Electric Circuits II

Magnetically Coupled Circuits

Dr. Firas Obeidat

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Introduction

When two loops with or without contacts between them affect each other through the magnetic field generated by one of them, they are said to be magnetically coupled.

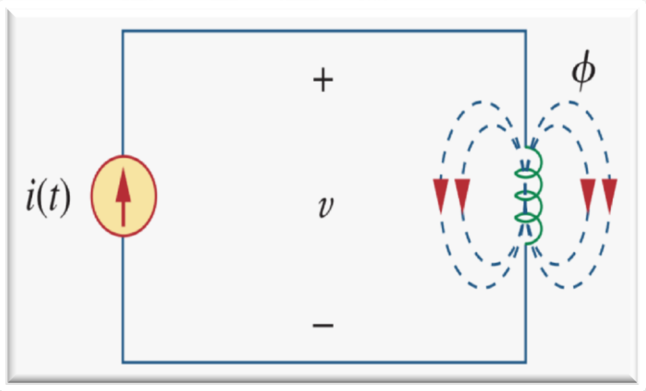
The transformer is an electrical device designed on the basis of the concept of magnetic coupling. It uses magnetically coupled coils to transfer energy from one circuit to another.

The transformers are used in power systems for stepping up or stepping down ac voltages or currents.

The transformers are used in electronic circuits such as radio and television receivers for such purposes as impedance matching, isolating one part of a circuit from another, and for stepping up or down ac voltages and currents.

Mutual Inductance

When two inductors (or coils) are in a close proximity to each other, the magnetic flux caused by current in one coil links with the other coil, thereby inducing voltage in the latter. This phenomenon is known as *mutual inductance*.



A coil with N turns, when current i flows through the coil, a magnetic flux is produced around it. According to Faraday's law:

$$v = N\frac{d\phi}{dt}$$

Any change in ϕ is caused by a change in the current.

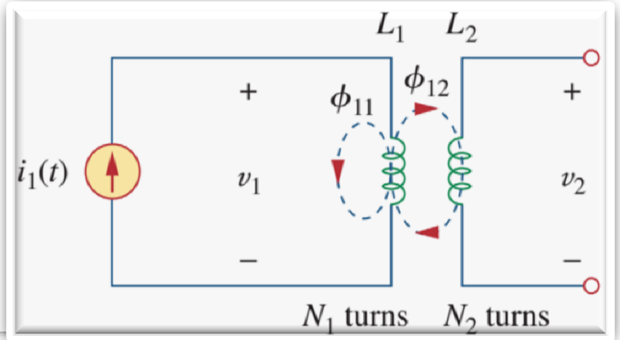
$$v = N\frac{d\phi}{di}\frac{di}{dt} \quad \text{or} \quad v = L\frac{di}{dt} \quad \text{where} \quad L = N\frac{d\phi}{di}$$

L is called *self-inductance*, because it relates the voltage induced in a coil by a time-varying current in the same coil.

When two coils with self-inductances L_1 and L_2 that are close to each other. Coil 1 has N_1 turns, while coil 2 has N_2 turns. Assume that the second inductor carries no current. The magnetic flux ϕ_1 emanating from coil 1 has two components: One component ϕ_{11} links only coil 1, and another component ϕ_{12} links both coils.

$$\phi_1 = \phi_{11} + \phi_{12}$$

Mutual Inductance



Entire flux ϕ_1 links coil 1, so the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\phi_1}{dt} \quad \textbf{or} \quad v_1 = N_1 \frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1 \frac{di_1}{dt}$$

Where $L_1=N_1 d\phi_1/di_1$ is the *self inductance* of coil 1

Only flux ϕ_{12} links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2 \frac{d\phi_{12}}{dt} \quad \textbf{or} \quad v_2 = N_2 \frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21} \frac{di_1}{dt} \quad \textbf{Where} \quad M_{21} = N_2 \frac{d\phi_{12}}{di_1}$$

M_{21} is known as the *mutual inductance* of coil 2 with respect to coil 1. Subscript 21 indicates that the inductance relates the voltage induced in coil 2 to the current in coil 1.

