Electric Circuits II

Three-Phase Circuits

Dr. Firas Obeidat

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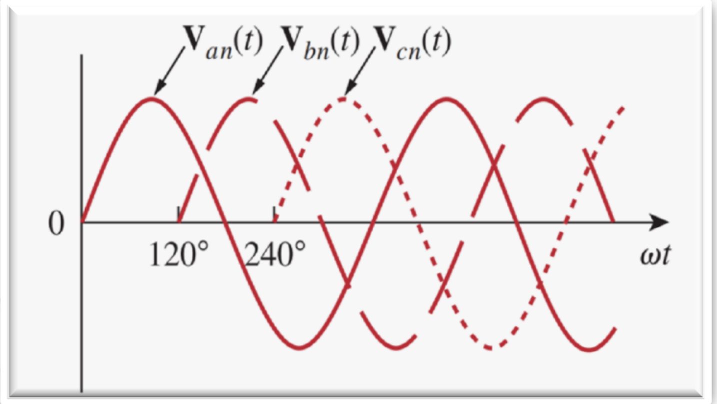
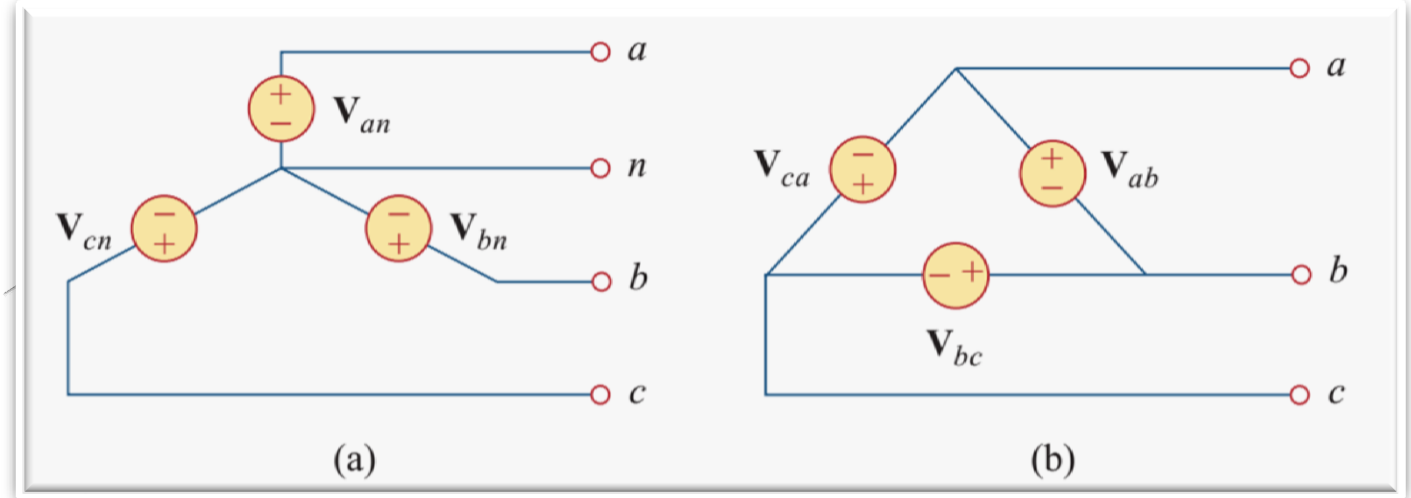
Balanced Three-Phase Voltages

- Three phase voltage sources produce three voltages which are equal in magnitude but out of phase by 120º.

- A typical three-phase system consists of three voltage sources connected to loads by three or four wires (or transmission lines).

- The voltage sources can be either wye-connected as in fig (a) or delta-connected as in fig (b).





Balanced Three-Phase Voltages

wye connection

- **The voltages V_{an}, V_{bn} and V_{cn} are called phase voltages.**
- **If the voltage sources have the same amplitude and frequency and are out of phase with each other by 120º the voltages are said to be balanced.**

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0$$

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$$

Balanced phase voltages are equal in magnitude and are out of phase with each other by 120º.

There are two possible combinations

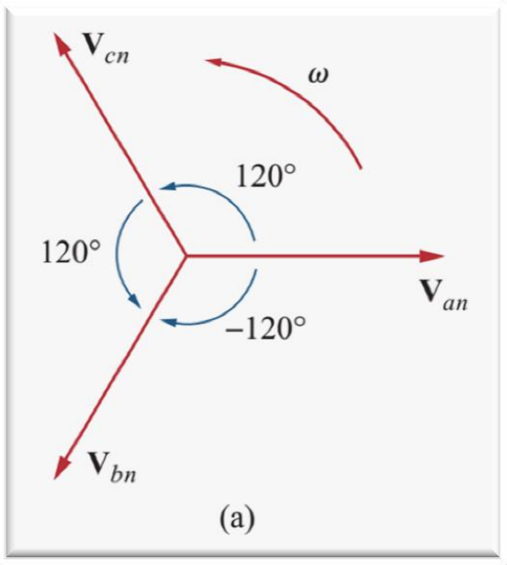
1- *abc or positive sequence*

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$

$$\mathbf{V}_{bn} = V_p\angle -120^\circ$$

$$\mathbf{V}_{cn} = V_p\angle -240^\circ = V_p\angle +120^\circ$$

V_p is the effective or *rms* value of the phase voltages

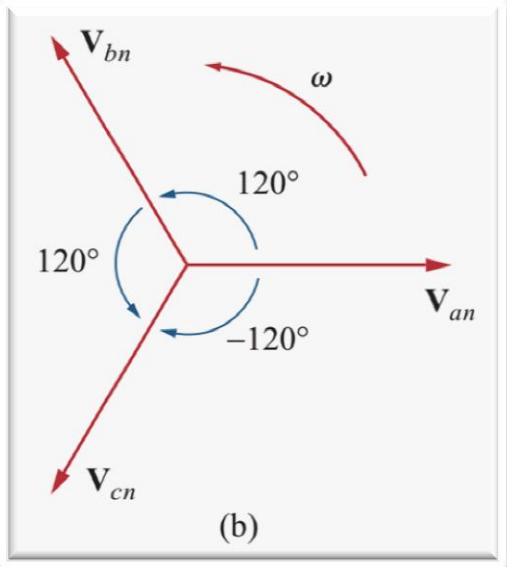


Balanced Three-Phase Voltages

wye connection

2- *acb or negative sequence*

$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{cn} = V_p\angle -120^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -240^\circ = V_p\angle +120^\circ$$



The voltages in the two cases satisfy

$$\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} = 0$$

$$|\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$$

$$\begin{aligned}\mathbf{V}_{an} + \mathbf{V}_{bn} + \mathbf{V}_{cn} &= V_p\angle 0^\circ + V_p\angle -120^\circ + V_p\angle +120^\circ \\ &= V_p(1.0 - 0.5 - j0.866 - 0.5 + j0.866) \\ &= 0\end{aligned}$$

The phase sequence is the time order in which the voltages pass through their respective maximum values.

