Theory of machinery

Chapter two

Position analysis

By

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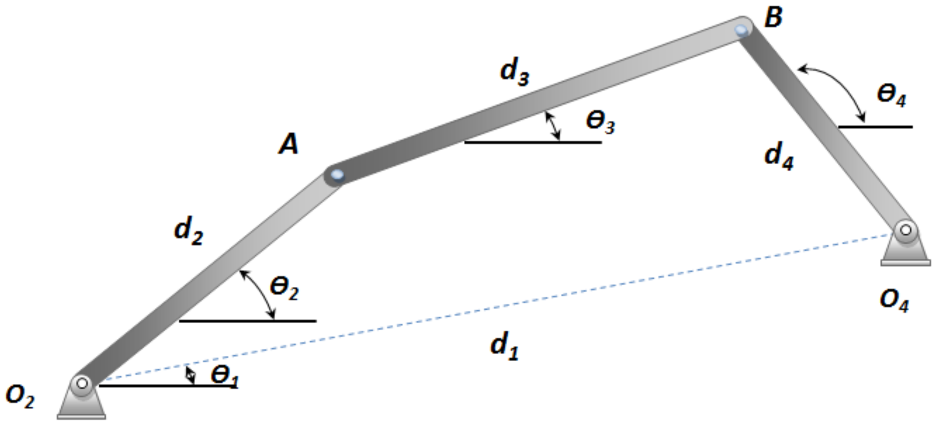
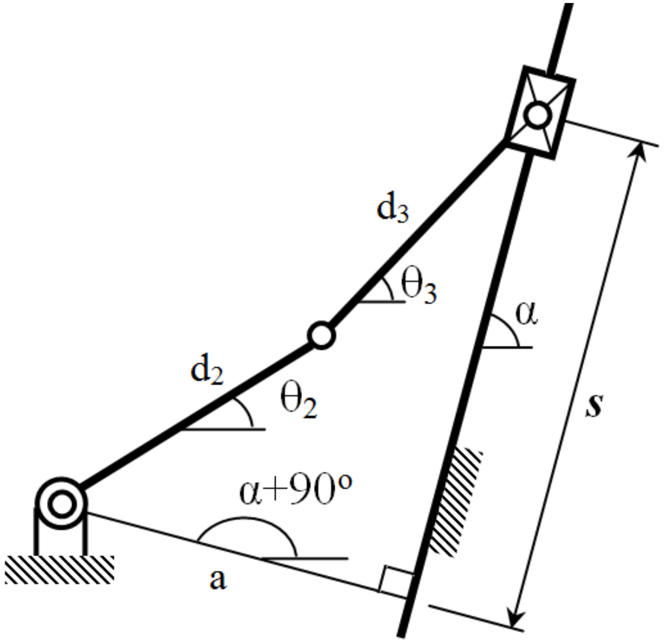
Position analysis

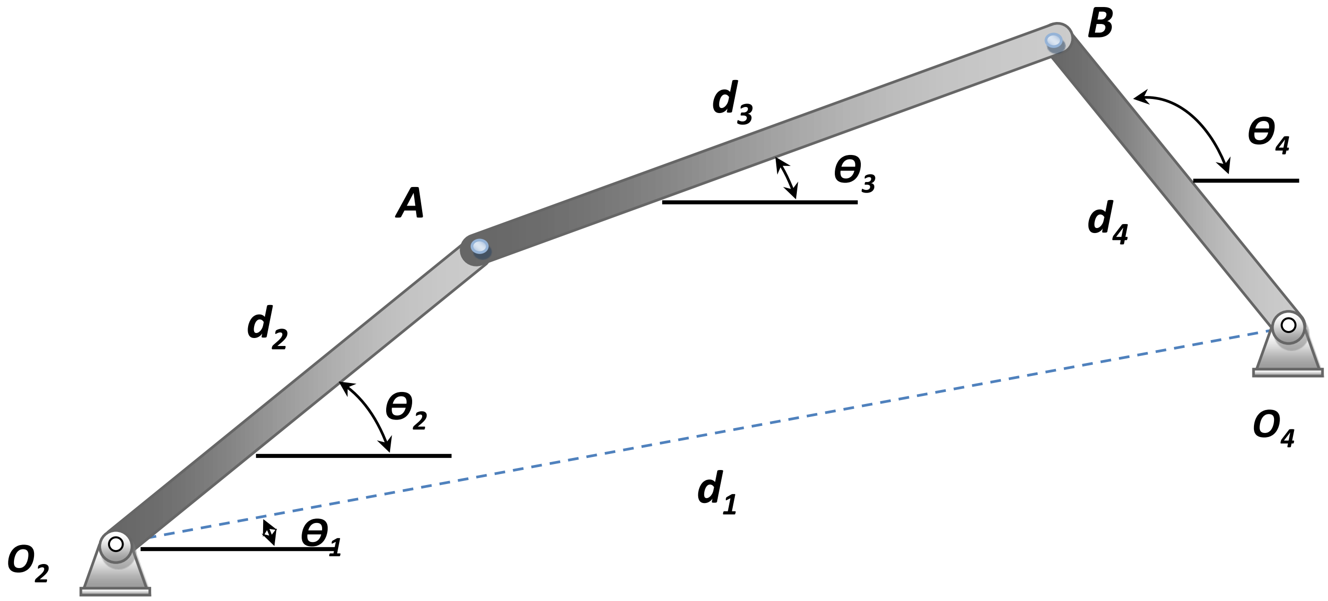
To perform position analysis, you must have the links dimensions and the location of fixed points and find the position relationships between all moving links.





