Electric Circuits II

Resonance

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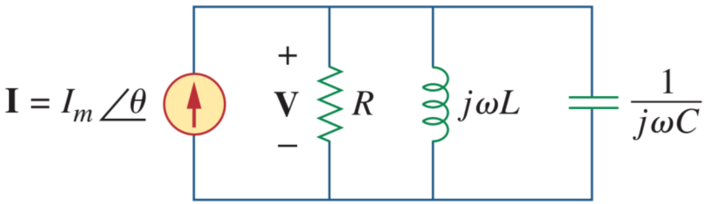
Parallel Resonance

The parallel *RLC* circuit is the dual of the series *RLC* circuit. The admittance is

$$\mathbf{Y} = H(\omega) = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

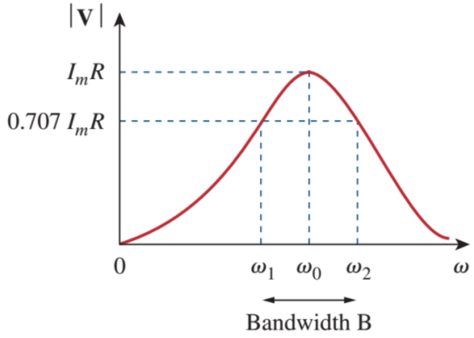
$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$



Resonance occurs when the imaginary part of **Y** is zero,

$$\omega C - \frac{1}{\omega L} = 0$$

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$



Using duality between series and parallel circuits. By replacing *R*, *L*, and *C* in the expressions for the series circuit with *1/R*, *C* and *L* respectively, we obtain for the parallel circuit

$$\boxed{\omega_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}}$$

$$\boxed{\omega_2 = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}}$$

Parallel Resonance

$$B = \omega_2 - \omega_1 = \frac{1}{RC}$$

$$Q = \frac{\omega_0}{B} = \omega_0 RC = \frac{R}{\omega_0 L}$$

The half power frequencies in terms of the quality factor is

$$\omega_1 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} - \frac{\omega_0}{2Q}, \qquad \omega_2 = \omega_0\sqrt{1 + \left(\frac{1}{2Q}\right)^2} + \frac{\omega_0}{2Q}$$

for high-Q circuits (Q≥10)

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

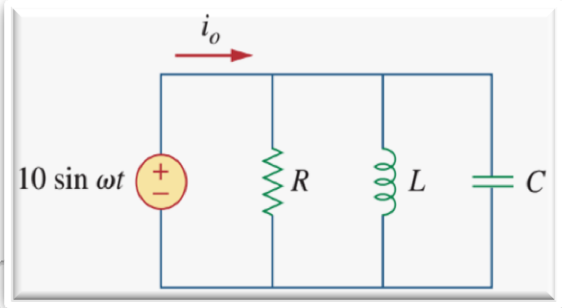
Parallel Resonance

Example: In the parallel *RLC* circuit, let R=8 kΩ, *L*=0.2 mH and C=8 μF.

(a) Calculate ω_o, Q, and B.

(b) Calculate ω_1 and ω_2.

(c) Determine the power dissipated at ω_o, ω_1 and ω_2.



(a)

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.2 \times 10^{-3} \times 8 \times 10^{-6}}} = \frac{10^5}{4} = 25 \text{ krad/s}$$

$$Q = \frac{R}{\omega_0 L} = \frac{8 \times 10^3}{25 \times 10^3 \times 0.2 \times 10^{-3}} = 1{,}600$$

$$B = \frac{\omega_0}{Q} = 15.625 \text{ rad/s}$$

(b) Due to the high value of Q, we can regard this as a high-Q circuit, Hence,

$$\omega_1 = \omega_0 - \frac{B}{2} = 25{,}000 - 7.812 = 24{,}992 \text{ rad/s}$$

$$\omega_2 = \omega_0 + \frac{B}{2} = 25{,}000 + 7.812 = 25{,}008 \text{ rad/s}$$

Parallel Resonance

(c) At $\omega = \omega_0$, $\mathbf{Y} = 1/R$ or $\mathbf{Z} = R = 8\ \text{k}\Omega$. Then

$$\mathbf{I}_o = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{10\underline{/-90^\circ}}{8{,}000} = 1.25\underline{/-90^\circ}\ \text{mA}$$