Balanced Three-Phase Voltages

Three-phase load

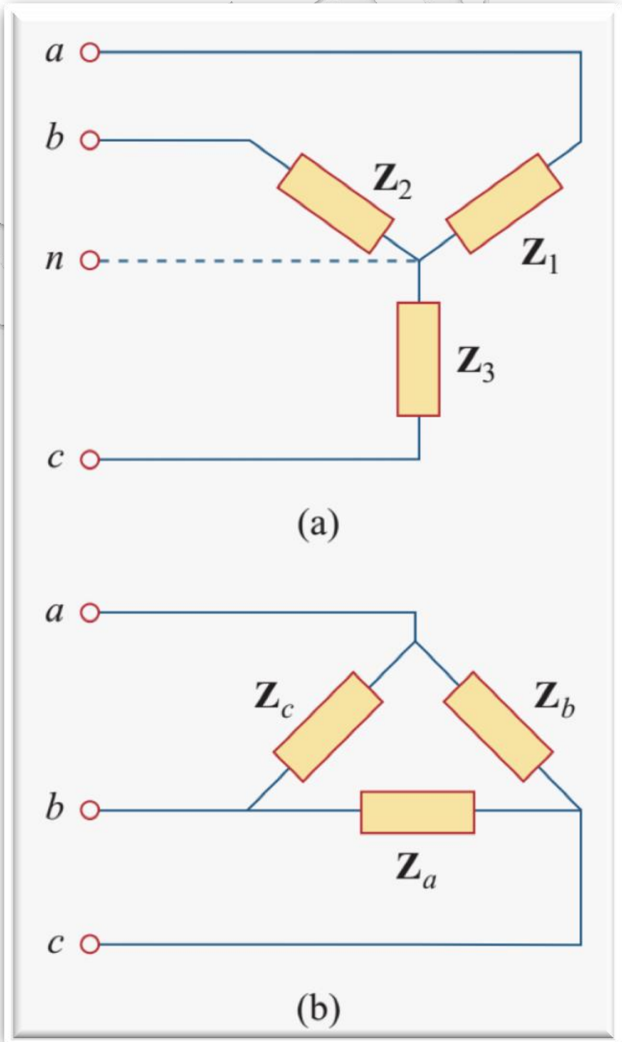
- A three-phase load can be either wye-connected as in fig(a) or delta-connected as in fig(b).
- The neutral line in fig(a) may or may not be there, depending on whether the system is four- or three-wire.
- A wye- or delta-connected load is said to be unbalanced if the phase impedances are not equal in magnitude or phase.
- For a balanced wye-connected load,

$$\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathbf{Z}_Y$$

- For a balanced delta-connected load,

$$\mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c = \mathbf{Z}_\Delta$$

$$\boxed{\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta}$$



Balanced Three-Phase Voltages

Since both the three-phase source and the three-phase load can be either wye- or delta-connected, there are four possible connections:

1) **Y-Y connection (i.e., Y-connected source with a Y-connected load).**
2) **Y-Δ connection (i.e., Y-connected source with a Δ -connected load).**
3) **Δ-Y connection (i.e., Δ -connected source with a Y-connected load).**
4) **Δ -Δ connection (i.e., Δ -connected source with a Δ -connected load).**

Example: Determine the phase sequence of the set of voltages

$$v_{an} = 200\cos(\omega t + 10^\circ)$$

$$v_{bn} = 200\cos(\omega t - 230^\circ), \qquad v_{cn} = 200\cos(\omega t - 110^\circ)$$

The voltages can be expressed in phasor form as

$$\mathbf{V}_{an} = 200\underline{/10^\circ}\text{ V}, \qquad \mathbf{V}_{bn} = 200\underline{/-230^\circ}\text{ V}, \qquad \mathbf{V}_{cn} = 200\underline{/-110^\circ}\text{ V}$$

V_{an} **leads** V_{cn} **by** 120^o **and** V_{cn} **leads** V_{bn} 120^o

Hence, the sequence is ***acb*** **sequence.**

Balanced Wye-Wye Connection

A balanced Y-Y system is a three-phase system with a balanced Y-connected source and a balanced Y-connected load.

For balanced four-wire Y-Y system

$$\mathbf{Z}_Y = \mathbf{Z}_s + \mathbf{Z}_\ell + \mathbf{Z}_L$$

Where

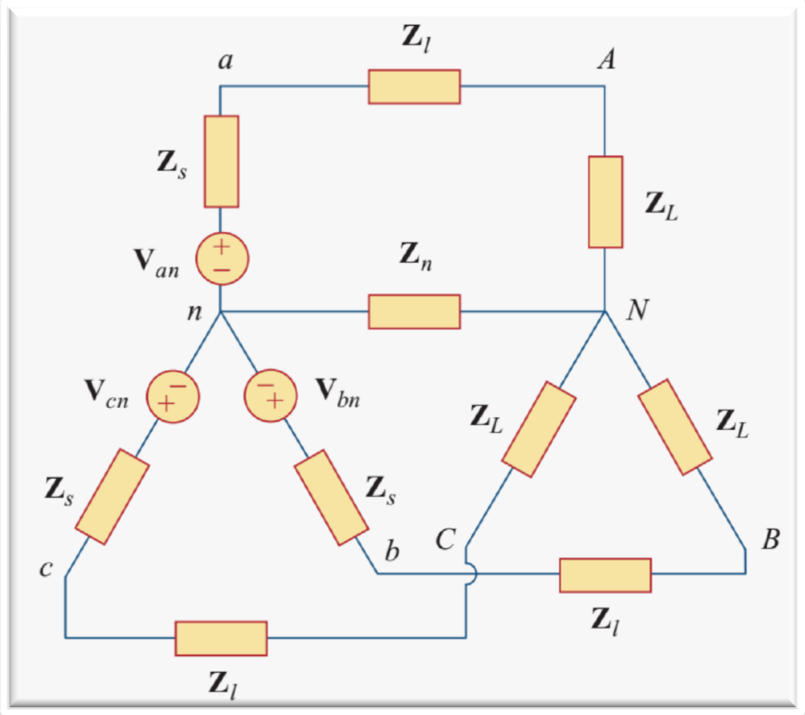
Z_Y: is the total load impedance per phase.

Z_s: is the source impedance.

Z_l: is the line impedance.

Z_L: is the load impedance for each phase.

Z_n: is the impedance of the neutral line.



Balanced Wye-Wye Connection

Assuming the positive sequence, the *phase* voltages (or line to neutral voltages) are

$$\mathbf{V}_{an} = V_p\underline{/0^\circ}$$

$$\mathbf{V}_{bn} = V_p\underline{/-120^\circ}, \qquad \mathbf{V}_{cn} = V_p\underline{/+120^\circ}$$

The *line-to-line* voltages or *line* voltages V_{ab}, V_{bc} and V_{ca} are related to the phase voltages.

$$\mathbf{V}_{ab} = \mathbf{V}_{an} + \mathbf{V}_{nb} = \mathbf{V}_{an} - \mathbf{V}_{bn} = V_p\underline{/0^\circ} - V_p\underline{/-120^\circ}$$

$$= V_p\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \sqrt{3}V_p\underline{/30^\circ}$$

$$\mathbf{V}_{bc} = \mathbf{V}_{bn} - \mathbf{V}_{cn} = \sqrt{3}V_p\underline{/-90^\circ}$$

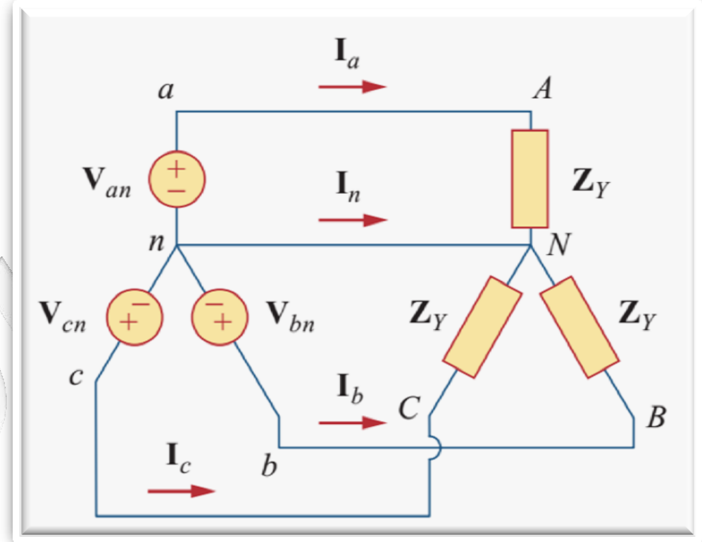
$$\mathbf{V}_{ca} = \mathbf{V}_{cn} - \mathbf{V}_{an} = \sqrt{3}V_p\underline{/-210^\circ}$$