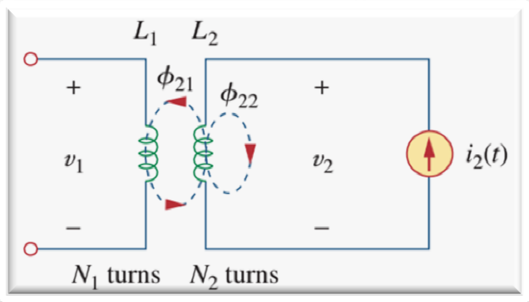
The open-circuit mutual voltage (or induced voltage) across coil 2 is

$$\boxed{v_2 = M_{21} \frac{di_1}{dt}}$$

Suppose the current flow in coil 2, while coil 1 carries no current. The magnetic flux ϕ_2 emanating from coil 2 comprises flux ϕ_{22} that links only coil 2 and flux ϕ_{21} that links both coils.

Mutual Inductance

$$\phi_2 = \phi_{21} + \phi_{22}$$



The entire flux ϕ_2 links coil 2, so the voltage induced in coil 2 is

$$v_2 = N_2 \frac{d\phi_2}{dt} = N_2 \frac{d\phi_2}{di_2}\frac{di_2}{dt} = L_2 \frac{di_2}{dt}$$

Where $L_2 = N_2 d\phi_2/di_2$ is the *self inductance* of coil 2

Only flux ϕ_{21} links coil 1, so the voltage induced in coil 1 is

$$v_1 = N_1 \frac{d\phi_{21}}{dt} = N_1 \frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12} \frac{di_2}{dt}$$

Where $M_{12} = N_1 \frac{d\phi_{21}}{di_2}$

M_{12} is known as the mutual inductance of coil 1 with respect to coil 2.

The open-circuit mutual voltage (or induced voltage) across coil 1 is

$$\boxed{v_1 = M_{12} \frac{di_2}{dt}}$$

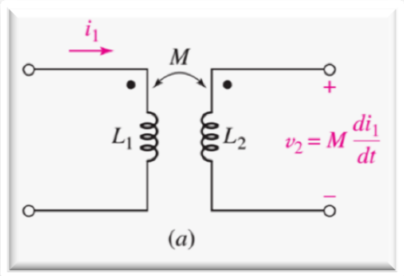
$$M_{12} = M_{21} = M$$

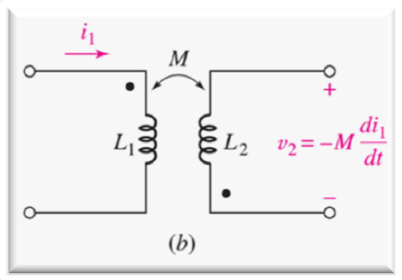
Mutual inductance **is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).**

Mutual Inductance - Dot Convention

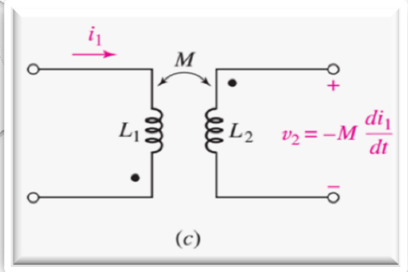
Although mutual inductance *M* is always a positive quantity, the mutual voltage *Mdi/dt* may be negative or positive, The dots are used along with the dot convention to determine the polarity of the mutual voltage.

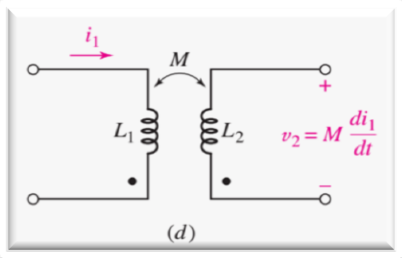
If a current enters the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is positive at the dotted terminal of the second coil.





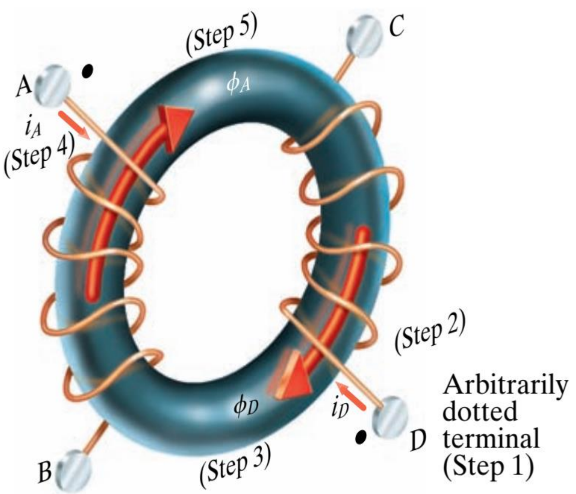
If a current leaves the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is negative at the dotted terminal of the second coil.





