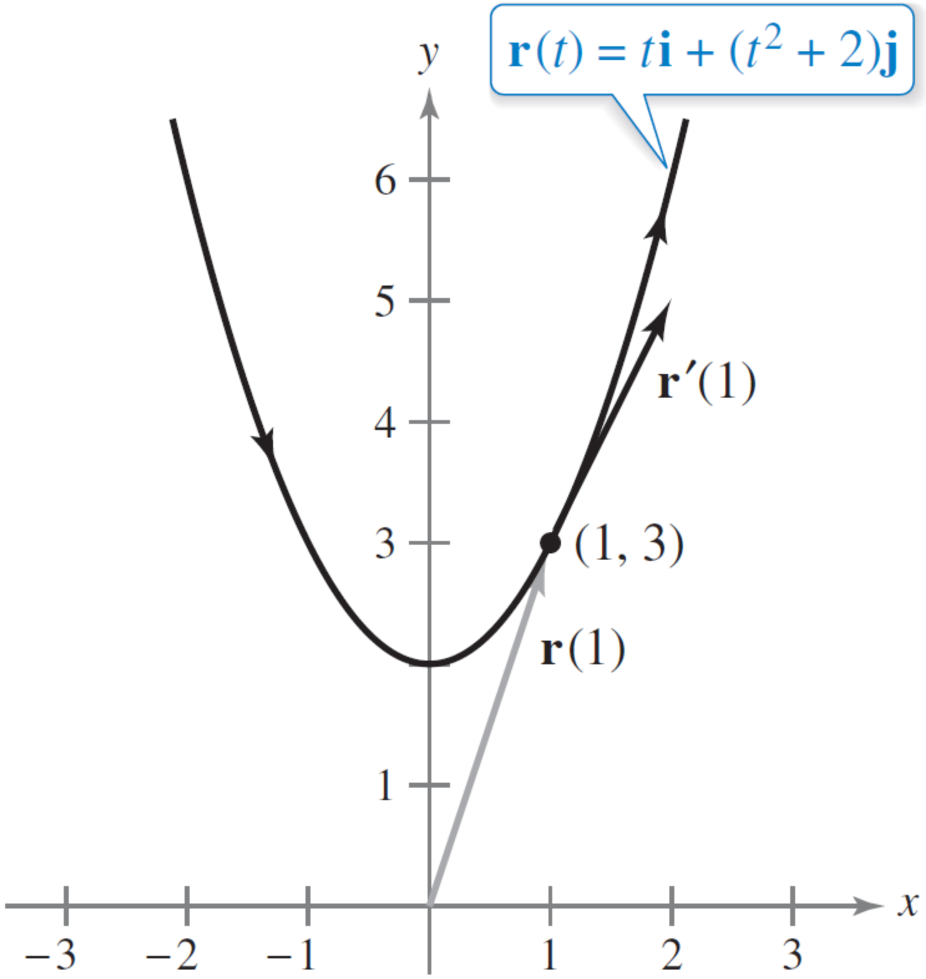


DERIVATIVES

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Example For the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2)\mathbf{j}$, find $\mathbf{r}'(1)$.



$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j}$$

DERIVATIVES

Example For the vector-valued function $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\mathbf{k}$, find:

1. $\mathbf{r}'(t)$ $\qquad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + 2\mathbf{k}$

2. $\mathbf{r}''(t)$ $\qquad \mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$

3. $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ $\qquad \mathbf{r}'(t) \cdot \mathbf{r}''(t) = \sin t \cos t - \cos t \sin t = 0$

4. $\mathbf{r}'(t) \times \mathbf{r}''(t)$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} = 2\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j} + \mathbf{k}$$

DERIVATIVE RULES

1. $\dfrac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

2. $\dfrac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

3. $\dfrac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$

4. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

5. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

6. $\dfrac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$

7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

DERIVATIVE RULES

Example For $\mathbf{u}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\,\mathbf{k}$ and $\mathbf{v}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$ then:

1

$$\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}(t) \cdot \mathbf{v}'(t) + \mathbf{u}'(t) \cdot \mathbf{v}(t)$$

$$= \left\langle \frac{1}{t}, -1, \ln t \right\rangle \cdot \langle 2t, -2, 0 \rangle + \left\langle \frac{-1}{t^2}, 0, \frac{1}{t} \right\rangle \cdot \langle t^2, -2t, 1 \rangle$$

$$= (2 + 2 + 0) + \left(-1 + 0 + \frac{1}{t} \right)$$

$$= 3 + \frac{1}{t}$$

DERIVATIVE RULES

Example For $\mathbf{u}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\,\mathbf{k}$ and $\mathbf{v}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$ then:

2

$$\frac{d}{dt}[\mathbf{v}(t) \times \mathbf{v}'(t)] = \mathbf{v}(t) \times \mathbf{v}''(t) + \mathbf{v}'(t) \times \mathbf{v}'(t)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0}$$

$$= 2\mathbf{j} + 4t\mathbf{k}$$

TANGENT LINES TO GRAPHS OF VECTOR-VALUED FUNCTIONS

Example Find parametric equations of the tangent line to the circular helix $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$ at the point where $t = \pi$.

$t = \pi$

POINT

$$(\cos \pi, \sin \pi, \pi) = (-1, 0, \pi)$$

TANGENT VECTOR

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}'(\pi) = -\mathbf{j} + \mathbf{k}$$

∴ The parametric equations of the tangent line are

$$x = -1$$

$$y = -t$$

$$z = \pi + t$$

DEFINITE AND INDEFINITE INTEGRALS OF VECTOR-VALUED FUNCTIONS

In general, we have

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b x(t)\,dt\right)\mathbf{i} + \left(\int_a^b y(t)\,dt\right)\mathbf{j} + \left(\int_a^b z(t)\,dt\right)\mathbf{k}$$

Example Let $\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} - 2\cos(\pi t)\,\mathbf{k}$. Then

$$\int_0^1 \mathbf{r}(t)\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 e^t\,dt\right)\mathbf{j} - \left(\int_0^1 2\cos\pi t\,dt\right)\mathbf{k}$$

$$= \frac{t^3}{3}\Big]_0^1 \mathbf{i} + e^t\Big]_0^1 \mathbf{j} - \frac{2}{\pi}\sin\pi t\Big]_0^1 \mathbf{k} = \frac{1}{3}\mathbf{i} + (e-1)\mathbf{j}$$

DEFINITE AND INDEFINITE INTEGRALS OF VECTOR-VALUED FUNCTIONS

Example $$\int (2t\mathbf{i} + 3t^2\mathbf{j})\,dt = \left(\int 2t\,dt\right)\mathbf{i} + \left(\int 3t^2\,dt\right)\mathbf{j}$$

$$= (t^2 + C_1)\mathbf{i} + (t^3 + C_2)\mathbf{j}$$

$$= (t^2\mathbf{i} + t^3\mathbf{j}) + (C_1\mathbf{i} + C_2\mathbf{j}) = (t^2\mathbf{i} + t^3\mathbf{j}) + \mathbf{C}$$

DEFINITE AND INDEFINITE INTEGRALS OF VECTOR-VALUED FUNCTIONS

Example Find $\mathbf{r}(t)$ given that $\mathbf{r}'(t) = \langle 3, 2t \rangle$ and $\mathbf{r}(1) = \langle 2, 5 \rangle$.

$$\mathbf{r}(t) = \int \mathbf{r}'(t)dt = \int \langle 3,2t \rangle dt = \langle 3t, t^2 \rangle + \mathbf{C}$$

But

$$\mathbf{r}(1) = \langle 2, 5 \rangle$$

$$\langle 3,1 \rangle + \mathbf{C} = \langle 2, 5 \rangle$$

$$\mathbf{C} = \langle -1, 4 \rangle$$

So

$$\mathbf{r}(t) = \langle 3t, t^2 \rangle + \langle -1,4 \rangle$$

$$\mathbf{r}(t) = \langle 3t - 1, t^2 + 4 \rangle$$

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Chapter: [12]

VECTOR-VALUED FUNCTIONS

Section: [12.3]

CHANGE OF PARAMETER; ARC LENGTH

SMOOTH PARAMETRIZATIONS

- We will say that a curve represented by $\mathbf{r}(t)$ is *smoothly parametrized* by $\mathbf{r}(t)$, or that $\mathbf{r}(t)$ is a smooth function of t if:
  - ✓ $\mathbf{r}'(t)$ is continuous, and
  - ✓ $\mathbf{r}'(t) \neq \mathbf{0}$ for any allowable value of t.
- Geometrically, this means that a smoothly parametrized curve can have no abrupt (مفاجئ) changes in direction as the parameter increases.