Since the entire current flows through R at resonance, the average power dissipated at $\omega = \omega_0$ is

$$P = \frac{1}{2}|\mathbf{I}_o|^2 R = \frac{1}{2}(1.25 \times 10^{-3})^2(8 \times 10^3) = 6.25\ \text{mW}$$

or

$$P = \frac{V_m^2}{2R} = \frac{100}{2 \times 8 \times 10^3} = 6.25\ \text{mW}$$

At $\omega = \omega_1, \omega_2$,

$$P = \frac{V_m^2}{4R} = 3.125\ \text{mW}$$

Parallel Resonance

Example: A parallel resonance circuit has a resistance of 2 kΩ and half-power frequencies of 86 kHz and 90 kHz. Determine:

(a) the capacitance
(b) the inductance
(c) the resonant frequency
(d) the bandwidth
(e) the quality factor

(a) $B = \omega_2 - \omega_1 = 2\pi(f_2 - f_1) = 2\pi(90 - 86)x10^3 = 8\pi krad/s$

$$\omega_o = \frac{1}{2}(\omega_1 + \omega_2) = 2\pi(88)x10^3 = 176\pi X10^3$$

$$B = \frac{1}{RC} \longrightarrow C = \frac{1}{BR} = \frac{1}{8\pi x10^3 x2x10^3} = \underline{19.89nF}$$

(b) $\omega_o = \frac{1}{\sqrt{LC}} \longrightarrow L = \frac{1}{\omega^2{}_o C} = \frac{1}{(176\pi X10^3)^2 x19.89x10^{-9}} = \mathbf{\underline{164.45\ \mu H}}$

(c) $\omega_o = 176\pi = \underline{552.9krad/s}$

(d) $B = 8\pi = \underline{25.13krad/s}$

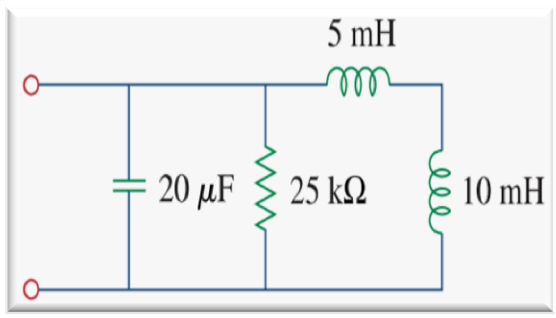
(e) $Q = \frac{\omega_o}{B} = \frac{176\pi}{8\pi} = \underline{22}$

Parallel Resonance

Example: For the circuit:

(a) Calculate the resonant frequency ω_o, the quality factor Q, and the bandwidth B.

(b) What value of capacitance must be connected in series with the 20 μF capacitor in order to double the bandwidth?.



(a) $L = 5 + 10 = 15\ \text{mH}$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{15\text{x}10^{-3}\text{x}20\text{x}10^{-6}}} = \underline{\mathbf{1.8257\ k\ rad/sec}}$$

$$Q = \omega_0 RC = 1.8257\text{x}10^{3}\text{x}25\text{x}10^{3}\text{x}20\text{x}10^{-6} = \underline{\mathbf{912.8}}$$

$$B = \frac{1}{RC} = \frac{1}{25\text{x}10^{3}20\text{x}10^{-6}} = \underline{\mathbf{2\ rad/s}}$$

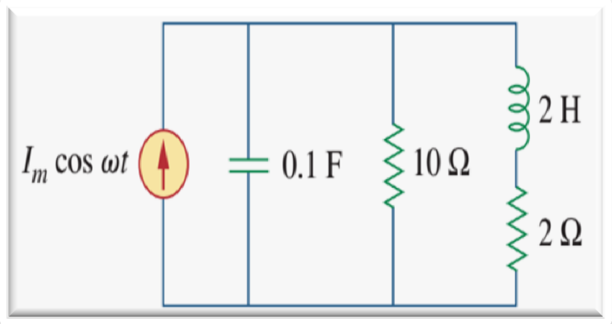
(b) To increase B by 100% means that B' = 4.

$$C' = \frac{1}{RB'} = \frac{1}{25\text{x}10^{3}\text{x}4} = \underline{\mathbf{10\ \mu F}}$$

Since $C' = \dfrac{C_1 C_2}{C_1 + C_2} = 10\mu\text{F}$ and $C_1 = 20\ \mu\text{F}$, we then obtain $C_2 = 20\ \mu\text{F}$.

Series-Parallel Resonance

Example: Determine the resonant frequency of the circuit.