Mutual Inductance - Dot Convention

The Procedure for Determining Dot Markings



a) Arbitrarily select one terminal—say, the *D* terminal—of one coil and mark it with a dot.

b) Assign a current into the dotted terminal and label it i_D.

c) Use the right-hand rule to determine the direction of the magnetic field established by i_D inside the coupled coils and label this field ϕ_D.

d) Arbitrarily pick one terminal of the second coil—say, terminal *A*—and assign a current into this terminal, showing the current as i_A.

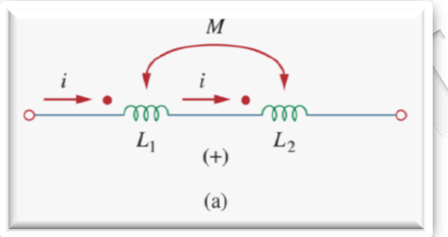
e) Use the right-hand rule to determine the direction of the flux established by i_A inside the coupled coils and label this flux ϕ_A.

f) Compare the directions of the two fluxes ϕ_D and ϕ_A. If the fluxes have the same reference direction, place a dot on the terminal of the second coil where the test current (i_A) enters. (The fluxes ϕ_D and ϕ_A have the same reference direction, and therefore a dot goes on terminal *A*). If the fluxes have different reference directions, place a dot on the terminal of the second coil where the test current leaves.

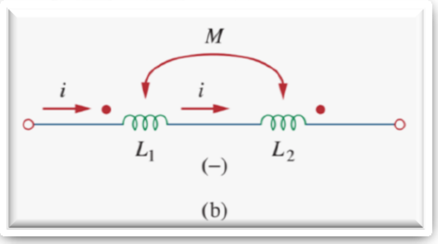
Analyze Circuits Involving Mutual Inductance

The total inductance for coupled coils in series is

$$L = L_1 + L_2 + 2M \qquad \text{(Series-aiding connection)}$$



$$L = L_1 + L_2 - 2M \qquad \text{(Series-opposing connection)}$$

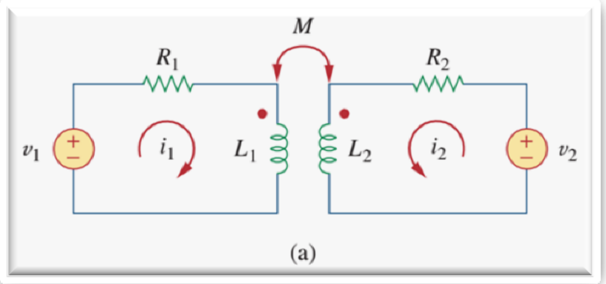


For the time domain circuit shown in fig.(a). Applying KVL to coil 1 gives

$$v_1 = i_1R_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

For coil 2, KVL gives

$$v_2 = i_2R_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$



The above two equations can be written in the frequency domain as

$$\mathbf{V}_1 = (R_1 + j\omega L_1)\mathbf{I}_1 + j\omega M\mathbf{I}_2$$

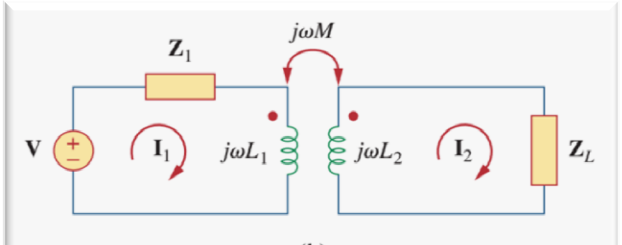
$$\mathbf{V}_2 = j\omega M\mathbf{I}_1 + (R_2 + j\omega L_2)\mathbf{I}_2$$