SMOOTH PARAMETRIZATIONS

Example Determine whether the following vector-valued functions are smooth.

1. $$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k} \qquad a > 0, c > 0$$

$$\mathbf{r}'(t) = -a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + c\mathbf{k}$$

✓ The components are continuous functions, and

✓ there is no value of t for which all three of them are zero.

✓ So $\mathbf{r}(t)$ is a smooth function.

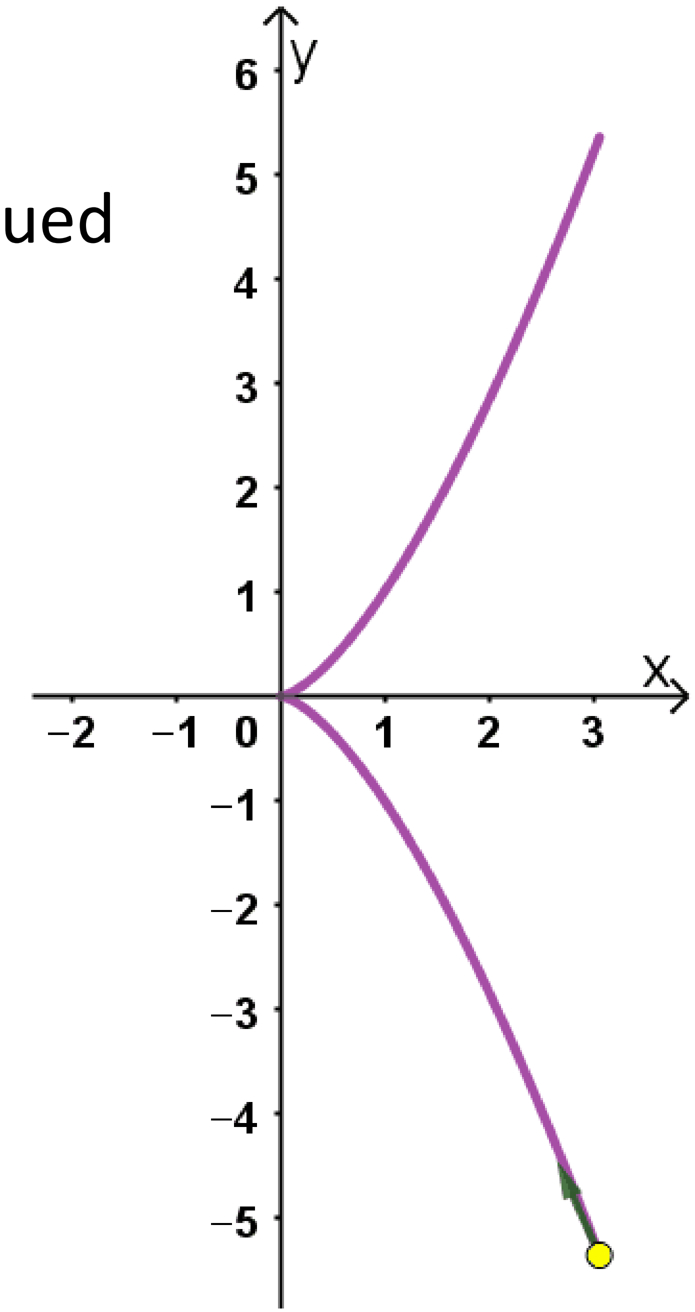
SMOOTH PARAMETRIZATIONS

Example Determine whether the following vector-valued functions are smooth.

2. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

- ✓ The components are continuous functions, and
- ✓ they are both equal to zero if $t = 0$.
- ✓ So, $\mathbf{r}(t)$ is NOT a smooth function.



ARC LENGTH FROM THE VECTOR VIEWPOINT

If C is the graph of a smooth vector-valued function $\mathbf{r}(t)$, then its arc length ℓ from $t = a$ to $t = b$ is

$$\ell = \int_a^b \left\| \frac{d\mathbf{r}}{dt} \right\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example Find the arc length of that portion of the circular helix $\mathbf{r}(t) = \langle \cos t\,, \sin t\,, t \rangle$ from $t = 0$ to $t = \pi$.

ARC LENGTH FROM THE VECTOR VIEWPOINT

$$\ell = \int_a^b \left\|\frac{d\mathbf{r}}{dt}\right\| dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Example Find the arc length of that portion of the circular helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ from $t = 0$ to $t = \pi$.

