Boolean Algebra

Basic Definition.

- Boolean algebra like any other deductive mathematical system may be defined with a set of elements, a set of operators, and a number of unproved axioms or postulates.

- Boolean algebra is an algebraic structure defined on a set of elements B together with two binary operators + and · provided the following postulates are satisfied:

Postulate 1: Set and Operators

•***(Definition)***: A Boolean algebra is a **closed algebraic system** containing a set *K* of two or more elements and the two operators · and + which refer to logical "AND" and logical "OR".

1(a) Closure with respect to the operator +.

(b) Closure with respect to the operator ·.

Postulate 2: Identity Elements

- There exist 0 and 1 elements in K, such that for every element a in K
  - a + 0 = a
  - a · 1 = a

Note:
a · 0 = 0
a + 1 = 1

- Definitions:
  - 0 is the identity element for + operation
  - 1 is the identity element for · operation

Postulate 3: Commutativity

- Binary operators + and · are commutative.
- That is, for any elements a and b in K:
  - a + b = b + a
  - a · b = b · a

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Postulate 4: Distributivity

- Binary operator (+) is distributive over (·) and (·) is distributive over (+).
- That is, for any elements a, b and c in K:
  - a + (b · c) = (a + b) · (a + c)
  - a · (b + c) = (a · b) + (a · c)

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Postulate 5: Complement

- A unary operation, *complementation*, exists for every element of K.
- That is, for any elements a in K:

$$x + x' = 1$$
$$x \cdot x' = 0$$

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Basic Theorems

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Duality Principle

- Each postulate of Boolean algebra contains a pair of expressions or equations such that one is transformed into the other and vice-versa by interchanging the operators, + ↔ ·, and identity elements, 0 ↔ 1.
- The two expressions are called the duals of each other.
- Example: duals

A + (BC) = (A+B)(A+C) ↔ A (B+C) = AB + AC

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Properties of Boolean Algebra

- Properties stated as theorems.
- The postulates are basic axioms of the algebraic structure and need no proof.
- The theorems must be proven from the postulates.

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Postulates and Theorems of Boolean Algebra

Post. 2	(a) $x + 0 = x$	(b) $x \cdot 1 = x$
Post. 5	(a) $x + x' = 1$	(b) $x \cdot x' = 0$
Theorem 1	(a) $x + x = x$	(b) $x \cdot x = x$
Theorem 2	(a) $x + 1 = 1$	(b) $x \cdot 0 = 0$
Theorem 3 involution:	$(x')' = x$	
Post. 3 commutative:	(a) $x + y = y + x$	(b) $xy = yx$
Theorem 4 associative:	(a) $x + (y + z) = (x + y) + z$	(b) $x(yz) = (xy)z$
Post. 4 distributive:	(a) $x(y + z) = xy + xz$	(b) $x + yz = (x + y)(x + z)$
Theorem 5 DeMorgan:	(a) $(x + y)' = x'y'$	(b) $(xy)' = x' + y'$
Theorem 6 absorption:	(a) $x + xy = x$	(b) $x(x + y) = x$

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Theorem 1: Idempotency

- For all elements a in K: x + x = x;
  x.x = x.
- Proof:

x + x	=	(x + x)1,	(identity element)
	=	(x + x)(x + x\`),	(complement)
	=	x + x x\`,	(distributivity)
	=	x + 0,	(complement)
	=	x,	(complement)

Similar proof for x.x = x

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Theorem 2: Null Elements Exist

- $x + 1 = 1$ ….. for (+) operator.
- $x \cdot 0 = 0$ ….. for (·) operator.
- Proof: $x + 1 =$ $(x + 1)1$, (identity element)

$$
\begin{aligned}
x + 1 &= (x + 1)1, && \text{(identity element)} \\
&= 1(x + 1), && \text{(commutativity)} \\
&= (x + x`)(x + 1), && \text{(complement)} \\
&= x + x`.\,1, && \text{(distributivity)} \\
&= x + x`, && \text{(identity element)} \\
&= 1, && \text{(complement)}
\end{aligned}
$$

Similar proof for $x.\,0 = 0$.