Analyze Circuits Involving Mutual Inductance

For the frequency domain circuit shown in fig.(b). Applying KVL to coil 1 gives

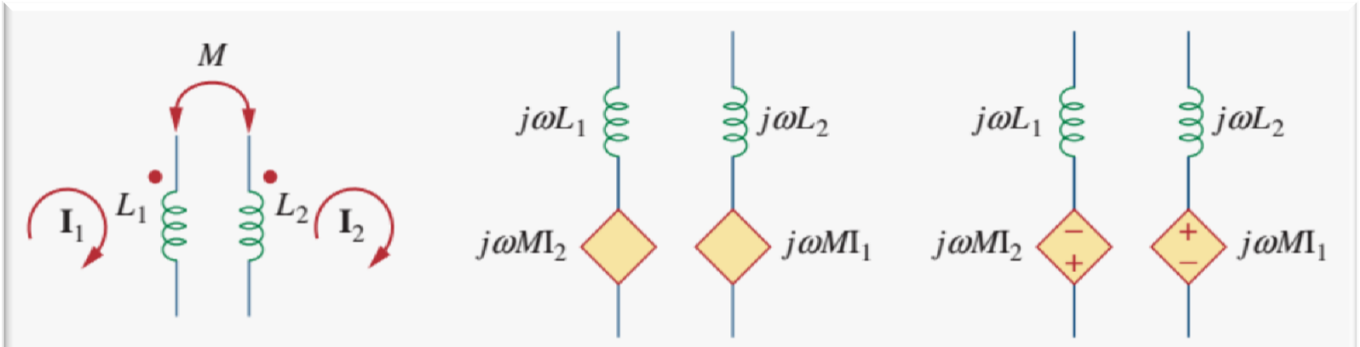
$$\mathbf{V} = (\mathbf{Z}_1 + j\omega L_1)\mathbf{I}_1 - j\omega M\mathbf{I}_2$$

$$0 = -j\omega M\mathbf{I}_1 + (\mathbf{Z}_L + j\omega L_2)\mathbf{I}_2$$



(b)

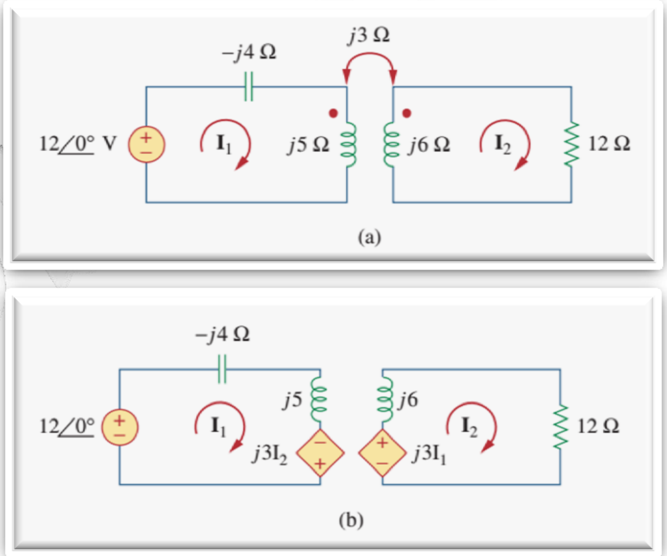
Model that makes analysis of mutually coupled easier to solve



(a) (b) (c)

Analyze Circuits Involving Mutual Inductance

Example: Calculate the phasor currents I_1 and I_2 in the circuit



For loop 1, KVL gives

$$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0$$

$$j\mathbf{I}_1 - j3\mathbf{I}_2 = 12 \qquad (1)$$

For loop 2, KVL gives

$$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0$$

$$\mathbf{I}_1 = \frac{(12 + j6)\mathbf{I}_2}{j3} = (2 - j4)\mathbf{I}_2 \qquad (2)$$