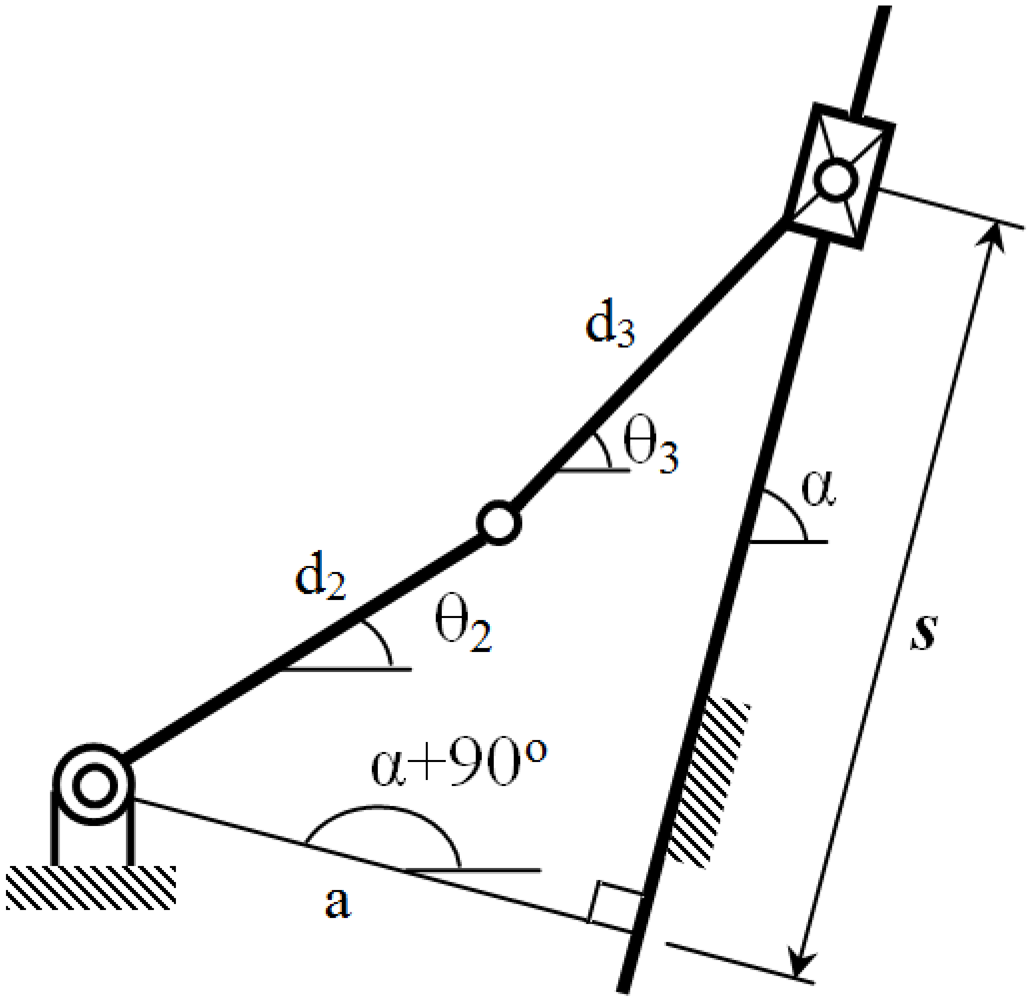
Position analysis

The mechanism shown in figure is assumed to have d_1, d_2, d_3, d_4 and Θ_1 (the angle between the grounded links) as given data (position of fixed points and dimensions of links) and Θ_2 as input and the required is to find both Θ_3 and Θ_4.



Position analysis