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1}$$

$$= \sqrt{2}$$

$$\ell = \int_0^\pi \|\mathbf{r}'(t)\| dt$$

$$= \int_0^\pi \sqrt{2}\, dt$$

$$= \sqrt{2}\,\pi$$

Course: Calculus (3)

Chapter: [12]

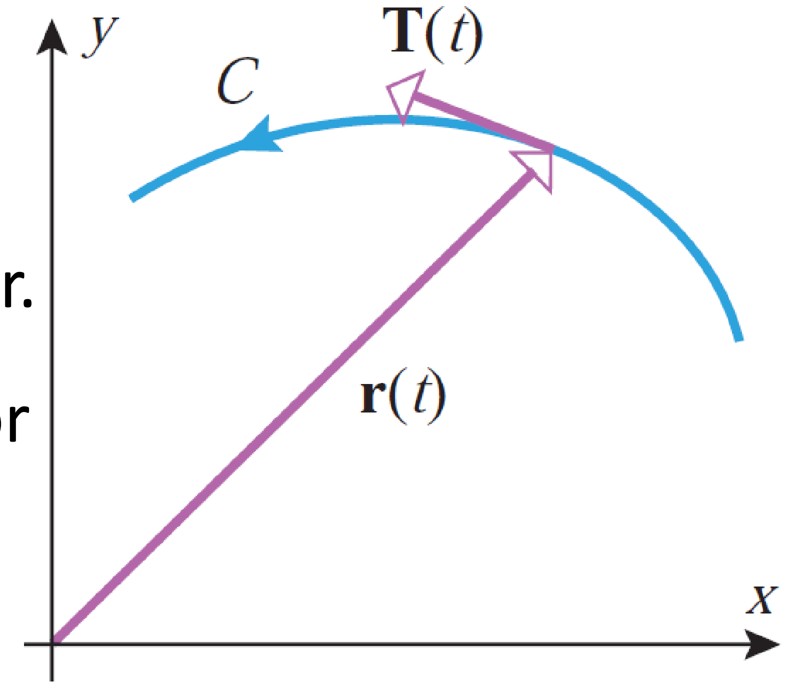
VECTOR-VALUED FUNCTIONS

Section: [12.4]

UNIT TANGENT, NORMAL, AND BINORMAL VECTORS

UNIT TANGENT VECTORS

- Recall that if C is the graph of a *smooth* vector-valued function $\mathbf{r}(t)$, then the vector $\mathbf{r}'(t)$ is:
  - ✓ nonzero, tangent to C, and
  - ✓ points in the direction of increasing parameter.



- Thus, by normalizing $\mathbf{r}'(t)$ we obtain a unit vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

that is tangent to C and points in the direction of increasing parameter.

- We call $\mathbf{T}(t)$ the unit tangent vector to C at t.