Substitute eq(1) in eq(2) gives

$$(j2 + 4 - j3)\mathbf{I}_2 = (4 - j)\mathbf{I}_2 = 12$$

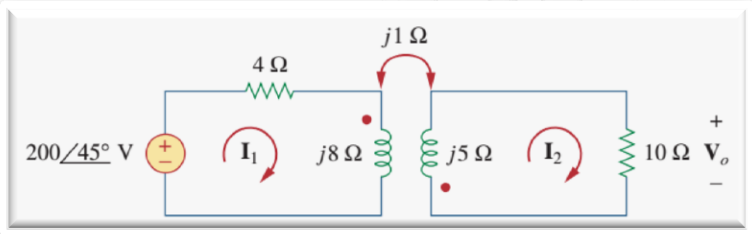
$$\mathbf{I}_2 = \frac{12}{4 - j} = 2.91\underline{/14.04^\circ} \text{ A}$$

Substitute I_2 in eq(2) gives

$$\mathbf{I}_1 = (2 - j4)\mathbf{I}_2 = (4.472\underline{/-63.43^\circ})(2.91\underline{/14.04^\circ})$$
$$= 13.01\underline{/-49.39^\circ} \text{ A}$$

Analyze Circuits Involving Mutual Inductance

Example: Determine the voltage V_o in the circuit



For loop 1, KVL gives

$$-200\angle45^o + 4I_1 + j8I_1 + jI_2 = 0$$

$$(4 + j8)I_1 + jI_2 = 200\angle45^o \qquad (1)$$

For loop 2, KVL gives

$$j5I_2 + 10I_2 + jI_1 = 0$$

$$(10 + j5)I_2 + jI_1 = 0$$

$$(10 + j5)I_2 = -jI_1$$

$$I_1 = \frac{(10+j5)I_2}{-j} = (-5+j10)I_2 \qquad (2)$$

Substitute eq(2) in eq(1) gives

$$(4 + j8)(-5+j10)I_2 + jI_2 = 200\angle45^o$$

$$(-20 + j40 - j40 - 80)I_2 + jI_2 = 200\angle45^o$$

Analyze Circuits Involving Mutual Inductance

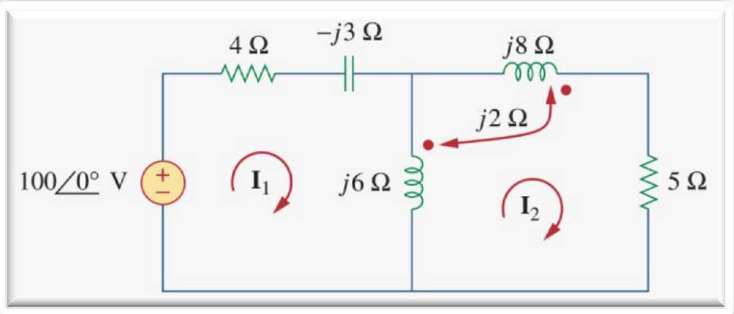
$$(-20 + j40 - j40 - 80)I_2 + jI_2 = 200\angle 45^o$$

$$(-100 + j1)I_2 = 200\angle 45^o$$

$$I_2 = \frac{200\angle 45^o}{-100 + j1} = \frac{200\angle 45^o}{100\angle 179.4} = 2\angle -134.4$$

$$V_o = 10 \times I_2 = 10 \times 2\angle -134.4 = 20\angle -134.4$$

Example: Calculate the mesh currents in the circuit

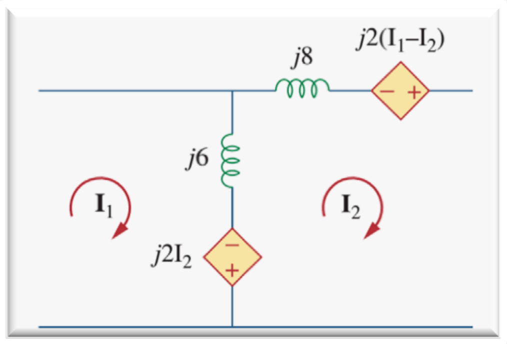


for mesh 1, KVL gives

$$-100 + \mathbf{I}_1(4 - j3 + j6) - j6\mathbf{I}_2 - j2\mathbf{I}_2 = 0$$

$$100 = (4 + j3)\mathbf{I}_1 - j8\mathbf{I}_2$$

for mesh 2 , KVL gives



$$0 = -2j\mathbf{I}_1 - j6\mathbf{I}_1 + (j6 + j8 + j2 \times 2 + 5)\mathbf{I}_2$$

$$0 = -j8\mathbf{I}_1 + (5 + j18)\mathbf{I}_2$$

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 + j3 & -j8 \\ -j8 & 5 + j18 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