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Theorem 3: Involution Holds

- $(x`)` = x$
- Proof: $x + x` = 1$ and $x.x` = 0$, (complements)

  or $x` + x = 1$ and $x`.x = 0$, (commutativity)

  i.e., x is complement of x`

  Therefore, $x`` = x$

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Theorem 6: Absorption

- a + a b = a
- a (a + b) = a
- Proof: a + a b = a 1 + a b, (identity element)
  - = a(1 + b), (distributivity)
  - = a 1, (Theorem 2)
  - = a, (identity element)

Similar proof for a (a + b) = a.

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Theorem 5: De Morgan

(a) $(x + y)' = x'y'$

(b) $(xy)' = x' + y'$

- The theorems of Boolean algebra can be shown to hold true by means of truth tables.

For example, the truth table for the first De Morgan's theorem $\boxed{(x + y)' = x'y'}$ is shown below.

x	y	$x + y$	$(x + y)'$	x'	y'	$x'y'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

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Theorem 4: Associativity

- Binary operators + and · are associative.
- That is, for any elements a, b and c in K:
  - $a + (b + c) = (a + b) + c$
  - $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

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Operator precedence

- The operator precedence for evaluating Boolean expressions is:
  (1) paren theses,
  (2) NOT,
  (3) AND,
  (4) OR.

- **Example:**
  (x+y)`
  X`.y`

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Boolean Functions

Boolean functions

- Binary variable takes values 0 or 1.
- Boolean function is an expression formed with: binary variables, binary operators (AND, OR), and unary operator (NOT), parentheses, equal sign.
- Function can be 1 or 0.

EX:

F1= xyz`….. When does F1=1?

Boolean function representation

Boolean functions can be represented by :

1. Algebraic expression:
2. Truth table:
   - We need 2^n combinations of 1's and 0's.
   - A column showing the combinations where function = 1 or 0.

     EX: if we have F=xyz, *then* :

     we have **2^3=8** combinations
3. Logical diagram (using AND,OR,NOT gates).

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- Consider the following functions:

  F1=xyz\`　　　F2=x+y\`z

  F3=x\`y\`z+x\`yz+xy\`　　　F4=xy\`+x\`z

The Truth table is:

x	y	z	F_1	F_2	F_3	F_4
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

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- We can see that F3 = F4
- The question: is it possible to find 2 algebraic expressions that specify the same function?

  Answer is yes.

From Table above we find that F_4 is the same as F_3, since both have identical 1's and 0's for each combination of values of the three binary variables. In general, two functions of n binary variables are said to be equal if they have the same value for all possible 2^n combinations of the n variables.

- Manipulation of Boolean expression is applied to **find simpler expressions for the same function**

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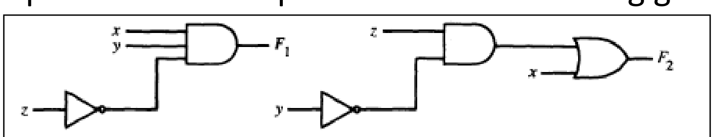
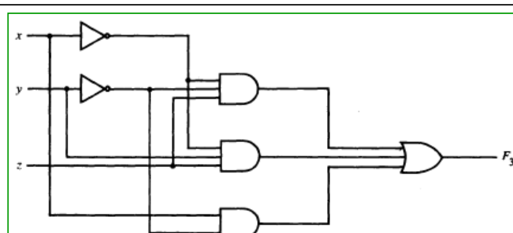
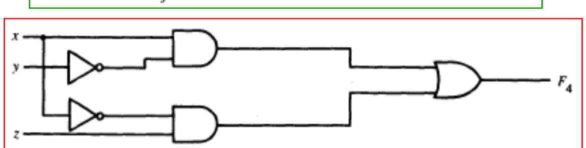
- Implementation of previous functions using gates:



(a) $F_1 = xyz'$ (b) $F_2 = x + y'z$



(c) $F_3 = x'y'z + x'yz + xy'$



(d) $F_4 = xy' + x'z$

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Simpler is better

• From the diagrams it is obvious that the implementation of F_4 requires less gates and less inputs than F_3. Since F_4 and F_3 are equal Boolean functions, it is more economical to implement the F_4 form than the F_3 form. To find simpler circuits, one must know how to manipulate Boolean functions to obtain equal and simpler expressions. What constitutes the best form of a Boolean function depends on the particular application. In this section, consideration is given to the criterion of equipment minimization.

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Algebraic Manipulation

- A *literal* is a primed or unprimed variable.
- each literal in the function designates an input to a gate,
- each term is implemented with a gate.
- The minimization of the number of literals and the number of terms will result in a circuit with less equipment.

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Function Minimization using Boolean Algebra

- ***Examples***:
  - $a + ab =$ $a(1+b)=a$
  - $a(a + b) =$ $a.a +ab=a+ab=a(1+b)=a$.
  - $a + a'b =$ $(a + a')(a + b)=1(a + b) =a+b$
  - $a(a' + b) =$ $a. a' +ab=0+ab=ab$

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Try

- F = abc + abc’ + a’c
  =(a.b(c+c’)) + a’.c
  = ab.(1)+ a’c
  =ab+ a’c

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The other type of question

Show that;

1- $ab + ab' = a$

2- $(a + b)(a + b') = a$

1- $ab + ab' = a(b+b') = a.1=a$

2- $(a + b)(a + b') = a.a + a.b' + a.b + b.b'$

$= a + a.b' + a.b + 0$

$= a + a.(b' + b) + 0$

$= a + a.1 + 0$

$= a + a = a$

More Examples

- Show that;
  (a) $ab + ab'c = ab + ac$
  (b) $(a + b)(a + b' + c) = a + bc$

(a) $ab + ab'c = a(b + b'c)$

$= a((b+b').(b+c))=a(b+c)=ab+ac$

(b) $(a + b)(a + b' + c)$

$= (a.a + a.b' + a.c + ab + b.b' + bc)$

$= \dots$

Complement of a Function

- The complement of a function F is F'
- may be derived algebraically through De Morgan's theorem.
- De Morgan's can be extended to three or more variables.

- $(A + B + C)' = (A + X)'$ Let $B + C = X$
  - $= A'X'$ by theorem 5(a) (De Morgan)
  - $= A' \cdot (B + C)'$ substitute $B + C = X$
  - $= A' \cdot (B'C')$ by theorem 5(a) (De Morgan)
  - $= A'B'C'$ by theorem 4(b) (associative)

- De Morgan's theorems for any number of variables are similar

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Find the complement of the functions:

$$F_1 = x'yz' + x'y'z$$
$$F_2 = x(y'z' + yz).$$

$$\begin{aligned} F_1' &= (x'yz' + x'y'z)' \\ &= (x'yz')' \cdot (x'y'z)' \\ &= (x + y' + z)(x + y + z') \end{aligned}$$

$$\begin{aligned} F_2' &= [x(y'z' + yz)]' \\ &= x' + (y'z' + yz)' \\ &= x' + (y'z')' \cdot (yz)' \\ &= x' + (y + z)(y' + z') \end{aligned}$$

- A simpler procedure for deriving the complement of a function
  1) take the dual of the function.
  2) complement each literal.

- Find the complement of the function $F_1 = x'yz' + x'y'z$ by taking their dual and complementing each literal.

1) The dual is: $(x' + y + z')(x' + y' + z)$
2) Complement each literal: $(x + y' + z)(x + y + z') = F_1'$

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