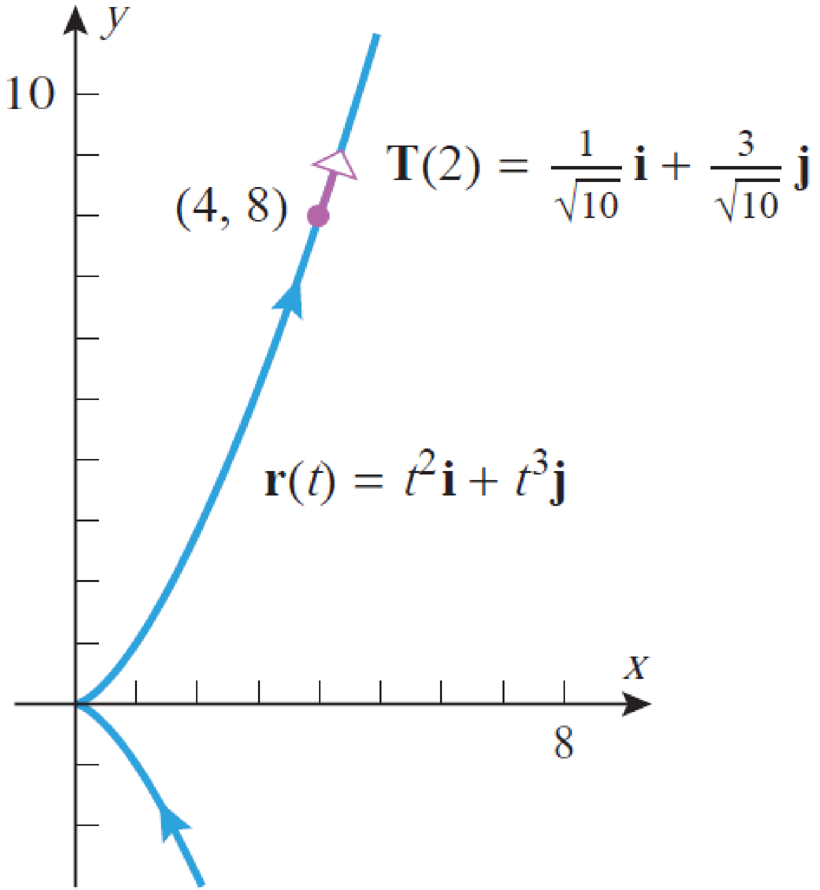
UNIT TANGENT VECTORS

Example Find the unit tangent vector to the graph of $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$ at the point where $t = 2$.

$$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$$

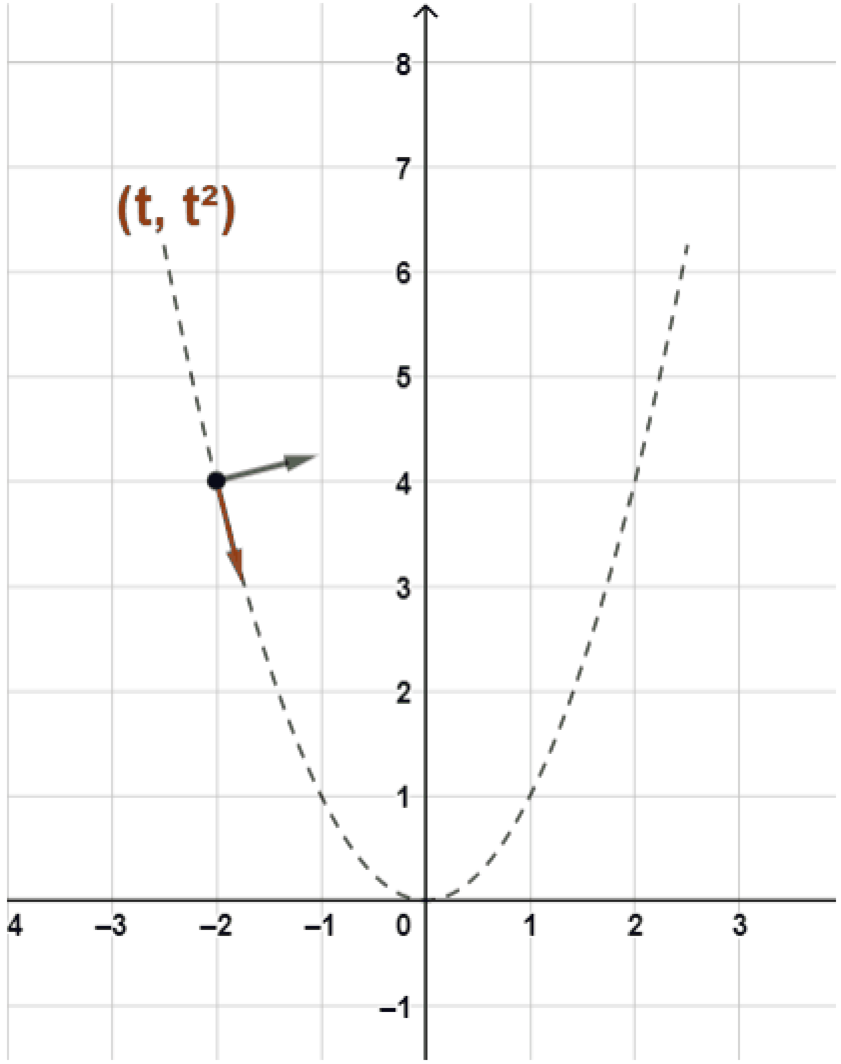
$$\mathbf{r}'(2) = 4\mathbf{i} + 12\mathbf{j}$$

$$\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{\|\mathbf{r}'(2)\|} = \frac{4\mathbf{i} + 12\mathbf{j}}{\sqrt{160}} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$$



UNIT NORMAL VECTORS

- Recall if $\|\mathbf{r}(t)\| = c$, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal vectors.
- $\mathbf{T}(t)$ has constant norm 1, so $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal vectors.
- This implies that $\mathbf{T}'(t)$ is perpendicular to the tangent line to C at t, so we say that $\mathbf{T}'(t)$ is *normal* to C at t.