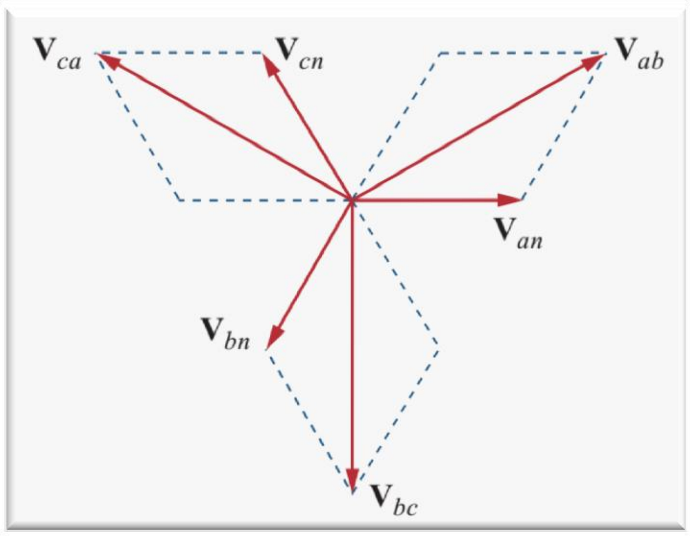
- **The set of line-to-line voltages leads the set of line-to-neutral voltages by 30°.**
- **The relationship between the magnitude of the line voltage (V_L) to the magnitude of the phase voltage (V_p) is**

$$\boxed{V_L = \sqrt{3}V_p}$$

$$V_p = |\mathbf{V}_{an}| = |\mathbf{V}_{bn}| = |\mathbf{V}_{cn}|$$

$$V_L = |\mathbf{V}_{ab}| = |\mathbf{V}_{bc}| = |\mathbf{V}_{ca}|$$



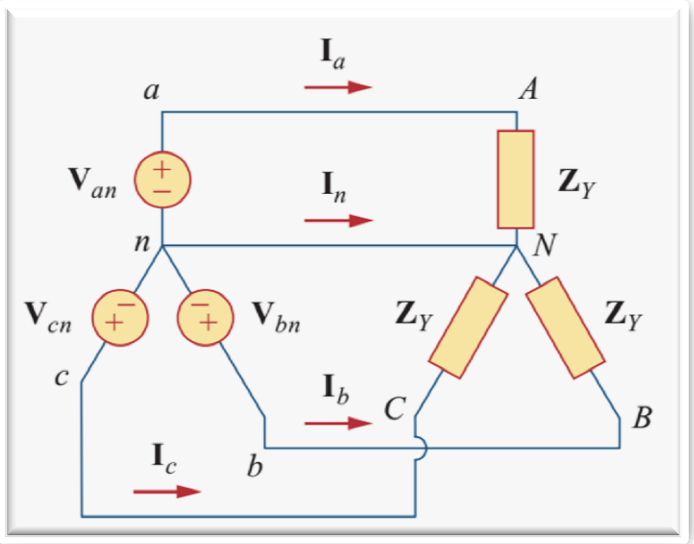


Balanced Wye-Wye Connection

Apply KVL to each phase to get line current

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}, \qquad \mathbf{I}_b = \frac{\mathbf{V}_{bn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\underline{/-120^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\underline{/-120^\circ}$$

$$\mathbf{I}_c = \frac{\mathbf{V}_{cn}}{\mathbf{Z}_Y} = \frac{\mathbf{V}_{an}\underline{/-240^\circ}}{\mathbf{Z}_Y} = \mathbf{I}_a\underline{/-240^\circ}$$



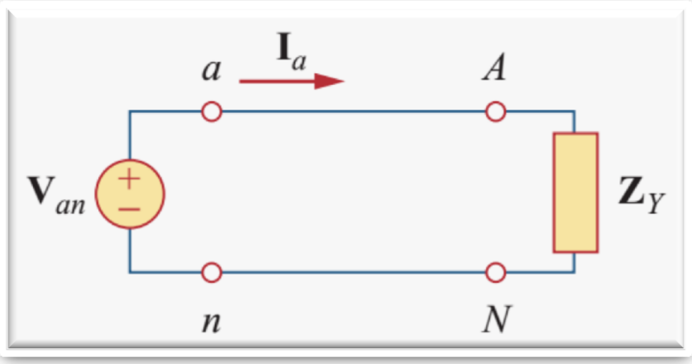
The summation of line currents is equal to zero

$$\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c = 0$$

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = 0$$

$$\mathbf{V}_{nN} = \mathbf{Z}_n\mathbf{I}_n = 0$$

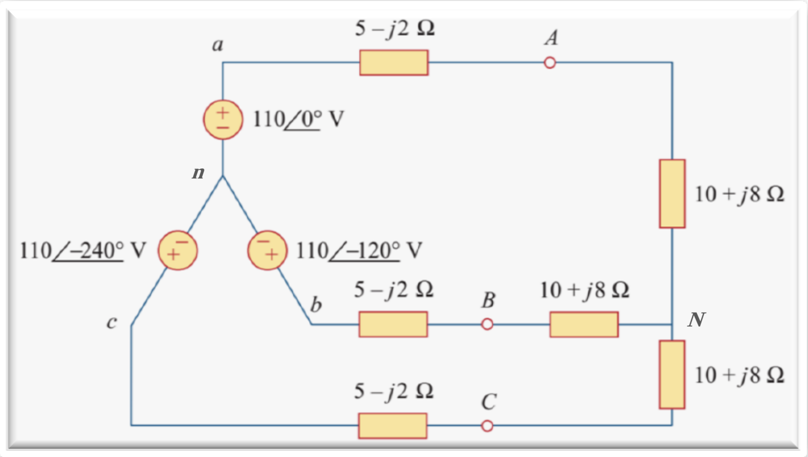
In the Y-Y system, the line current is the same as the phase current.



$$\boxed{\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y}}$$

Balanced Wye-Wye Connection

Example: For three-wire Y-Y system Calculate: the line currents, the phase voltages across the loads V_{AN}, A_{BN} and V_{CN}, the line voltages V_{AB}, A_{BC} and V_{CA}, the voltages V_{An}, A_{Bn} and V_{Cn}, the line voltages V_{ab}, A_{bc} and V_{ca}.



$$I_a = \frac{V_{an}}{Z_Y}$$

$$Z_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\angle 21.8^o$$

$$I_a = \frac{110\angle 0^o}{16.155\angle 21.8^o} = 6.81\angle\text{-}21.8^o \text{ A}$$

$$I_b = Ia\angle - 120 = 6.81\angle\text{-}141.8^o \text{ A}$$

$$I_b = Ic\angle - 240 = 6.81\angle{-261.8^o} \text{ A} = 6.81\angle 98.2^o \text{ A}$$

$$V_{AN} = (10 + j8)6.81\angle - 21.8^o = (12.8\angle 38.66^o)6.81\angle - 21.8^o = 87.17\angle 16.86^o$$

$$V_{BN} = 87.17\angle - 103.14^o \qquad V_{CN} = 87.17\angle - 223.14^o = 87.17\angle 136.86^o$$

Balanced Wye-Wye Connection

$V_{AB} = \sqrt{3}\angle 30^o\, V_{AN}$=150.98$\angle 46.86^o$

V_{BC}=63.46$\angle - 73.14^o$

V_{CA}=63.46$\angle - 193.14^o$

$V_{Aa} = (5 - j2)6.81\angle - 21.8^o = (5.38\angle - 21.8^o)6.81\angle - 21.8^o = 36.64\angle - 43.6^o$

$V_{Bb} = 36.64\angle - 163.6^o$ $\quad V_{Cc} = 36.64\angle - 283.6^o = 36.64\angle 76.4^o$

$V_{An} = 110\angle 0^o - V_{Aa} = 110 - 17 + j25.28 = 93 + j25.28 = 96.37\angle 15.21^o$

$V_{Bn} = 96.37\angle\text{-}104.79^o$ $\quad V_{Cn} = 96.37\angle\text{-}224.79^o$

$V_{ab} = \sqrt{3}\angle 30^o V_{an} = 190.5\angle 30^o$

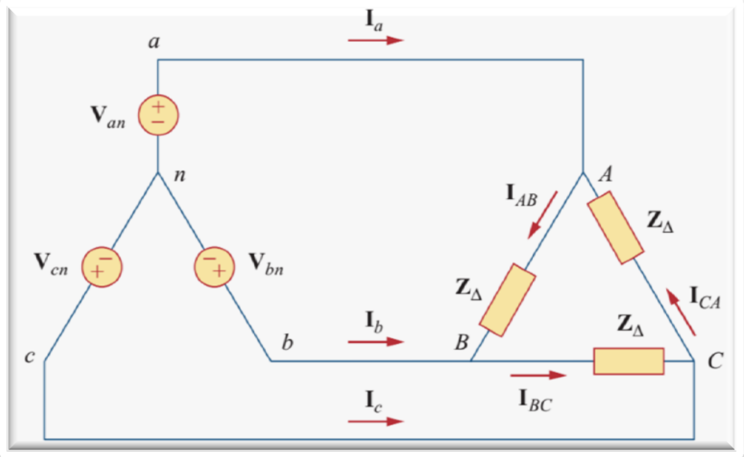
$V_{bc} = \sqrt{3}\angle 30^o V_{bn} = 190.5\angle\text{-}90^o$

$V_{ca} = \sqrt{3}\angle 30^o V_{cn} = 190.5\angle\text{-}210^o$

Balanced Wye-Delta Connection

A **balanced Y-Δ system** consists of a balanced Y-connected source feeding a balanced Δ-connected load.