Analyze Circuits Involving Mutual Inductance

$$\Delta = \begin{vmatrix} 4 + j3 & -j8 \\ -j8 & 5 + j18 \end{vmatrix} = 30 + j87$$

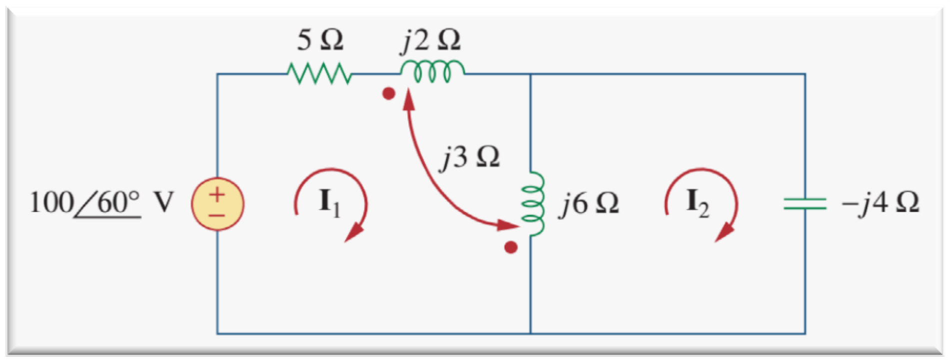
$$\Delta_1 = \begin{vmatrix} 100 & -j8 \\ 0 & 5 + j18 \end{vmatrix} = 100(5 + j18)$$

$$\Delta_2 = \begin{vmatrix} 4 + j3 & 100 \\ -j8 & 0 \end{vmatrix} = j800$$

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{100(5 + j18)}{30 + j87} = \frac{1{,}868.2\angle 74.5^\circ}{92.03\angle 71^\circ} = 20.3\angle 3.5^\circ \text{ A}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j800}{30 + j87} = \frac{800\angle 90^\circ}{92.03\angle 71^\circ} = 8.693\angle 19^\circ \text{ A}$$

H.W.: Determine the phasor currents I_1 and I_2 in the circuit

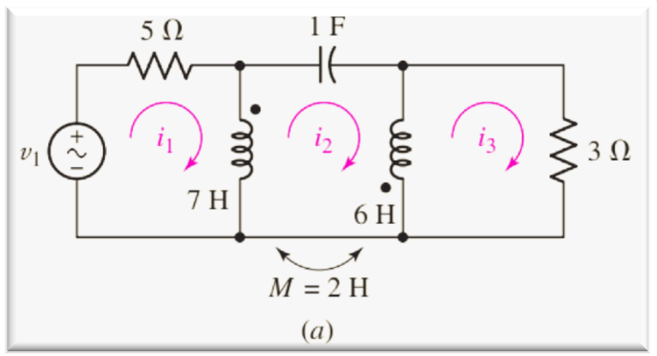


Answer: $I_1 = 17.889\angle 86.57^\circ$ A, $I_2 = 26.83\angle 86.57^\circ$ A.

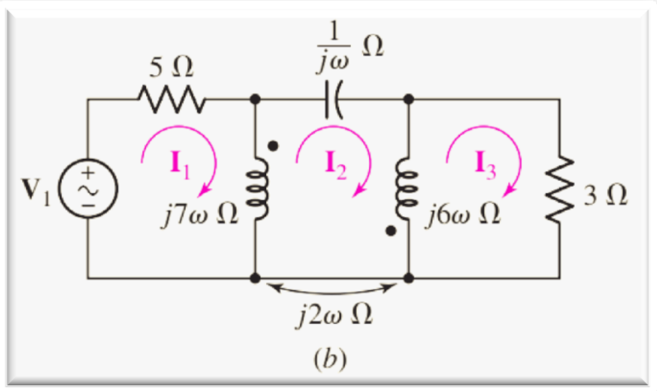
Analyze Circuits Involving Mutual Inductance

Example: Write a complete set of phasor mesh equations for the circuit of fig. (a).

Replace both the mutual inductance and the two self-inductances with their corresponding impedances as in fig. (b).