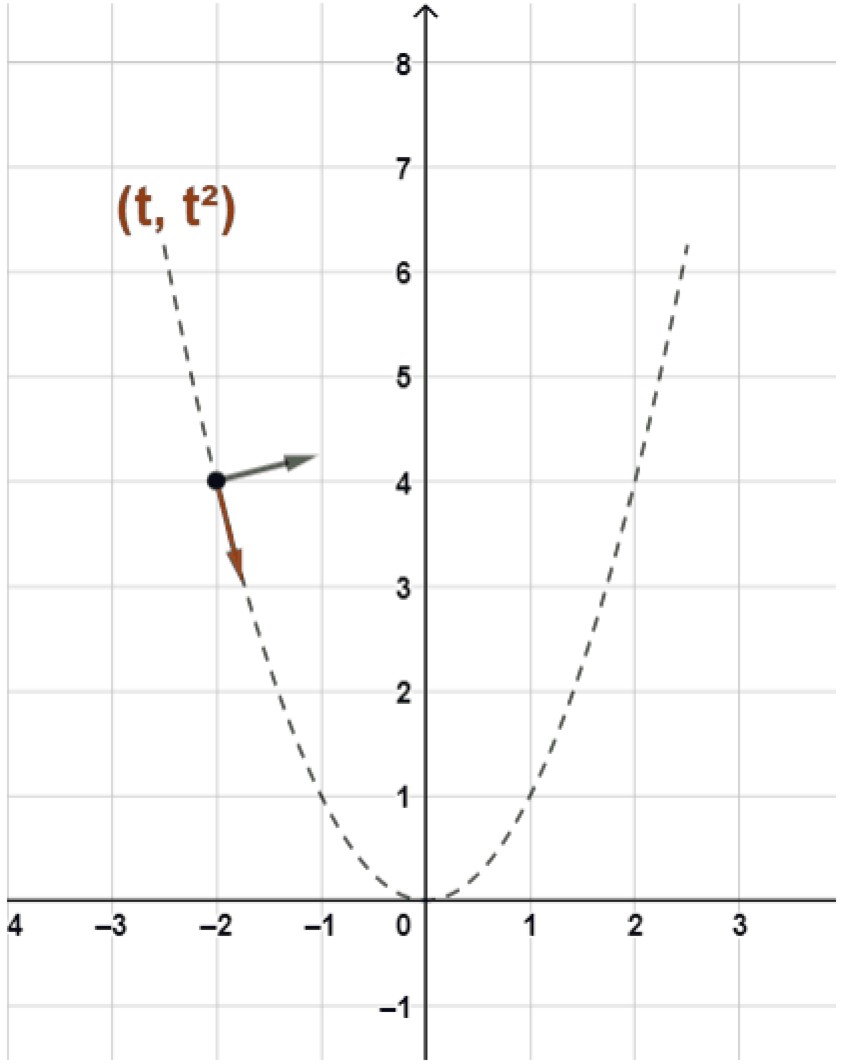


UNIT NORMAL VECTORS

- It follows that if $\mathbf{T}'(t) \neq \mathbf{0}$, and if we normalize $\mathbf{T}'(t)$, then we obtain a unit vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

that is normal to C and points in the same direction as $\mathbf{T}'(t)$.



UNIT NORMAL VECTORS

- We call $\mathbf{N}(t)$ the *principal unit normal vector to C at t*, or more simply, the *unit normal vector*.
- Observe that the unit normal vector is defined only at points where $\mathbf{T}'(t) \neq \mathbf{0}$. Unless stated otherwise, we will assume that this condition is satisfied.
- In particular, this excludes straight lines.

UNIT NORMAL VECTORS

Example Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for the circular helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t \rangle$.

$$\mathbf{r}'(t) = \langle -3\sin t, 3\cos t, 4 \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$$

$$\mathbf{T}(t) = \frac{\langle -3\sin t, 3\cos t, 4 \rangle}{5} = \left\langle \frac{-3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5} \right\rangle$$

UNIT NORMAL VECTORS

Example Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for the circular helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

$$\mathbf{T}(t) = \frac{\langle -3\sin t, 3\cos t, 4\rangle}{5} = \left\langle \frac{-3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{-3}{5}\cos t, \frac{-3}{5}\sin t, 0\right\rangle$$