Assuming the positive sequence, the phase voltages are



$$\mathbf{V}_{an} = V_p\angle 0^\circ$$
$$\mathbf{V}_{bn} = V_p\angle -120^\circ, \qquad \mathbf{V}_{cn} = V_p\angle +120^\circ$$

The line voltages are

$$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \sqrt{3}V_p\angle -90^\circ = \mathbf{V}_{BC}$$
$$\mathbf{V}_{ca} = \sqrt{3}V_p\angle -150^\circ = \mathbf{V}_{CA}$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta}$$

These currents have the same magnitude but are out of phase with each other by 120^o.

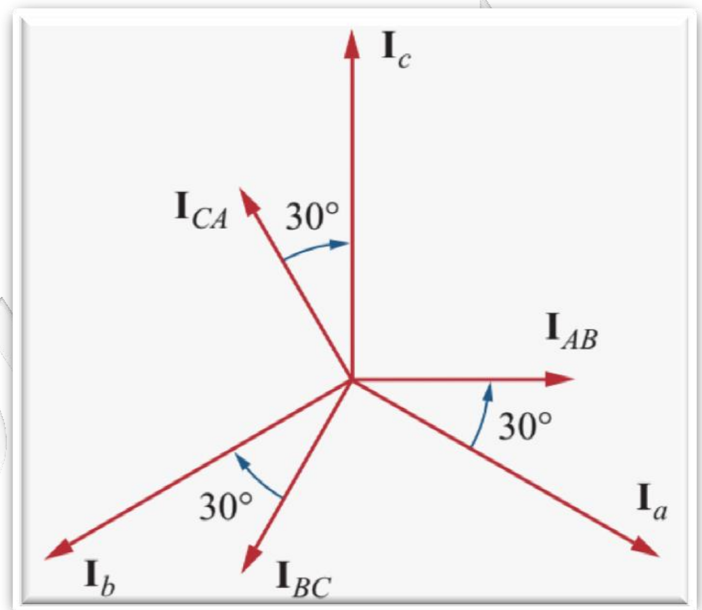
The line currents are obtained from the phase currents by applying KCL at nodes A, B, and C.

Balanced Wye-Delta Connection

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/-240^\circ}$$

$$\begin{aligned}\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA} &= \mathbf{I}_{AB}(1 - 1\underline{/-240^\circ}) \\ &= \mathbf{I}_{AB}(1 + 0.5 - j0.866) = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ}\end{aligned}$$



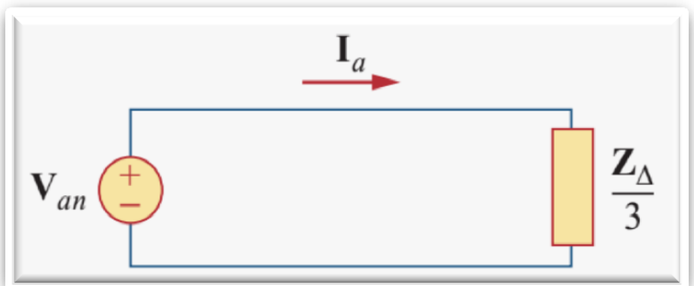
Showing that the magnitude of the line current is *sqrt(3)* times the magnitude of the phase current, or

$$\boxed{I_L = \sqrt{3}I_p}$$

$$I_L = |\mathbf{I}_a| = |\mathbf{I}_b| = |\mathbf{I}_c|$$
$$I_p = |\mathbf{I}_{AB}| = |\mathbf{I}_{BC}| = |\mathbf{I}_{CA}|$$

Another way of analyzing the circuit is to transform the Δ-connected load to an equivalent Y-connected load. Using the Following formula

$$\boxed{\mathbf{Z}_Y = \frac{\mathbf{Z}_\Delta}{3}}$$



Balanced Wye-Delta Connection

Example: A balanced *abc*-sequence Y-connected source with $V_{an}=100\angle 10^o$ is connected to a Δ-connected balanced load 8+j4Ω per phase. Calculate the phase and line currents.

$$\mathbf{Z}_\Delta = 8 + j4 = 8.944\angle 26.57^\circ\ \Omega$$

$$\mathbf{V}_{ab} = \mathbf{V}_{an}\sqrt{3}\angle 30^\circ = 100\sqrt{3}\angle 10^\circ + 30^\circ = \mathbf{V}_{AB}$$

$$\mathbf{V}_{AB} = 173.2\angle 40^\circ\ \text{V}$$

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{173.2\angle 40^\circ}{8.944\angle 26.57^\circ} = 19.36\angle 13.43^\circ\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\angle -120^\circ = 19.36\angle -106.57^\circ\ \text{A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\angle +120^\circ = 19.36\angle 133.43^\circ\ \text{A}$$

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ = \sqrt{3}(19.36)\angle 13.43^\circ - 30^\circ = 33.53\angle -16.57^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 33.53\angle -136.57^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ = 33.53\angle 103.43^\circ\ \text{A}$$

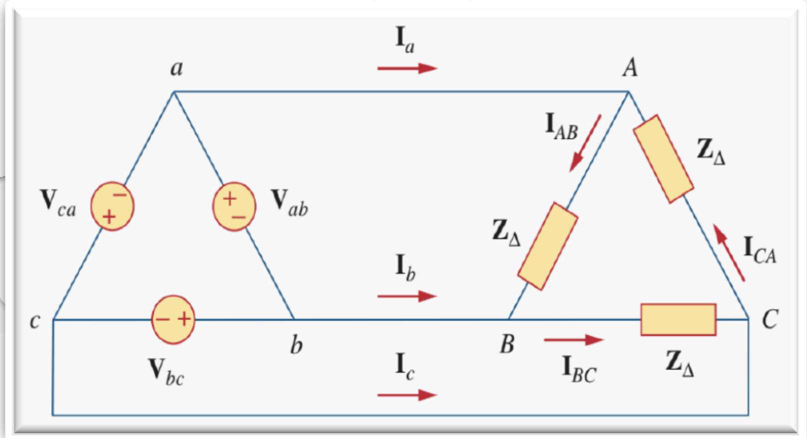
Balanced Delta-Delta Connection

A **balanced Δ-Δ system** is one in which both the balanced source and balanced load are Δ-connected.

