Electric Circuits I

RL Circuits

Dr. Firas Obeidat

The Source Free RL Circuit

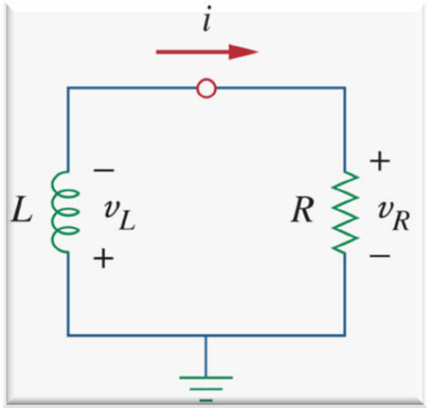
- **The analysis of circuits containing inductors and/or capacitors is dependent upon the formulation and solution of the integrodifferential equations that characterize the circuits. We will call the special type of equation we obtain a homogeneous linear differential equation.**
- **They call the solution of a homogeneous linear differential equation a complementary function.**
- **When we consider independent sources acting on a circuit, part of the response will resemble the nature of the particular source (or forcing function) used; this part of the response, called the *particular solution*, the *steady-state response*, or the *forced response*.**
- **The source-free response may be called the *natural response*, the *transient response*, the *free response*, or the *complementary function.***

The *complete response* of the circuit will then be given by the sum of the *complementary function* and the *particular solution*.

The *natural response* of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The Source Free RL Circuit

- **Let us designate the time-varying current as *i(t)*; we will represent the value of *i(t)* at *t = 0* as I_o; in other words, *i(0)* = I_o. We therefore have.**



$$Ri + v_L = Ri + L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

- **Our goal is an expression for *i(t)* which satisfies this equation and also has the value *Io* at *t = 0*.**

$$\frac{di}{i} = -\frac{R}{L}\,dt$$

$$\int_{I_0}^{i(t)} \frac{di'}{i'} = \int_0^t -\frac{R}{L}\,dt'$$

$$\ln i'\Big|_{I_0}^{i} = -\frac{R}{L}t'\Big|_0^t$$

$$\ln i - \ln I_0 = -\frac{R}{L}(t - 0)$$

$$i(t) = I_0 e^{-Rt/L}$$

$$\tau = \frac{L}{R}$$

$$i(t) = I_0 e^{-t/\tau}$$

The Source Free RL Circuit

The voltage across the resistor is

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

$$w_R = \int_0^\infty p_R \, dt = I_0^2 R \int_0^\infty e^{-2Rt/L} \, dt$$

$$= I_0^2 R \left(\frac{-L}{2R}\right) e^{-2Rt/L}\Big|_0^\infty = \frac{1}{2} L \, I_0^2$$

The energy absorbed by the resistor is

$$w_R(t) = \int_0^t p(\lambda) d\lambda = \int_0^t I_0^2 e^{-2\lambda/\tau} \, d\lambda = -\frac{\tau}{2} I_0^2 R e^{-2\lambda/\tau}\Big|_0^t, \qquad \tau = \frac{L}{R}$$

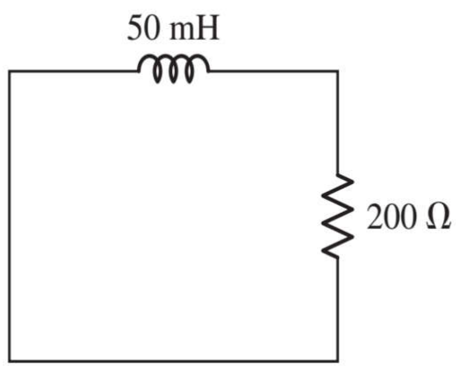
$$w_R(t) = \frac{1}{2} L \, I_0^2 (1 - e^{-2t/\tau})$$

The *time constant* of a circuit is the time required for the response to decay to a factor of 36.8 percent of its initial value

Note that as $t \to \infty$, $w_R(\infty) \to \frac{1}{2} L \, I_0^2$

The Source Free RL Circuit

Exampe: If the inductor of shown figure has a current $i_L = 2\,A$ at t=0, find an expression for $i_L(t)$ valid for t > 0, and its value at t=200 μs.