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{9}{25}\cos^2 t + \frac{9}{25}\sin^2 t + 0} = \frac{3}{5}$$

UNIT NORMAL VECTORS

Example Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for the circular helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

$$\mathbf{T}'(t) = \left\langle \frac{-3}{5}\cos t, \frac{-3}{5}\sin t, 0 \right\rangle$$

$$\|\mathbf{T}'(t)\| = \sqrt{\frac{9}{25}\cos^2 t + \frac{9}{25}\sin^2 t + 0} = \frac{3}{5}$$

$$\mathbf{N}(t) = \frac{\left\langle \frac{-3}{5}\cos t, \frac{-3}{5}\sin t, 0 \right\rangle}{\frac{3}{5}} = \langle -\cos t, -\sin t, 0\rangle$$

UNIT NORMAL VECTORS

Example Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, \frac{1}{3}t^3 \right\rangle$ at $t = 1$.

$$\mathbf{r}'(t) = \langle t, t^2 \rangle \qquad \mathbf{r}'(1) = \langle 1,1 \rangle$$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 1,1 \rangle}{\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{t^2 + t^4}} \langle t, t^2 \rangle = (t^2 + t^4)^{-1/2} \langle t, t^2 \rangle$$

$$\mathbf{T}'(t) = (t^2 + t^4)^{-1/2} \langle 1,2t \rangle - \frac{1}{2}(2t + 4t^3)(t^2 + t^4)^{-3/2} \langle t, t^2 \rangle$$

UNIT NORMAL VECTORS

Example Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ for $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, \frac{1}{3}t^3 \right\rangle$ at $t = 1$.

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$$\mathbf{T}'(1) = \frac{\langle 1,2\rangle}{\sqrt{2}} - \frac{3\langle 1,1\rangle}{2\sqrt{2}} = \left\langle \frac{-1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right\rangle$$

$$\|\mathbf{T}'(1)\| = \sqrt{\left(\frac{-1}{2\sqrt{2}}\right)^2 + \left(\frac{1}{2\sqrt{2}}\right)^2} = \frac{1}{2}$$

$$\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|} = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Course: Calculus (3)

Chapter: [12]

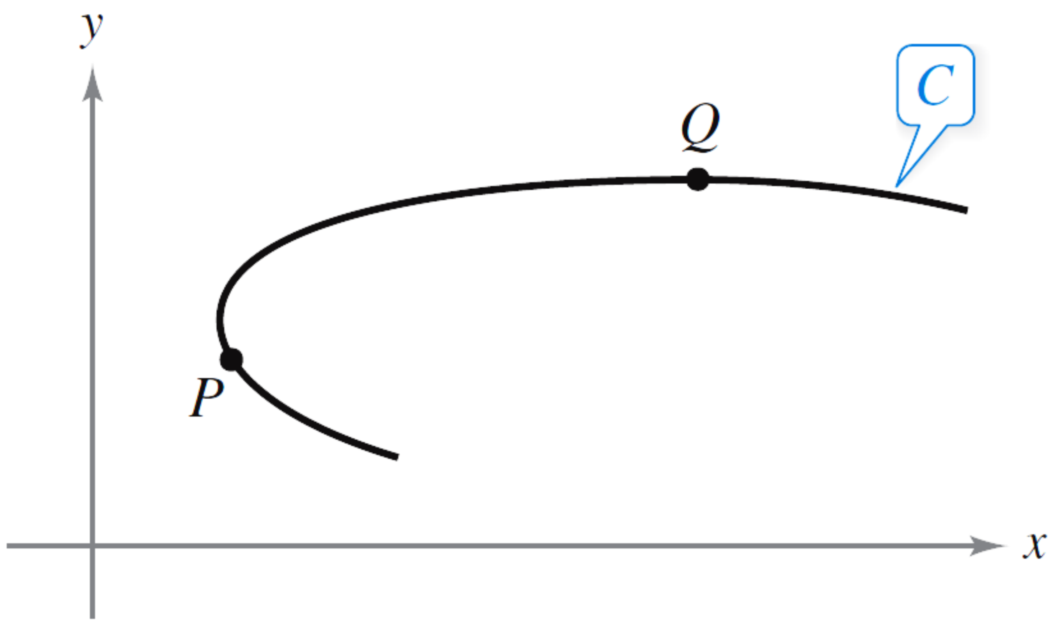
VECTOR-VALUED FUNCTIONS

Section: [12.5]