Assuming a positive sequence, the phase voltages for a delta-connected source are

$$\mathbf{V}_{ab} = V_p\angle 0^\circ$$

$$\mathbf{V}_{bc} = V_p\angle -120^\circ, \qquad \mathbf{V}_{ca} = V_p\angle +120^\circ$$



Assuming there is no line impedances, the phase voltages of the delta connected source are equal to the voltages across the impedances.

$$\mathbf{V}_{ab} = \mathbf{V}_{AB}, \qquad \mathbf{V}_{bc} = \mathbf{V}_{BC}, \qquad \mathbf{V}_{ca} = \mathbf{V}_{CA}$$

The phase currents are

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_\Delta}, \qquad \mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_\Delta}$$

$$\mathbf{I}_{CA} = \frac{\mathbf{V}_{CA}}{\mathbf{Z}_\Delta} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_\Delta}$$

The line currents are obtained from the phase currents by applying KCL at nodes A, B, and C

$$\mathbf{I}_a = \mathbf{I}_{AB} - \mathbf{I}_{CA}, \qquad \mathbf{I}_b = \mathbf{I}_{BC} - \mathbf{I}_{AB}, \qquad \mathbf{I}_c = \mathbf{I}_{CA} - \mathbf{I}_{BC}$$

Balanced Delta-Delta Connection

Each line current lags the corresponding phase current by 30º. the magnitude I_L of the line current is *sqrt(3)* times the magnitude I_p of the phase current.

$$\boxed{I_L = \sqrt{3} I_p}$$

Example: A balanced Δ-connected load having an impedance *20-j15Ω* is connected to a Δ-connected, positive-sequence generator having $V_{ab}=330\angle 0^o$. Calculate the phase currents of the load and the line currents.

$$\mathbf{Z}_\Delta = 20 - j15 = 25\underline{/-36.87^\circ}\ \Omega$$

$$\mathbf{V}_{AB} = \mathbf{V}_{ab}$$

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{\mathbf{Z}_\Delta} = \frac{330\underline{/0^\circ}}{25\underline{/-36.87}} = 13.2\underline{/36.87^\circ}\ \text{A}$$

$$\mathbf{I}_{BC} = \mathbf{I}_{AB}\underline{/-120^\circ} = 13.2\underline{/-83.13^\circ}\ \text{A}$$

$$\mathbf{I}_{CA} = \mathbf{I}_{AB}\underline{/+120^\circ} = 13.2\underline{/156.87^\circ}\ \text{A}$$

For a delta load, the line current always lags the corresponding phase current by 30º and has a magnitude *sqrt(3)* times that of the phase current. Hence, the line currents are

$$\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\underline{/-30^\circ} = (13.2\underline{/36.87^\circ})(\sqrt{3}\underline{/-30^\circ}) = 22.86\underline{/6.87^\circ}\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\underline{/-120^\circ} = 22.86\underline{/-113.13^\circ}\ \text{A}$$

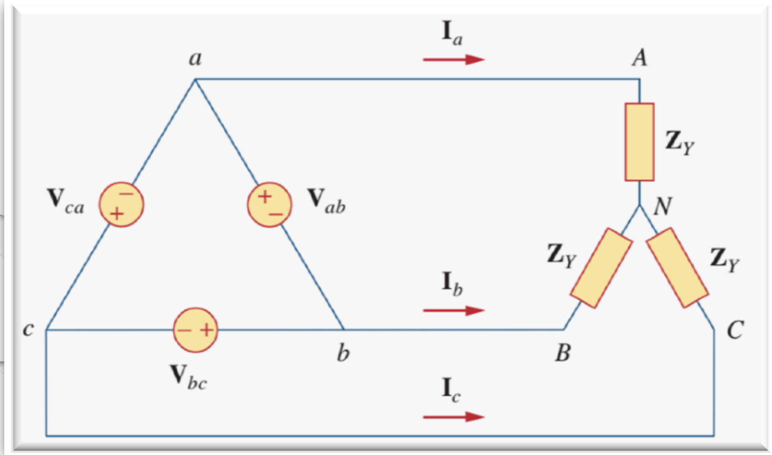
$$\mathbf{I}_c = \mathbf{I}_a\underline{/+120^\circ} = 22.86\underline{/126.87^\circ}\ \text{A}$$

Balanced Delta-Wye Connection

A **balanced Δ-Y system** consists of a balanced Δ-connected source feeding a balanced Y-connected load.

Assuming the *abc* sequence, the phase voltages of a delta-connected source are

$$\mathbf{V}_{ab} = V_p\angle 0^\circ, \qquad \mathbf{V}_{bc} = V_p\angle -120^\circ$$
$$\mathbf{V}_{ca} = V_p\angle +120^\circ$$



Apply KVL to loop ***aANBba***

$$-\mathbf{V}_{ab} + \mathbf{Z}_Y\mathbf{I}_a - \mathbf{Z}_Y\mathbf{I}_b = 0$$
$$\mathbf{Z}_Y(\mathbf{I}_a - \mathbf{I}_b) = \mathbf{V}_{ab} = V_p\angle 0^\circ$$
$$\mathbf{I}_a - \mathbf{I}_b = \frac{V_p\angle 0^\circ}{\mathbf{Z}_Y}$$

But I_b lags I_a by 120º for the abc sequence

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$$
$$\mathbf{I}_a - \mathbf{I}_b = \mathbf{I}_a(1 - 1\angle -120^\circ)$$
$$= \mathbf{I}_a\left(1 + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = \mathbf{I}_a\sqrt{3}\angle 30^\circ$$

Equating the last two equations gives

$$\boxed{\mathbf{I}_a = \frac{V_p/\sqrt{3}\angle -30^\circ}{\mathbf{Z}_Y}}$$

Balanced Delta-Wye Connection

Equating the last two equations gives

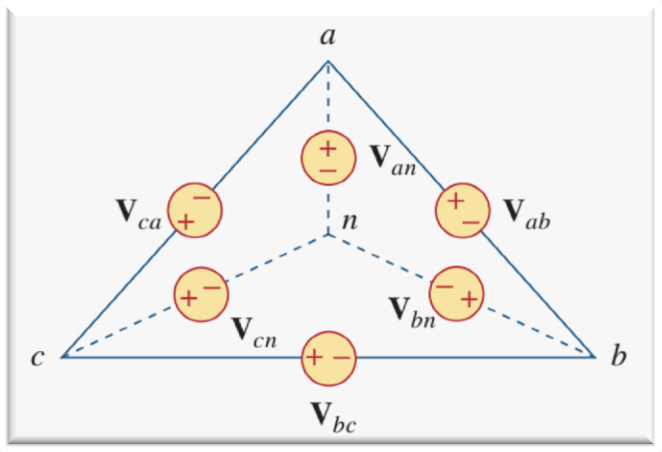
$$\mathbf{I}_b = \mathbf{I}_a \angle{-120^\circ} \qquad \mathbf{I}_c = \mathbf{I}_a \angle{-240^\circ}$$

The phase currents are equal to the line currents

The equivalent wye-connected source has the phase voltages

$$\mathbf{V}_{an} = \frac{V_p}{\sqrt{3}} \angle{-30^\circ}$$

$$\mathbf{V}_{bn} = \frac{V_p}{\sqrt{3}} \angle{-150^\circ}, \qquad \mathbf{V}_{cn} = \frac{V_p}{\sqrt{3}} \angle{+90^\circ}$$



The equivalent delta-connected load has the phase voltages

$$\mathbf{V}_{AN} = \mathbf{I}_a \mathbf{Z}_Y = \frac{V_p}{\sqrt{3}} \angle{-30^\circ}$$

$$\mathbf{V}_{BN} = \mathbf{V}_{AN} \angle{-120^\circ}, \qquad \mathbf{V}_{CN} = \mathbf{V}_{AN} \angle{+120^\circ}$$

Balanced Delta-Wye Connection

Example: A balanced Y-connected load with a phase impedance of *400+j25Ω* is supplied by a balanced, positive sequence Δ-connected source with a line voltage of 210 V. Calculate the phase currents. Use V_{ab} as a reference..

$$\mathbf{Z}_Y = 40 + j25 = 47.17\angle 32^\circ\ \Omega$$

$$\mathbf{V}_{ab} = 210\angle 0^\circ\ \text{V}$$

When the Δ-connected source is transformed to a Y-connected source

$$\mathbf{V}_{an} = \frac{\mathbf{V}_{ab}}{\sqrt{3}}\angle -30^\circ = 121.2\angle -30^\circ\ \text{V}$$