This is the identical type of circuit just considered, so we expect an inductor current of the form

$$i_L = I_0 e^{-Rt/L}$$

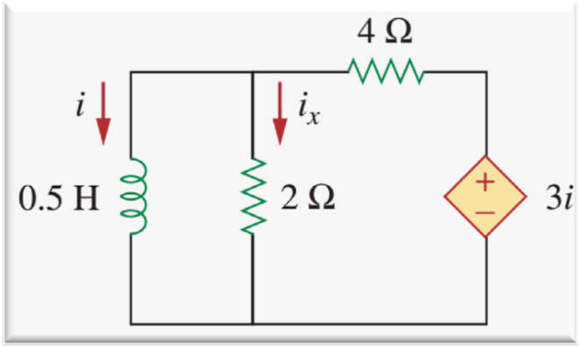
where $R = 200\ \Omega$, $L = 50$ mH and I_0 is the initial current flowing through the inductor at $t = 0$. Thus,

$$i_L(t) = 2e^{-4000t}$$

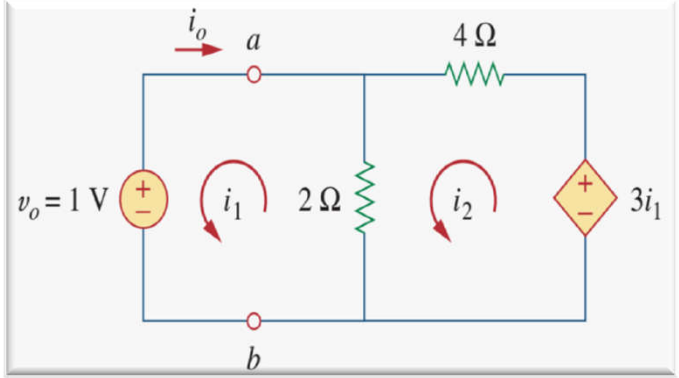
Substituting $t = 200 \times 10^{-6}$ s, we find that $i_L(t) = 898.7$ mA, less than half the initial value.

The Source Free RL Circuit

Example: Assuming that *i(0)=10A*, calculate *i(t)* and $i_x(t)$ in the circuit of shown figure.



Applying KVL to the two loops results in



$$2(i_1 - i_2) + 1 = 0 \quad \Rightarrow \quad i_1 - i_2 = -\frac{1}{2}$$

$$6i_2 - 2i_1 - 3i_1 = 0 \quad \Rightarrow \quad i_2 = \frac{5}{6}i_1$$

$$i_1 = -3 \text{ A}, \qquad i_o = -i_1 = 3 \text{ A}$$

$$R_{eq} = R_{Th} = \frac{v_o}{i_o} = \frac{1}{3}\,\Omega$$

$$\tau = \frac{L}{R_{eq}} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}\text{ s}$$

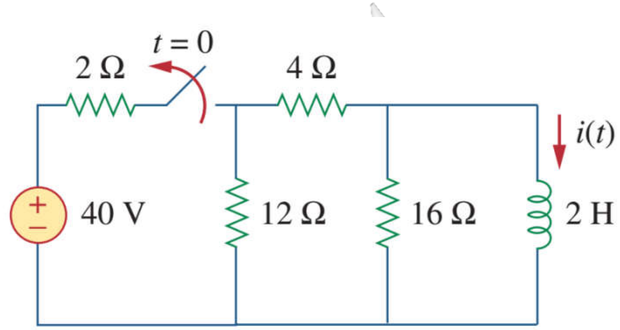
$$i(t) = i(0)e^{-t/\tau} = 10e^{-(2/3)t} \text{ A}, \qquad t > 0$$

$$v = L\frac{di}{dt} = 0.5(10)\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{10}{3}e^{-(2/3)t} \text{ V}$$

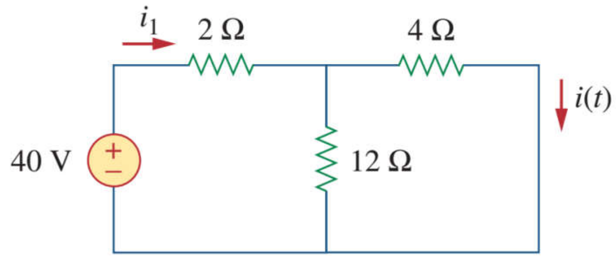
$$i_x(t) = \frac{v}{2} = -1.6667e^{-(2/3)t} \text{ A}, \qquad t > 0$$

The Source Free RL Circuit

Example: The switch in the circuit of shown figure has been closed for a long time. At t=0 the switch is opened. Calculate i(t) for t>0.