The slider mechanism shown in figure can be processed in position analysis if we know the dimensions d_2, d_3, a and Θ_1 with Θ_2 as input to find both Θ_3 and S.

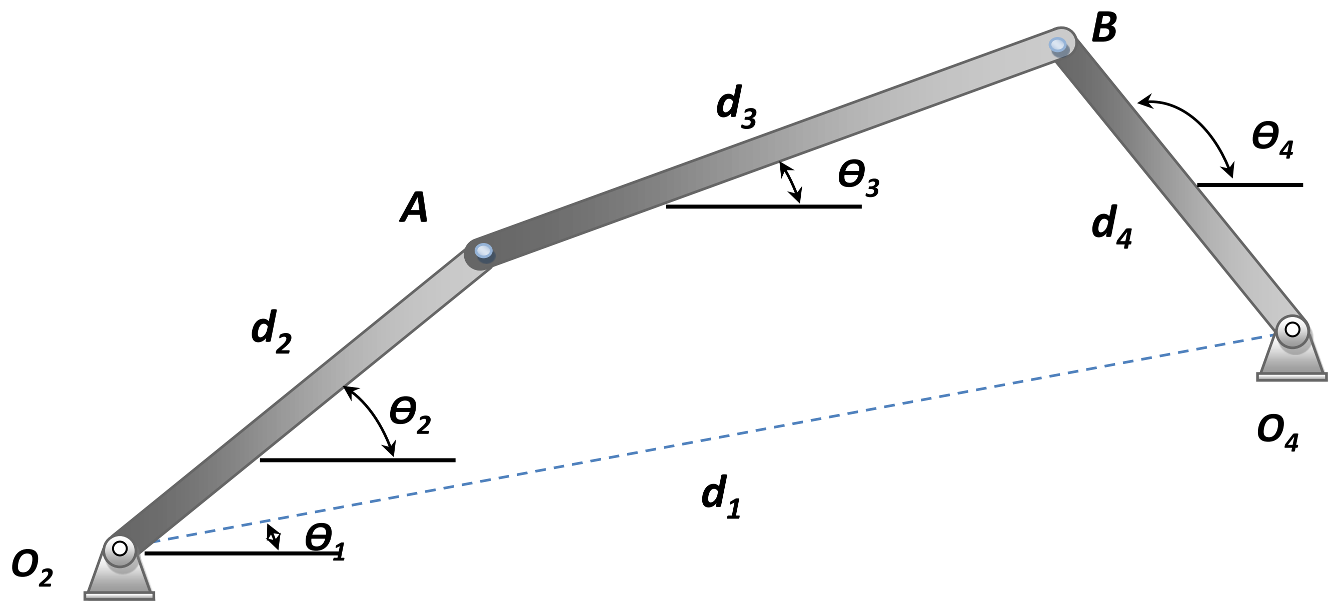


Theory of machinery

Position analysis

4-bar mechanism

Assume the there is the following 4-bar mechanism where d_1, d_2, d_3, d_4 and Θ_1 are given data and Θ_2 is input and the required is to find both Θ_3 and Θ_4