The line currents are

$$\mathbf{I}_a = \frac{\mathbf{V}_{an}}{\mathbf{Z}_Y} = \frac{121.2\angle -30^\circ}{47.12\angle 32^\circ} = 2.57\angle -62^\circ\ \text{A}$$

$$\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ = 2.57\angle -178^\circ\ \text{A}$$

$$\mathbf{I}_c = \mathbf{I}_a\angle 120^\circ = 2.57\angle 58^\circ\ \text{A}$$

Which are the same as the phase currents

balanced three-phase systems

Connection	Phase voltages/currents	Line voltages/currents
Y-Y	$\mathbf{V}_{an} = V_p\angle 0^\circ$ $\mathbf{V}_{bn} = V_p\angle -120^\circ$ $\mathbf{V}_{cn} = V_p\angle +120^\circ$ Same as line currents	$\mathbf{V}_{ab} = \sqrt{3}V_p\angle 30^\circ$ $\mathbf{V}_{bc} = \mathbf{V}_{ab}\angle -120^\circ$ $\mathbf{V}_{ca} = \mathbf{V}_{ab}\angle +120^\circ$ $\mathbf{I}_a = \mathbf{V}_{an}/\mathbf{Z}_Y$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Y-Δ	$\mathbf{V}_{an} = V_p\angle 0^\circ$ $\mathbf{V}_{bn} = V_p\angle -120^\circ$ $\mathbf{V}_{cn} = V_p\angle +120^\circ$ $\mathbf{I}_{AB} = \mathbf{V}_{AB}/\mathbf{Z}_\Delta$ $\mathbf{I}_{BC} = \mathbf{V}_{BC}/\mathbf{Z}_\Delta$ $\mathbf{I}_{CA} = \mathbf{V}_{CA}/\mathbf{Z}_\Delta$	$\mathbf{V}_{ab} = \mathbf{V}_{AB} = \sqrt{3}V_p\angle 30^\circ$ $\mathbf{V}_{bc} = \mathbf{V}_{BC} = \mathbf{V}_{ab}\angle -120^\circ$ $\mathbf{V}_{ca} = \mathbf{V}_{CA} = \mathbf{V}_{ab}\angle +120^\circ$ $\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Δ-Δ	$\mathbf{V}_{ab} = V_p\angle 0^\circ$ $\mathbf{V}_{bc} = V_p\angle -120^\circ$ $\mathbf{V}_{ca} = V_p\angle +120^\circ$ $\mathbf{I}_{AB} = \mathbf{V}_{ab}/\mathbf{Z}_\Delta$ $\mathbf{I}_{BC} = \mathbf{V}_{bc}/\mathbf{Z}_\Delta$ $\mathbf{I}_{CA} = \mathbf{V}_{ca}/\mathbf{Z}_\Delta$	Same as phase voltages $\mathbf{I}_a = \mathbf{I}_{AB}\sqrt{3}\angle -30^\circ$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$
Δ-Y	$\mathbf{V}_{ab} = V_p\angle 0^\circ$ $\mathbf{V}_{bc} = V_p\angle -120^\circ$ $\mathbf{V}_{ca} = V_p\angle +120^\circ$ Same as line currents	Same as phase voltages $\mathbf{I}_a = \dfrac{V_p\angle -30^\circ}{\sqrt{3}\mathbf{Z}_Y}$ $\mathbf{I}_b = \mathbf{I}_a\angle -120^\circ$ $\mathbf{I}_c = \mathbf{I}_a\angle +120^\circ$

Summary of phase and line voltages/currents for balanced three-phase systems (*abc* sequence is assumed).

Power in a Balanced System

For a Y-connected load, the phase voltages are

$$v_{AN} = \sqrt{2}V_p \cos \omega t, \qquad v_{BN} = \sqrt{2}V_p \cos(\omega t - 120°)$$
$$v_{CN} = \sqrt{2}V_p \cos(\omega t + 120°)$$

If $Z_Y = Z\angle\theta$, the phase currents lag their corresponding phase voltages by θ

$$i_a = \sqrt{2}I_p \cos(\omega t - \theta), \qquad i_b = \sqrt{2}I_p \cos(\omega t - \theta - 120°)$$
$$i_c = \sqrt{2}I_p \cos(\omega t - \theta + 120°)$$

The total instantaneous power in the load is the sum of the instantaneous powers in the three phases

$$\begin{aligned} p &= p_a + p_b + p_c = v_{AN}i_a + v_{BN}i_b + v_{CN}i_c \\ &= 2V_pI_p[\cos \omega t \cos(\omega t - \theta) \\ &\qquad + \cos(\omega t - 120°)\cos(\omega t - \theta - 120°) \\ &\qquad + \cos(\omega t + 120°)\cos(\omega t - \theta + 120°)] \end{aligned}$$

Applying the trigonometric identity

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

Power in a Balanced System

Instantaneous power equation becomes

$$p = V_p I_p[3\cos\theta + \cos(2\omega t - \theta) + \cos(2\omega t - \theta - 240^\circ) + \cos(2\omega t - \theta + 240^\circ)]$$

$$= V_p I_p[3\cos\theta + \cos\alpha + \cos\alpha\cos 240^\circ + \sin\alpha\sin 240^\circ + \cos\alpha\cos 240^\circ - \sin\alpha\sin 240^\circ]$$

$$\text{where } \alpha = 2\omega t - \theta$$

$$= V_p I_p\left[3\cos\theta + \cos\alpha + 2\left(-\frac{1}{2}\right)\cos\alpha\right] = 3V_p I_p\cos\theta$$

The total instantaneous power in a balanced three-phase system is constant; it does not change with time as the instantaneous power of each phase does. This result is true whether the load is Y- or Δ-connected.

The average power per phase for either the Δ-connected load or the Y-connected load is *p/3*

$$P_p = V_p I_p \cos\theta$$

The reactive power per phase is

$$Q_p = V_p I_p \sin\theta$$

The apparent power per phase is

$$S_p = V_p I_p$$

Power in a Balanced System

The complex power per phase is

$$\mathbf{S}_p = P_p + jQ_p = \mathbf{V}_p\mathbf{I}_p^*$$

Where V_p and I_p are the phase voltage and phase current with magnitudes V_p and I_p respectively. The total average power is the sum of the average powers in the phases:

$$P = P_a + P_b + P_c = 3P_p = 3V_pI_p\cos\theta = \sqrt{3}V_LI_L\cos\theta$$

For a Y-connected load, $I_L=I_p$ but $V_L=sqrt(3)V_p$ whereas for a Δ-connected load, $I_L=sqrt(3)I_p$ but $V_L=V_p$.

the total reactive power is

$$Q = 3V_pI_p\sin\theta = 3Q_p = \sqrt{3}V_LI_L\sin\theta$$

the total complex power is

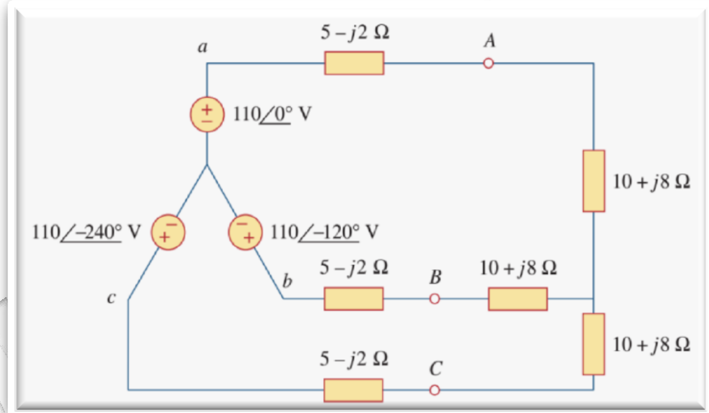
$$\mathbf{S} = 3\mathbf{S}_p = 3\mathbf{V}_p\mathbf{I}_p^* = 3I_p^2\mathbf{Z}_p = \frac{3V_p^2}{\mathbf{Z}_p^*}$$

Where $Z_p=Z_p\angle\theta$ is the load impedance per phase. (Z_p could be Z_Y or Z_Δ)

$$\boxed{\mathbf{S} = P + jQ = \sqrt{3}V_LI_L\underline{/\theta}}$$

Power in a Balanced System

Example: Determine the total average power, reactive power, and complex power at the source and at the load.