(a)



(b)

for mesh 1, KVL gives

$$5\mathbf{I}_1 + 7j\omega(\mathbf{I}_1 - \mathbf{I}_2) + 2j\omega(\mathbf{I}_3 - \mathbf{I}_2) = \mathbf{V}_1$$

$$(5 + 7j\omega)\mathbf{I}_1 - 9j\omega\mathbf{I}_2 + 2j\omega\mathbf{I}_3 = \mathbf{V}_1$$

for mesh 2, KVL gives

$$7j\omega(\mathbf{I}_2 - \mathbf{I}_1) + 2j\omega(\mathbf{I}_2 - \mathbf{I}_3) + \frac{1}{j\omega}\mathbf{I}_2 + 6j\omega(\mathbf{I}_2 - \mathbf{I}_3) + 2j\omega(\mathbf{I}_2 - \mathbf{I}_1) = 0$$

$$-9j\omega\mathbf{I}_1 + \left(17j\omega + \frac{1}{j\omega}\right)\mathbf{I}_2 - 8j\omega\mathbf{I}_3 = 0$$

for mesh 3, KVL gives

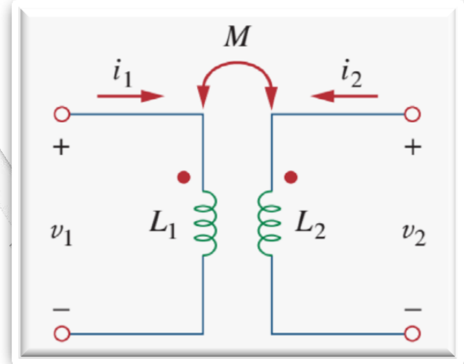
$$6j\omega(\mathbf{I}_3 - \mathbf{I}_2) + 2j\omega(\mathbf{I}_1 - \mathbf{I}_2) + 3\mathbf{I}_3 = 0$$

$$2j\omega\mathbf{I}_1 - 8j\omega\mathbf{I}_2 + (3 + 6j\omega)\mathbf{I}_3 = 0$$

Energy in a Coupled Circuit

The energy stored in an inductor is given by

$$w = \frac{1}{2}Li^2$$



For the circuit, assume that currents and are zero initially, so that the energy stored in the coils is zero. Let i_1 increase from zero to I_1 while maintaining i_2=0, the power in coil 1 is

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt}$$

The energy stored in the circuit is

$$w_1 = \int p_1\,dt = L_1 \int_0^{I_1} i_1\,di_1 = \frac{1}{2}L_1 I_1^2$$

If we now maintain $i_1=I_1$ and increase i_2 from zero to I_2, the mutual voltage induced in coil 1 is $M_{12}di_2/dt$ while the mutual voltage induced in coil 2 is zero, since I_1 does not change. The power in the coils is now

$$p_2(t) = i_1 M_{12}\frac{di_2}{dt} + i_2 v_2 = I_1 M_{12}\frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt}$$

The energy stored in the circuit is

$$w_2 = \int p_2\,dt = M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2\,di_2 = M_{12} I_1 I_2 + \frac{1}{2}L_2 I_2^2$$

Energy in a Coupled Circuit

The total energy stored in the coils when both i_1 and i_2 have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + M_{12}I_1I_2$$

If we reverse the order by which the currents reach their final values, that is, if we first increase i_2 from zero to I_2 and later increase i_1 from zero to I_1, the total energy stored in the coils is

$$w = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + M_{21}I_1I_2$$

Comparing the above two equations leads to

$$M_{12} = M_{21} = M$$

$$w = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2$$

The above equation was derived based on the assumption that the coil currents both entered the dotted terminals. If one current enters one dotted terminal while the other current leaves the other dotted terminal, the sign of the mutual energy term is reversed.

$$w = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 - MI_1I_2$$

Energy in a Coupled Circuit

Since I_1 and I_2 are arbitrary values, they may be replaced by i_1 and i_2 which gives the instantaneous energy stored in the circuit

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

The positive sign is selected for the mutual term if both currents enter or leave the dotted terminals of the coils; the negative sign is selected otherwise