$$\frac{4 \times 12}{4 + 12} = 3\ \Omega$$

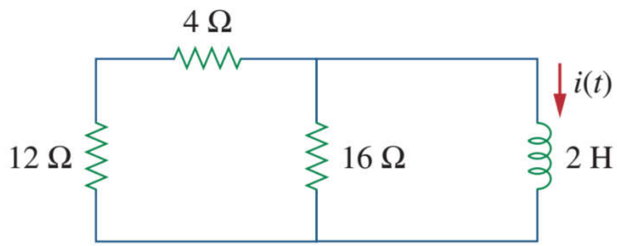


(a)

$$i_1 = \frac{40}{2 + 3} = 8\ \text{A}$$

$$i(t) = \frac{12}{12 + 4} i_1 = 6\ \text{A}, \qquad t < 0$$

$$i(0) = i(0^-) = 6\ \text{A}$$



(b)

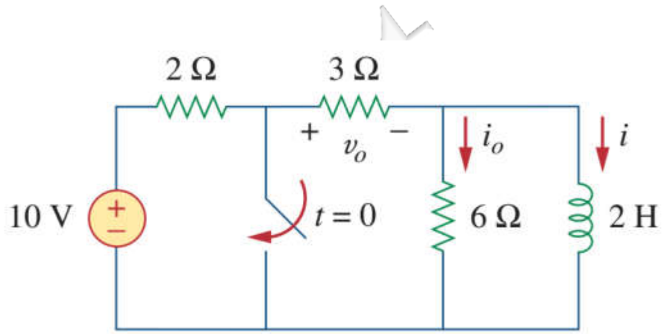
$$R_{\text{eq}} = (12 + 4) \parallel 16 = 8\ \Omega$$

$$\tau = \frac{L}{R_{\text{eq}}} = \frac{2}{8} = \frac{1}{4}\ \text{s}$$

$$i(t) = i(0)e^{-t/\tau} = 6e^{-4t}\ \text{A}$$

The Source Free RL Circuit

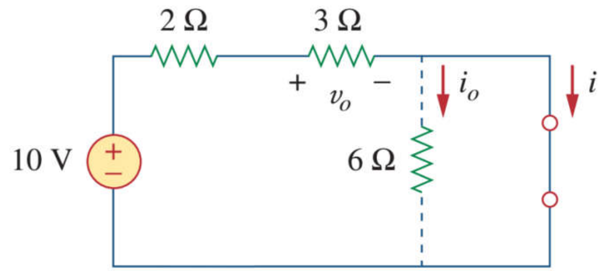
Example: The switch in the circuit of shown figure has been opened for a long time. At t=0 the switch is closed. Calculate i(t) for t>0.



$$i(t) = \frac{10}{2+3} = 2\ \text{A}, \qquad t < 0$$

$$v_o(t) = 3i(t) = 6\ \text{V}, \qquad t < 0$$

$$R_{\text{Th}} = 3 \parallel 6 = 2\ \Omega \qquad \tau = \frac{L}{R_{\text{Th}}} = 1\ \text{s}$$

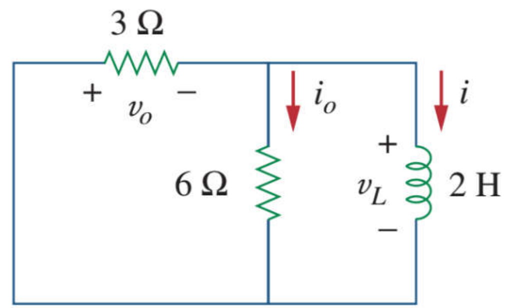


(a)

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \qquad t > 0$$

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \text{V}, \qquad t > 0$$

$$i_o(t) = \frac{v_L}{6} = -\frac{2}{3}e^{-t}\ \text{A}, \qquad t > 0$$



(b)

$$i_o(t) = \begin{cases} 0\ \text{A}, & t < 0 \\ -\frac{2}{3}e^{-t}\ \text{A}, & t > 0 \end{cases}, \qquad v_o(t) = \begin{cases} 6\ \text{V}, & t < 0 \\ 4e^{-t}\ \text{V}, & t > 0 \end{cases}$$

$$i(t) = \begin{cases} 2\ \text{A}, & t < 0 \\ 2e^{-t}\ \text{A}, & t \geq 0 \end{cases}$$