For phase a

$$I_a = \frac{V_{an}}{Z_Y}$$

$$Z_Y = (5 - j2) + (10 + j8) = 15 + j6 = 16.155\angle 21.8^o$$

$$I_a = \frac{110\angle 0^o}{16.155\angle 21.8^o} = 6.81\angle\text{-}21.8^o \text{ A}$$

$$S_s = 3V_p I_p^* = 3(110\angle 0^o)(6.81\angle 21.8^o) = 2247\angle 21.8^o = (2087 + j834.6)\text{VA}$$

The real or average power absorbed is 2087 W and the reactive power is 834.6 VAR.

$$S_L = 3|I_p|^2 Z_p = 3(6.81)^2(10+j8) = 3(6.81)^2(12.81\angle 38.66^o) = 1782\angle 38.66^o = (1392 + j1113)\text{ VA}$$

The real or average power absorbed is 1392 W and the reactive power is 1113 VAR.

$$S_l = 3|I_p|^2 Z_l = 3(6.81)^2(5-j2) = (695.6 - j278.3)\text{ VA}$$

Power in a Balanced System

Example: A three-phase motor can be regarded as a balanced Y-load. A three phase motor draws 5.6 kW when the line voltage is 220 V and the line current is 18.2 A. Determine the power factor of the motor.

The apparent power is

$$S = \sqrt{3}V_L I_L = \sqrt{3}(220)(18.2) = 6935.13 \text{ VA}$$

The real power is

$$P = S\cos\theta = 5600 \text{ W}$$

The power factor is

$$\text{pf} = \cos\theta = \frac{P}{S} = \frac{5600}{6935.13} = 0.8075$$

Y-Δ & Δ-Y Conversions

Y-Δ Conversions

$$\mathbf{Z}_a = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$$

$$\mathbf{Z}_b = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

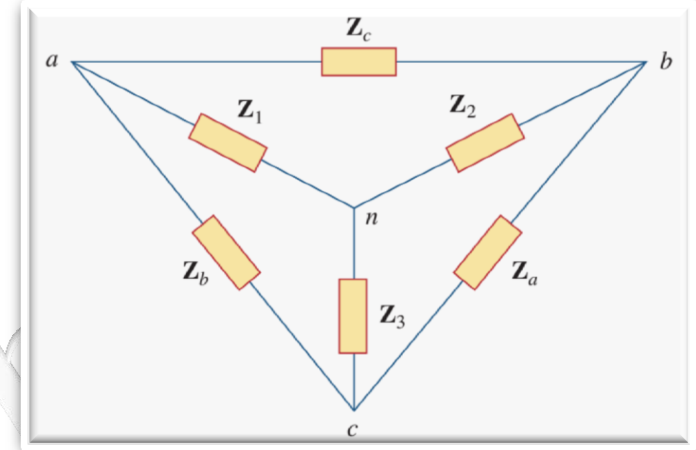
$$\mathbf{Z}_c = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

Δ-Y Conversions

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_b\mathbf{Z}_c}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_2 = \frac{\mathbf{Z}_c\mathbf{Z}_a}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$

$$\mathbf{Z}_3 = \frac{\mathbf{Z}_a\mathbf{Z}_b}{\mathbf{Z}_a + \mathbf{Z}_b + \mathbf{Z}_c}$$



A delta or wye circuit is said to be balanced if it has equal impedances in all three branches

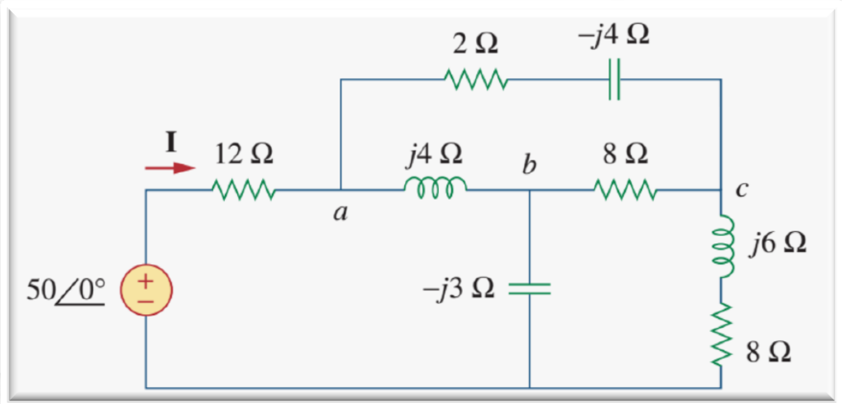
When Δ-Y is balanced

$$\mathbf{Z}_\Delta = 3\mathbf{Z}_Y \qquad \text{or} \qquad \mathbf{Z}_Y = \frac{1}{3}\mathbf{Z}_\Delta$$

$$\mathbf{Z}_Y = \mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 \text{ and } \mathbf{Z}_\Delta = \mathbf{Z}_a = \mathbf{Z}_b = \mathbf{Z}_c$$

Y-Δ & Δ-Y Conversions

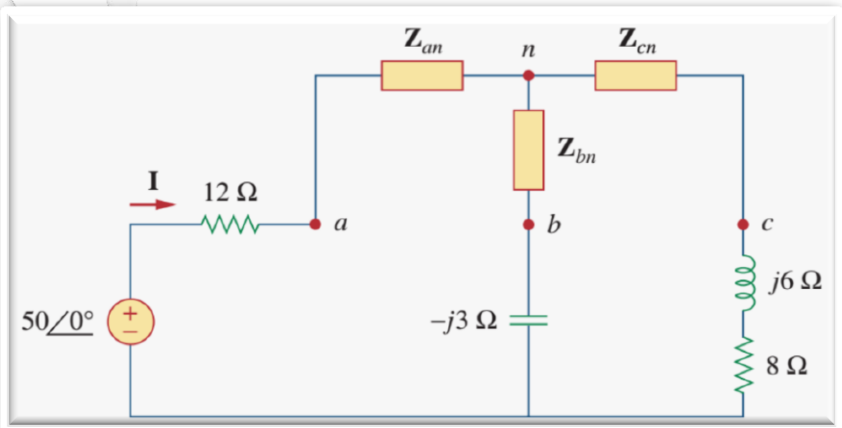
Example: Find the current I in the circuit.



The delta network connected to nodes *a*, *b*, and *c* can be converted to the Y network.

$$\mathbf{Z}_{an} = \frac{j4(2 - j4)}{j4 + 2 - j4 + 8} = \frac{4(4 + j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$\mathbf{Z}_{bn} = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad \mathbf{Z}_{cn} = \frac{8(2 - j4)}{10} = (1.6 - j3.2)\ \Omega$$



The total impedance at the source terminals is

$$\begin{aligned}
\mathbf{Z} &= 12 + \mathbf{Z}_{an} + (\mathbf{Z}_{bn} - j3) \parallel (\mathbf{Z}_{cn} + j6 + 8) \\
&= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8) \\
&= 13.6 + j0.8 + \frac{j0.2(9.6 + j2.8)}{9.6 + j3} \\
&= 13.6 + j1 = 13.64\underline{/4.204^\circ}\ \Omega
\end{aligned}$$

The current I is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{50\underline{/0^\circ}}{13.64\underline{/4.204^\circ}} = 3.666\underline{/-4.204^\circ}\ \text{A}$$

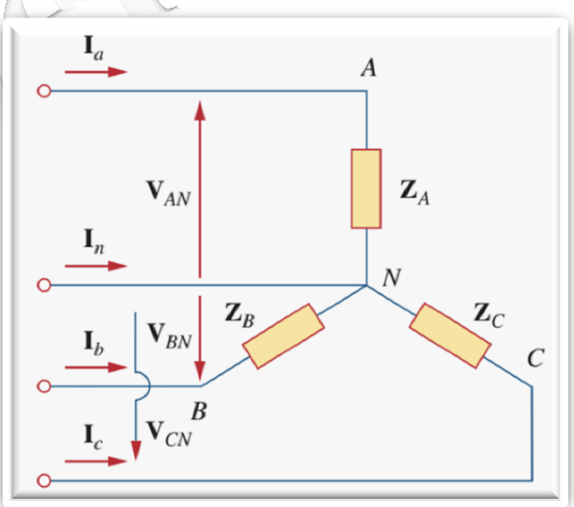
Unbalanced three-phase systems

An **unbalanced system** is due to unbalanced voltage sources or an unbalanced load.

For balanced source voltages, but an unbalanced load.