CURVATURE

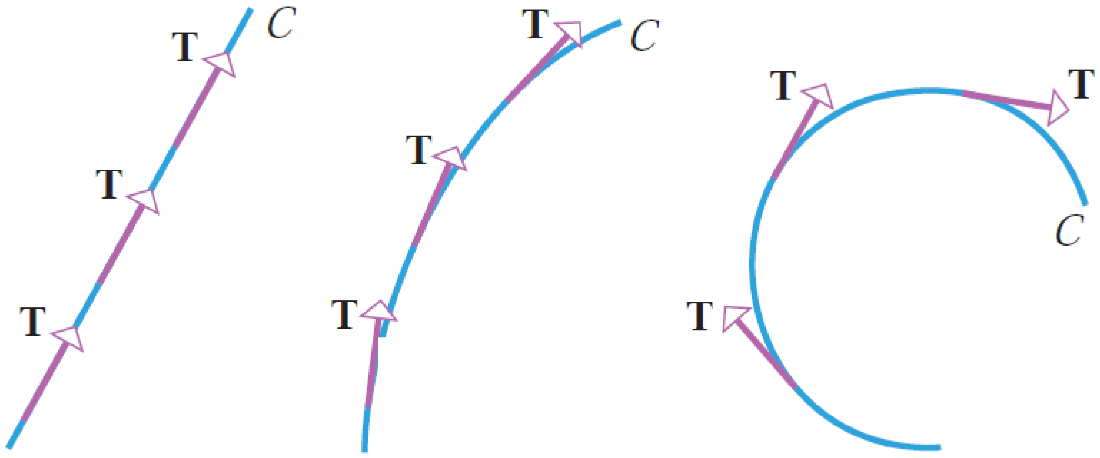
DEFINITION OF CURVATURE

- We will consider the problem of obtaining a *numerical measure of how sharply a curve bends*.
- For instance, in the figure, the curve bends more sharply at P than at Q and you can say that the curvature is greater at P than at Q.



DEFINITION OF CURVATURE

You can calculate curvature by calculating the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to the arc length s.



- If C is a *straight line (no bend)*, then the direction of $\mathbf{T}$ remains constant.
- If C *bends slightly*, then $\mathbf{T}$ undergoes a gradual change of direction.
- If C *bends sharply*, then $\mathbf{T}$ undergoes a rapid change of direction.

DEFINITION OF CURVATURE

If $\mathbf{r}(t)$ is a smooth vector-valued function, then for each value of t at which $\mathbf{T}'(t)$ and $\mathbf{r}''(t)$ exist, the curvature κ can be expressed as

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

Example Show that the curvature of a circle of radius R is $\kappa = \frac{1}{R}$.

1 $\quad \kappa(t) = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

DEFINITION OF CURVATURE

Example Show that the curvature of a circle of radius R is $\kappa = \frac{1}{R}$.

1

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \qquad \mathbf{r}(t) = R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j} \qquad t \in [0,2\pi]$$

$$\mathbf{r}'(t) = -R\sin t\,\mathbf{i} + R\cos t\,\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle -R\sin t, R\cos t\rangle}{\sqrt{(-R\sin t)^2 + (R\cos t)^2}} = \langle -\sin t, \cos t\rangle$$

$$\mathbf{T}'(t) = \langle -\cos t, -\sin t\rangle$$

$$\kappa(t) = \frac{\sqrt{(-\cos t)^2 + (-\sin t)^2}}{\sqrt{(-R\sin t)^2 + (R\cos t)^2}} = \frac{1}{R}$$

DEFINITION OF CURVATURE

Example Show that the curvature of a circle of radius R is $\kappa = \frac{1}{R}$.

2 $$\kappa(t) = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

$$\mathbf{r}(t) = R\cos t\,\mathbf{i} + R\sin t\,\mathbf{j} + 0\mathbf{k} \quad t \in [0,2\pi]$$

$$\mathbf{r}'(t) = -R\sin t\,\mathbf{i} + R\cos t\,\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{r}''(t) = -R\cos t\,\mathbf{i} - R\sin t\,\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R\sin t & R\cos t & 0 \\ -R\cos t & -R\sin t & 0 \end{vmatrix} = R^2\mathbf{k}$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = R^2$$

$$\|\mathbf{r}'(t)\| = R$$

$$\kappa(t) = \frac{R^2}{R^3} = \frac{1}{R}$$