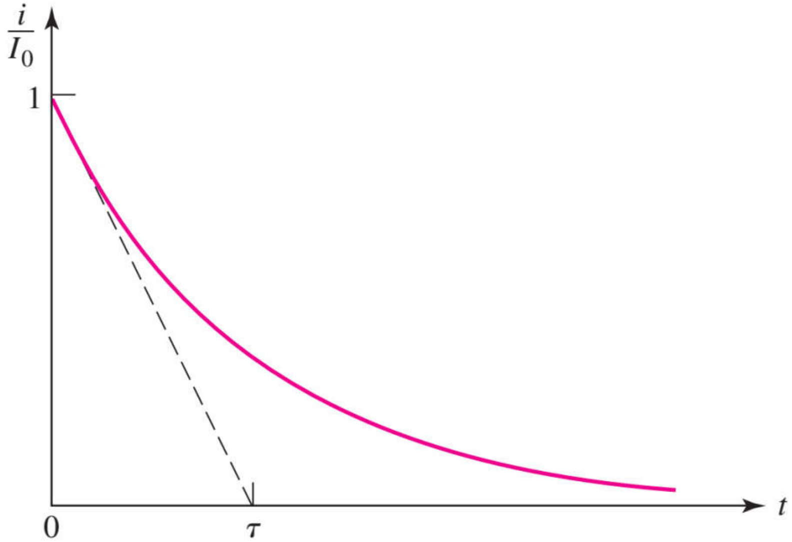
Properties of Exponential Response

At $t=0$, the current has value I_o, but as time increases, the current decreases and approaches zero.

$$i(t) = I_0 e^{-Rt/L}$$

The initial rate of decay is found by evaluating the derivative at zero time:

$$\left.\frac{d}{dt}\frac{i}{I_0}\right|_{t=0} = -\frac{R}{L}e^{-Rt/L}\Big|_{t=0} = -\frac{R}{L}$$



We designate the value of time it takes for i/I_o to drop from unity to zero, assuming a constant rate of decay

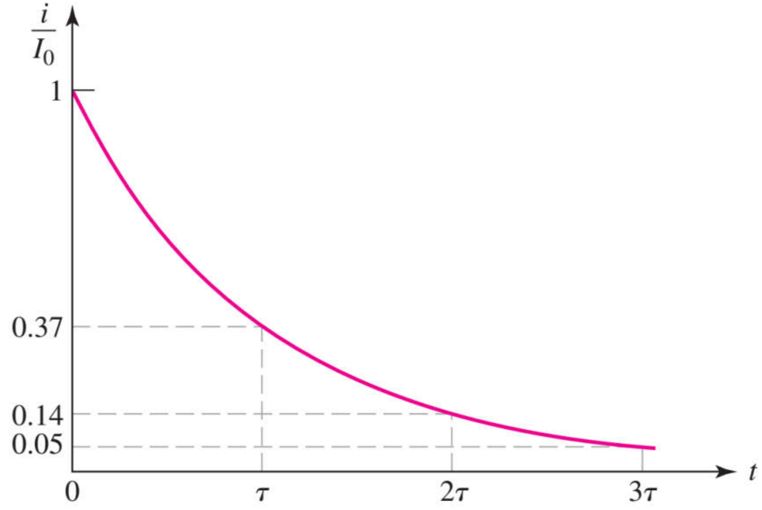
$$\left(\frac{R}{L}\right)\tau = 1 \qquad \tau = \frac{L}{R}$$

The time constant of a series RL circuit may be found graphically from the response curve; it is necessary only to draw the tangent to the curve at t = 0 and determine the intercept of this tangent line with the time axis.

Properties of Exponential Response

The value of *$i(t)/I_o$ at $t = \tau$* is.

$$\frac{i(\tau)}{I_0} = e^{-1} = 0.3679 \qquad \text{or} \qquad i(\tau) = 0.3679 I_0$$



In one time constant the response has dropped to 36.8 percent of its initial value; the value of τ may also be determined graphically from this fact. It is convenient to measure the decay of the current at intervals of one time constant, and recourse to a hand calculator shows that *$i(t)/I_o$* is 0.3679 at t = τ, 0.1353 at t = 2τ , 0.04979 at t = 3τ , 0.01832 at t = 4τ , and 0.006738 at t = 5τ.

Q. How long does it take for the current to decay to zero?
A. About five time constants.

19