When the load is unbalanced, Z_A, Z_B and Z_C are not equal. The line currents are determined by Ohm's law as

$$\boxed{\mathbf{I}_a = \frac{\mathbf{V}_{AN}}{\mathbf{Z}_A}, \qquad \mathbf{I}_b = \frac{\mathbf{V}_{BN}}{\mathbf{Z}_B}, \qquad \mathbf{I}_c = \frac{\mathbf{V}_{CN}}{\mathbf{Z}_C}}$$

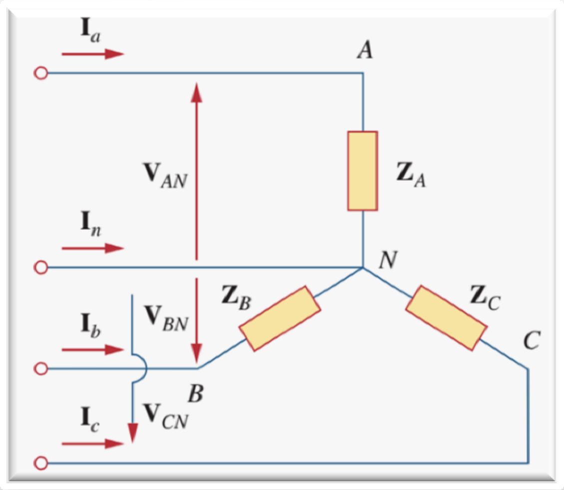


Applying KCL at node *N* gives the neutral line current as

$$\boxed{\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c)}$$

Unbalanced three-phase systems

Example: The unbalanced Y-load has balanced voltages of 100 V and the *acb* sequence. Calculate the line currents and the neutral current. Take Z_A =15 Ω, Z_B=10 +j5 Ω, Z_C=6+j8 Ω.



$$\mathbf{I}_a = \frac{100\angle 0^\circ}{15} = 6.67\angle 0^\circ \text{ A}$$

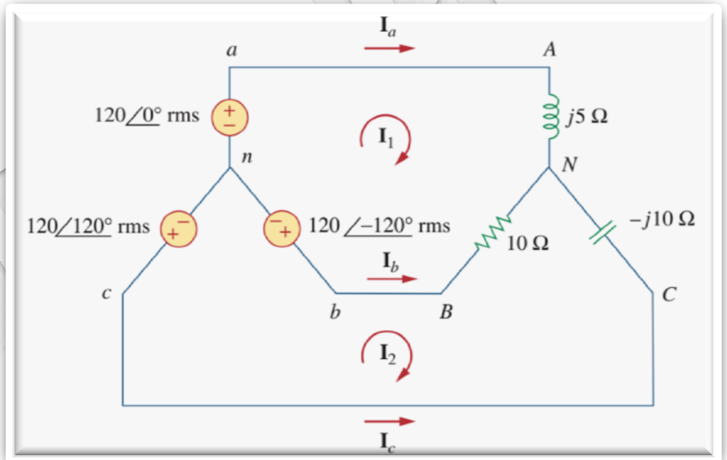
$$\mathbf{I}_b = \frac{100\angle 120^\circ}{10 + j5} = \frac{100\angle 120^\circ}{11.18\angle 26.56^\circ} = 8.94\angle 93.44^\circ \text{ A}$$

$$\mathbf{I}_c = \frac{100\angle -120^\circ}{6 - j8} = \frac{100\angle -120^\circ}{10\angle -53.13^\circ} = 10\angle -66.87^\circ \text{ A}$$

$$\mathbf{I}_n = -(\mathbf{I}_a + \mathbf{I}_b + \mathbf{I}_c) = -(6.67 - 0.54 + j8.92 + 3.93 - j9.2)$$
$$= -10.06 + j0.28 = 10.06\angle 178.4^\circ \text{ A}$$

Unbalanced three-phase systems

Example: For the unbalanced circuit, find: (a) the line currents, (b) the total complex power absorbed by the load, and (c) the total complex power absorbed by the source.



(a) Using mesh analysis to find the required currents. For mesh 1

$$120\angle -120° - 120\angle 0° + (10 + j5)\mathbf{I}_1 - 10\mathbf{I}_2 = 0$$

$$(10 + j5)\mathbf{I}_1 - 10\mathbf{I}_2 = 120\sqrt{3}\angle 30° \qquad (1)$$

For mesh 2

$$120\angle 120° - 120\angle -120° + (10 - j10)\mathbf{I}_2 - 10\mathbf{I}_1 = 0$$

$$-10\mathbf{I}_1 + (10 - j10)\mathbf{I}_2 = 120\sqrt{3}\angle -90° \qquad (2)$$

Write eq(1) & eq(2) as matrix equation

$$\begin{bmatrix} 10 + j5 & -10 \\ -10 & 10 - j10 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} 120\sqrt{3}\angle 30° \\ 120\sqrt{3}\angle -90° \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 10 + j5 & -10 \\ -10 & 10 - j10 \end{vmatrix} = 50 - j50 = 70.71\angle -45°$$

$$\Delta_1 = \begin{vmatrix} 120\sqrt{3}\angle 30° & -10 \\ 120\sqrt{3}\angle -90° & 10 - j10 \end{vmatrix} = 207.85(13.66 - j13.66)$$

$$= 4015\angle -45°$$

Unbalanced three-phase systems

$$\Delta_2 = \begin{vmatrix} 10 + j5 & 120\sqrt{3}\underline{/30^\circ} \\ -10 & 120\sqrt{3}\underline{/-90^\circ} \end{vmatrix} = 207.85(13.66 - j5)$$

$$= 3023.4\underline{/-20.1^\circ}$$

The mesh currents are

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{4015.23\underline{/-45^\circ}}{70.71\underline{/-45^\circ}} = 56.78 \text{ A}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{3023.4\underline{/-20.1^\circ}}{70.71\underline{/-45^\circ}} = 42.75\underline{/24.9^\circ} \text{ A}$$

The line currents are

$$\mathbf{I}_a = \mathbf{I}_1 = 56.78 \text{ A}, \qquad \mathbf{I}_c = -\mathbf{I}_2 = 42.75\underline{/-155.1^\circ} \text{ A}$$

$$\mathbf{I}_b = \mathbf{I}_2 - \mathbf{I}_1 = 38.78 + j18 - 56.78 = 25.46\underline{/135^\circ} \text{ A}$$

(b) Calculate the complex power absorbed by the load

$$\mathbf{S}_A = |\mathbf{I}_a|^2\mathbf{Z}_A = (56.78)^2(j5) = j16{,}120 \text{ VA}$$

$$\mathbf{S}_B = |\mathbf{I}_b|^2\mathbf{Z}_B = (25.46)^2(10) = 6480 \text{ VA}$$

$$\mathbf{S}_C = |\mathbf{I}_c|^2\mathbf{Z}_C = (42.75)^2(-j10) = -j18{,}276 \text{ VA}$$

Unbalanced three-phase systems

The total complex power absorbed by the load is

$$\mathbf{S}_L = \mathbf{S}_A + \mathbf{S}_B + \mathbf{S}_C = 6480 - j2156 \text{ VA}$$

(c) Calculate the power absorbed by the source

$$\mathbf{S}_a = -\mathbf{V}_{an}\mathbf{I}_a^* = -(120\angle 0^\circ)(56.78) = -6813.6 \text{ VA}$$

$$\mathbf{S}_b = -\mathbf{V}_{bn}\mathbf{I}_b^* = -(120\angle -120^\circ)(25.46\angle -135^\circ)$$
$$= -3055.2\angle 105^\circ = 790 - j2951.1 \text{ VA}$$

$$\mathbf{S}_c = -\mathbf{V}_{bn}\mathbf{I}_c^* = -(120\angle 120^\circ)(42.75\angle 155.1^\circ)$$
$$= -5130\angle 275.1^\circ = -456.03 + j5109.7 \text{ VA}$$

The total complex power absorbed by the three-phase source is

$$\mathbf{S}_s = \mathbf{S}_a + \mathbf{S}_b + \mathbf{S}_c = -6480 + j2156 \text{ VA}$$

Showing that $S_L + S_S = 0$


Q&A