Establishing an Upper Limit for *M*

The energy stored in passive circuit cannot be negative, so

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 \geq 0$$

Add and subtract the term $i_1 i_2 \sqrt{L_1 L_2}$

$$\frac{1}{2}(i_1\sqrt{L_1} - i_2\sqrt{L_2})^2 + i_1 i_2(\sqrt{L_1 L_2} - M) \geq 0$$

The squared term is never negative; at its least it is zero. Therefore, the second term must be greater than zero;

$$\sqrt{L_1 L_2} - M \geq 0 \qquad \textbf{Or} \qquad M \leq \sqrt{L_1 L_2}$$

Energy in a Coupled Circuit

The Coupling Coefficient

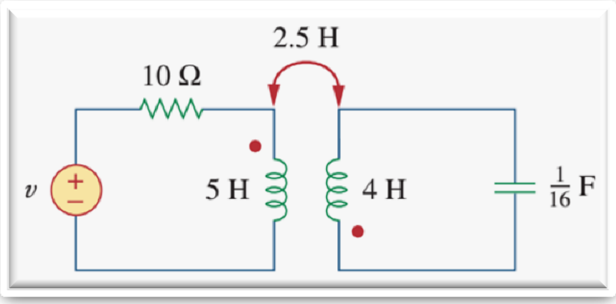
Coupling coefficient (k) is used to describe the degree of coupling between coils.

$$\boxed{M = k\sqrt{L_1 L_2}}$$

where $0 \leq k \leq 1$ or equivalently $0 \leq M \leq \sqrt{L_1 L_2}$

- **If the entire flux produced by one coil links another coil, then k=1 and we have 100 percent coupling, or the coils are said to be *perfectly coupled*.**
- **For k<0.5, the coils are said to be *loosely coupled*.**
- **For k>0.5, the coils are said to be *tightly coupled*.**

Example: Consider the circuit. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time t=1 s if v=60 cos(4t + 30°) V.

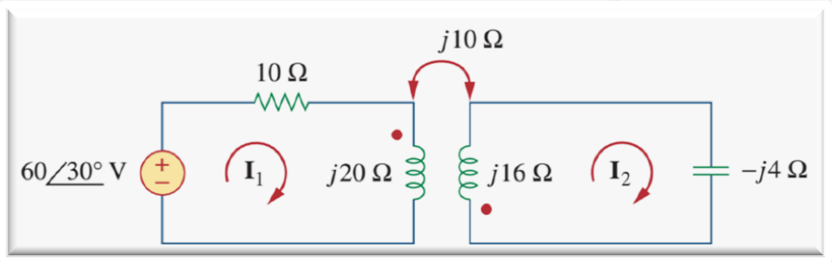


$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

$$\begin{aligned}
60\cos(4t + 30^\circ) \quad &\Rightarrow \quad 60\angle 30^\circ, \quad \omega = 4 \text{ rad/s} \\
5\text{ H} \quad &\Rightarrow \quad j\omega L_1 = j20\ \Omega \\
2.5\text{ H} \quad &\Rightarrow \quad j\omega M = j10\ \Omega \\
4\text{ H} \quad &\Rightarrow \quad j\omega L_2 = j16\ \Omega
\end{aligned}$$

Energy in a Coupled Circuit

$$\frac{1}{16}\text{ F} \quad \Rightarrow \quad \frac{1}{j\omega C} = -j4\ \Omega$$



For mesh 1

$$(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\angle 30^\circ \qquad (1)$$

For mesh 2

$$j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0$$

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 \qquad (2)$$

Substitute (2) in (1)

$$\mathbf{I}_2(-12 - j14) = 60\angle 30^\circ \quad \Rightarrow \quad \mathbf{I}_2 = 3.254\angle 160.6^\circ\text{ A}$$

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 = 3.905\angle -19.4^\circ\text{ A}$$

In time domain

$$i_1 = 3.905\cos(4t - 19.4^\circ), \qquad i_2 = 3.254\cos(4t + 160.6^\circ)$$

At time $t = 1$ s, $4t = 4$ rad $= 229.2^\circ$, and

$$i_1 = 3.905\cos(229.2^\circ - 19.4^\circ) = -3.389\text{ A}$$

$$i_2 = 3.254\cos(229.2^\circ + 160.6^\circ) = 2.824\text{ A}$$

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2 = \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73\text{ J}$$


Q
&
A