Different Modifications of BMO Functions

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Abstract :

The purpose of this paper is to suggest and justify different modifications of BMO functions. After a brief introduction of the statement of the problem different types of modifications of BMO norms for the classical representative of

the class of BMO functions have been evaluated for the function $\ln \frac{1}{x}$, which is an extremely function.

I- Introduction

The space of functions bounded mean oscillation (**BMO**) plays an important role in Harmonic Analysis [1].

It is the best replacement for $\mathbf{L}^{\infty}\left(\mathbf{R}^{n}\right)$

in many aspects such as embedding, interpolation and dual spaces.

The basic theory in the space of functions of bounded mean oscillation (BMO) was introduced by F.John and L.Nirenberg[2] and is also called the John-Nirenberg space.

The definition of BMO is that $\mathbf{f} \in \mathbf{BMO}$ if

$$\sup_{I\subset[0,1]} \frac{1}{|I|} \int_I |f(x)-f_I| \, dx = \|f\|_* < \infty , \qquad (1)$$

$$\text{where } f_I = \frac{1}{|I|} \int_I f(y) \, dy$$

And $\mathbf{I}$ is an interval of form $[0,1]$. It is important to know that

$L^\infty \subset BMO$, Let $f \in L^2[0,1]$, then

$$\frac{1}{|I|}\int_I (|f(x)-f_I|dx)^2 \le \frac{1}{|I|}\int_I |(f(x)-f_I)^2|dx =$$

$$\frac{1}{|I|}\int_I f^2(x)dx - 2(f^2{}_I) + f_I{}^2 = \frac{1}{|I|}\int f^2(x)dx - (f_I)^2 \le \frac{1}{|I|}\int_I f^2(x)dx \le \|f\|_\infty{}^2$$

the first inequality by Hölders inequality, the second inequality by Poincare inequality. It follows that $\|f\|_* \le \|f\|_\infty$.

We will consider different modifications of integral oscillation:

$$\Omega(f;I)=\frac{1}{|I|}\int_I |f(x)-f_I|\,dx;$$

- $$\bar{\Omega}(f;I)=\inf_C \frac{1}{|I|}\int_I |f(x)-C|\,dx;$$
-
- $$\bar{\bar{\Omega}}(f;I)=\frac{1}{|I|^2}\int_I\int_I |f(x)-f(y)|\,dx\,dy.$$

where the value of $\Omega(f;I)$, $\bar{\Omega}(f;I)$ and $\bar{\bar{\Omega}}(f;I)$ characterized the integral oscillations of the function f on the interval I .

We derive the following inequality for the relationships between the above different modifications of integral oscillations:

$$\overline{\overline{\Omega}}(f;I) \leq \Omega(f;I) \leq \overline{\Omega}(f;I) \qquad (2)$$

and even using certain techniques we can show that

$$\Omega(f;I) \leq 2\,\overline{\Omega}(f;I)$$
$$\overline{\overline{\Omega}}(f,I) \leq 2\,\Omega(f;I) \qquad (3)$$

Example:

If $f(x) = \mathrm{Sign}(x)$; $x \in [-1,1] \equiv I$

Then

$$\overline{\overline{\Omega}}(f\ ;I) = \frac{1}{|I|^2} \int_I \int_I |f\ (x) - f\ (y)|\ dx\ dy$$

$$= \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} |f\ (x) - f\ (y)|\ dx\ dy$$

$$= 1$$

where $(f\ )_I = \frac{1}{|I|} \int_I f\ (y)\, dy = 0$ **and** $\Omega(f\ ;I) = \frac{1}{|I|} \int_I |f\ (x)|\, dx = 1$

It follows that $\overline{\Omega} = \Omega = \overline{\overline{\Omega}}$

we also find another two examples at which we can see that the constant ''2'' in equation (3) cannot be minimized.

In many applied problems and even in some theoretical, we need to know the exact value of the BMO norms. We form the following definitions for the different modifications of BMO functions[3] for norms of the monotonic functions

$$\| f \|_{\bullet} = \operatorname*{Sup}_{I} \overline{\Omega}(f;I)$$

$$\| f \| = \operatorname*{Sup}_{I} \overline{\overline{\Omega}}(f;I) \qquad (4)$$

According to the equations (2) and (3) the following holds

$$\| f \|_{\bullet} \leq \|f\|_{*} \leq \|f\|$$

$$\| f \|_{*} \leq 2 \|f\|_{\bullet}$$

$$\| f \| \leq 2 \|f\|_{*}$$

BMO Norms Calculations :

We can say that in order to evaluate the BMO norms for the monotonic functions, it's a sufficient conditions to form the oscillation of the function of intervals of the forms that have one of the endpoints is either 0 or 1 (certain lemma have been proved)

Particularly, we evaluate the BMO norms for

$$f(x)=\ln\frac{1}{x} \quad ,(\quad \|f\|_{\bullet},\ \|f\|_{*}\ \text{and}\ \|f\| \quad)$$

using different methods of high level mathematics we obtain that

$$\|f\|_{*}=\frac{2}{e},\quad (......................)$$

$$\|f\|_{\bullet}= \quad ....$$

$$\|f\| \;= \quad ....$$

Conclusions :

- The results might be effectives in solving extremely problems that are related with the behavior of the functions of the BMO class
- We think following Berman(5) it will be more interesting if we approximate the function on each interval of the form [0 , 1] by polynomial of degree $\leq \mathbf{d}$ (fixed number).

References

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