The input admittance is

$$\mathbf{Y} = j\omega 0.1 + \frac{1}{10} + \frac{1}{2 + j\omega 2} = 0.1 + j\omega 0.1 + \frac{2 - j\omega 2}{4 + 4\omega^2}$$

At resonance, Im($\mathbf{Y}$) = 0 and

$$\omega_0 0.1 - \frac{2\omega_0}{4 + 4\omega_0^2} = 0$$

$$0.1\omega_o = \frac{2\omega_o}{4 + 4{\omega_o}^2}$$

$$0.4\omega_o + 0.4{\omega_o}^3 = 2\omega_o$$

$$0.4 + 0.4{\omega_o}^2 = 2$$

$$0.4{\omega_o}^2 = 2 - 0.4 = 1.6$$

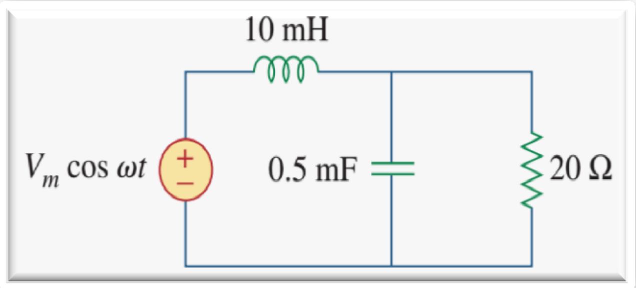
$${\omega_o}^2 = 1.6/0.4=4$$

$$\omega_o=\pm\sqrt{4} = \pm 2$$

ω_o=2 *rad/s*

Series-Parallel Resonance

Example: Determine the resonant frequency of the circuit.



$$Z = jX_L + \frac{-jX_C R}{R - jX_C} \times \frac{R + jX_C}{R + jX_C} = jX_L + \frac{RX_C{}^2 - jR^2 X_C}{R^2 + X_C{}^2}$$

$$Z = \frac{RX_C{}^2}{R^2 + X_C{}^2} + j\left(X_L - \frac{R^2 X_C}{R^2 + X_C{}^2}\right)$$

At resonance, Im(Z)=0.

$$X_L - \frac{R^2 X_C}{R^2 + X_C{}^2} \rightarrow X_L = \frac{R^2 X_C}{R^2 + X_C{}^2}$$

$$X_L R^2 + X_L X_C{}^2 = R^2 X_C$$

$$\omega_o L R^2 + \frac{\omega_o L}{\omega_o{}^2 C^2} = \frac{R^2}{\omega_o C}$$

$$\omega_o{}^2 C L R^2 + \frac{\omega_o{}^2 C L}{\omega_o{}^2 C^2} = R^2$$

$$\omega_o{}^2 C L R^2 + \frac{L}{C} = R^2$$

$$\omega_o{}^2 + \frac{1}{C^2 R^2} = \frac{R^2}{C L R^2}$$

$$\omega_o{}^2 + \frac{1}{C^2 R^2} = \frac{1}{LC}$$

$$\omega_o{}^2 = \frac{1}{CL} - \frac{1}{C^2 R^2} \rightarrow \omega_o = \pm\sqrt{\frac{1}{LC} - \frac{1}{C^2 R^2}}$$

$$\omega_o = \pm\sqrt{\frac{1}{0.5 \times 10^{-3} \times 10 \times 10^{-3}} - \frac{1}{(0.5 \times 10^{-3})^2 \times 20^2}}$$

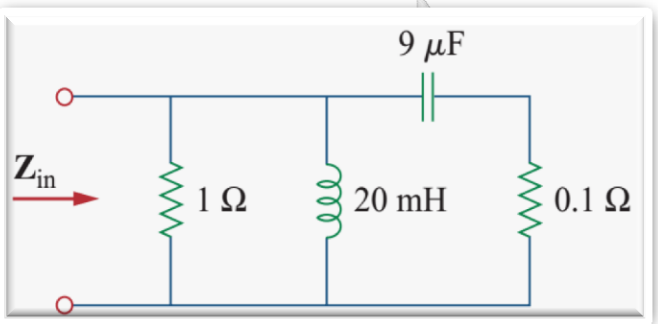
$$\omega_o = \pm\sqrt{0.2 \times 10^6 - 0.01 \times 10^6}$$

$$\omega_o = \pm\sqrt{0.19 \times 10^6} = \pm 0.4359 \times 10^3$$

$$\omega_o = 435.9\ rad/s$$

Series-Parallel Resonance

Example: For the circuit, find the resonant frequency ω_o and $Z_{in}(\omega_o)$.



$$\mathbf{Z}_{in} = (R_1 \,||\, j\omega L) \,||\, (R_2 + 1/j\omega C)$$

$$\mathbf{Z}_{in} = \left(\frac{R_1 j\omega L}{R_1 + j\omega L}\right) \Big|\Big| \left(R_2 + \frac{1}{j\omega C}\right)$$

$$\mathbf{Z}_{in} = \frac{\dfrac{j\omega R_1 L}{R_1 + j\omega L} \cdot \left(R_2 + \dfrac{1}{j\omega C}\right)}{R_2 + \dfrac{1}{j\omega C} + \dfrac{jR_1\omega L}{R_1 + j\omega L}}$$

$$\mathbf{Z}_{in} = \frac{j\omega R_1 L(1 + j\omega R_2 C)}{(R_1 + j\omega L)(1 + j\omega R_2 C) - \omega^2 LCR_1}$$

$$\mathbf{Z}_{in} = \frac{-\omega^2 R_1 R_2 LC + j\omega R_1 L}{R_1 - \omega^2 LCR_1 - \omega^2 LCR_2 + j\omega(L + R_1 R_2 C)}$$

$$\mathbf{Z}_{in} = \frac{(-\omega^2 R_1 R_2 LC + j\omega R_1 L)[R_1 - \omega^2 LCR_1 - \omega^2 LCR_2 - j\omega(L + R_1 R_2 C)]}{(R_1 - \omega^2 LCR_1 - \omega^2 LCR_2)^2 + \omega^2 (L + R_1 R_2 C)^2}$$

At resonance, $\text{Im}(\mathbf{Z}_{in}) = 0$

$$0 = \omega^3 R_1 R_2 LC(L + R_1 R_2 C) + \omega R_1 L(R_1 - \omega^2 LCR_1 - \omega^2 LCR_2)$$

$$0 = \omega^3 R_1^2 R_2^2 LC^2 + R_1^2 \omega L - \omega^3 R_1^2 L^2 C$$

$$0 = \omega^2 R_2^2 C^2 + 1 - \omega^2 LC$$

$$\omega^2 (LC - R_2^2 C^2) = 1$$

Series-Parallel Resonance

$$\omega_0 = \frac{1}{\sqrt{LC - R_2^2 C^2}}$$

$$\omega_0 = \frac{1}{\sqrt{(0.02)(9\times10^{-6}) - (0.1)^2(9\times10^{-6})^2}}$$

$$\omega_0 = \mathbf{2.357\ krad/s}$$

At $\omega = \omega_0 = 2.357\ \text{krad/s}$,

$$j\omega L = j(2.357\times10^{3})(20\times10^{-3}) = j47.14$$

$$R_1 \parallel j\omega L = \frac{j47.14}{1 + j47.14} = 0.9996 + j0.0212$$

$$R_2 + \frac{1}{j\omega C} = 0.1 + \frac{1}{j(2.357\times10^{3})(9\times10^{-6})} = 0.1 - j47.14$$

$$\mathbf{Z}_{in}(\omega_0) = (R_1 \parallel j\omega L) \parallel (R_2 + 1/j\omega C)$$

$$\mathbf{Z}_{in}(\omega_0) = \frac{(0.9996 + j0.0212)(0.1 - j47.14)}{(0.9996 + j0.0212) + (0.1 - j47.14)}$$

$$\mathbf{Z}_{in}(\omega_0) = \mathbf{1\,\Omega}$$

Series and Parallel Resonance

Summary of the characteristics of resonant *RLC* circuits.

Characteristic	Series circuit	Parallel circuit
Resonant frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Quality factor, Q	$\dfrac{\omega_0 L}{R}$ or $\dfrac{1}{\omega_0 RC}$	$\dfrac{R}{\omega_0 L}$ or $\omega_0 RC$
Bandwidth, B	$\dfrac{\omega_0}{Q}$	$\dfrac{\omega_0}{Q}$
Half-power frequencies, ω_1, ω_2	$\omega_0\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{\omega_0}{2Q}$	$\omega_0\sqrt{1+\left(\dfrac{1}{2Q}\right)^2} \pm \dfrac{\omega_0}{2Q}$
For $Q \geq 10$, ω_1, ω_2	$\omega_0 \pm \dfrac{B}{2}$	$\omega_0 \pm \dfrac{B}{2}$


Q
&
A