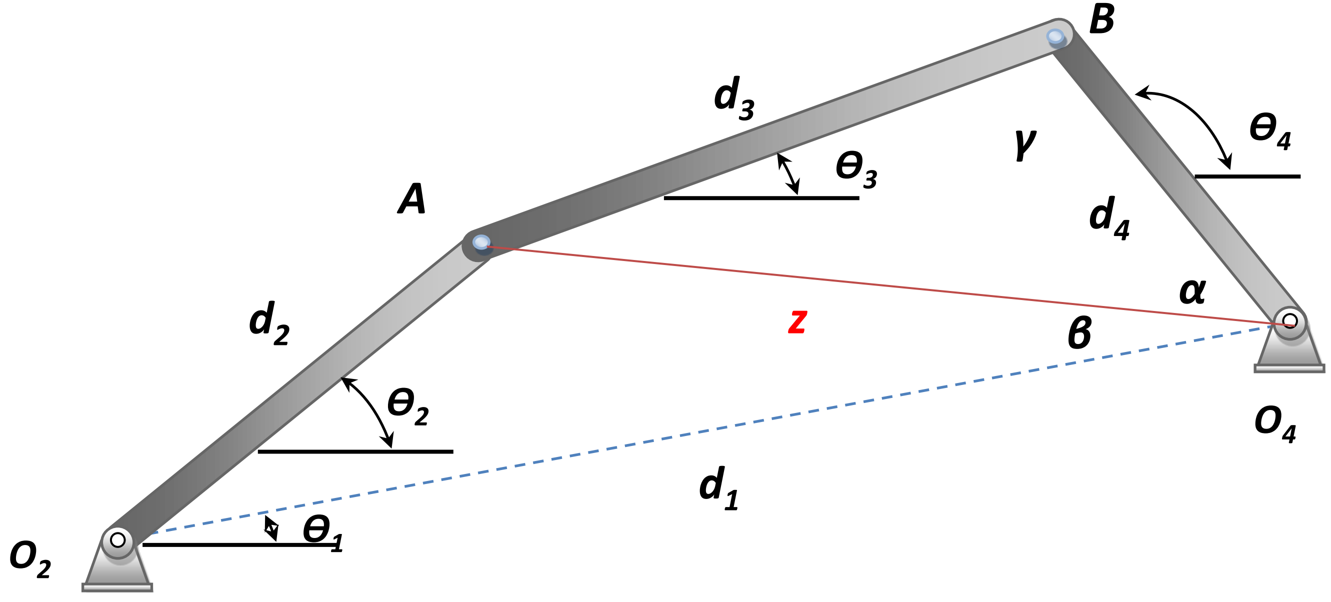


Position analysis

4-bar mechanism

Solution:

- Draw a dividing line from points ***A*** and O_4. this line has a length equal ***z***.
- A new two angels developed: **α** and **β**.
- Name the angle between links 2 and 3 as **γ**



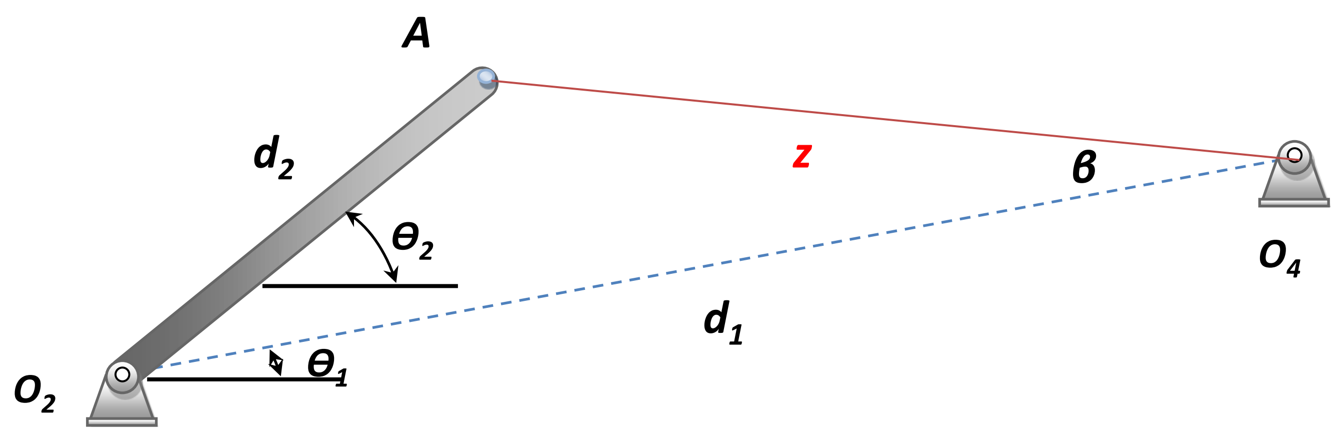
Position analysis

4-bar mechanism

Solution:

- Take the triangle $o_4 - A - o_2$
- **Apply the cosine low:** $z^2 = d_1^2 + d_2^2 - 2d_1 d_2 \cos(\theta_2 - \theta_1) --- (1)$
- **Apply cosine low again to find β:** $d_2^2 = z^2 + d_1^2 - 2zd_1 \cos(\beta) --- (2)$
- **Rearrange Eq.2**

$$\beta = \cos^{-1}\left(\frac{d_2^2 - z^2 - d_1^2}{-2zd_1}\right)$$



T h e o r y o f m a c h i n e r y

Position analysis

4-bar mechanism

Solution:

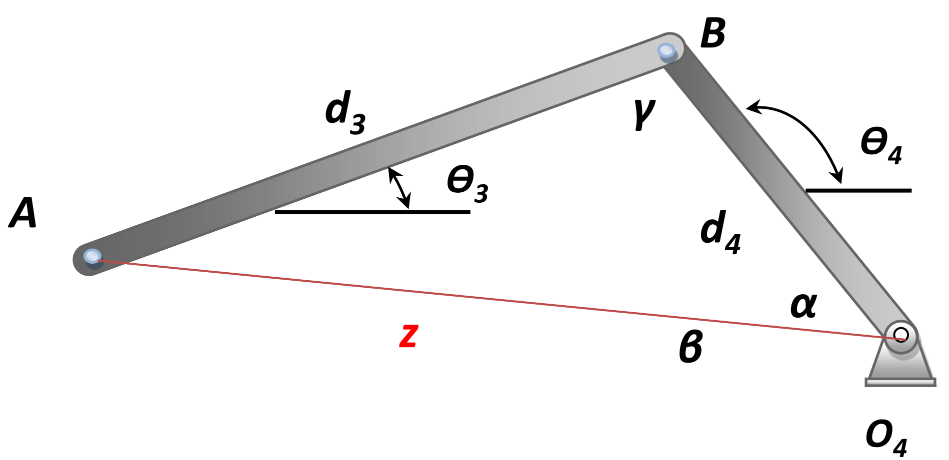
- Take the triangle $o_4 - A - B$
- **Apply the cosine low:**

$$z^2 = d_3^2 + d_4^2 - 2d_3d_4\cos(\gamma) --- (3)$$

- **Rearrange Eq.3**

$$\gamma = \cos^{-1}\left[\frac{z^2 - d_3^2 - d_4^2}{-2d_3d_4}\right]$$

- **Remember: z is obtained from Eq.1.**



Position analysis

4-bar mechanism

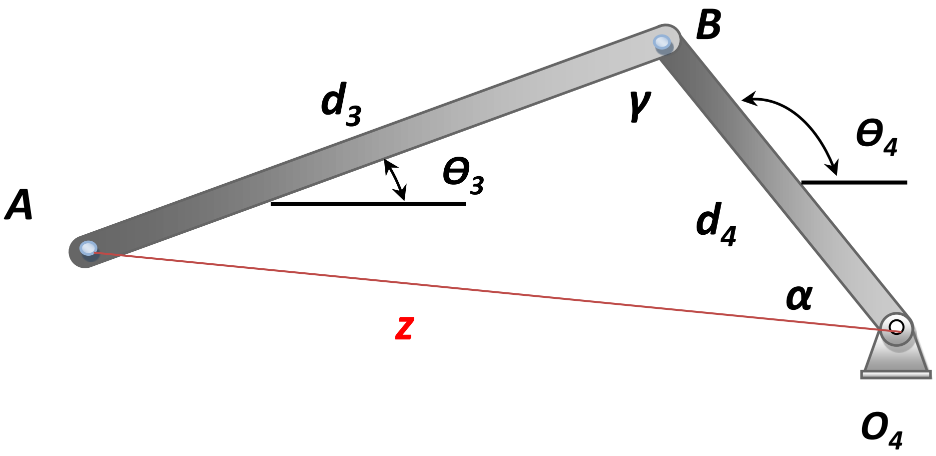
Solution:

- Take the same triangle $o_4 - A - B$
- **Apply the cosine low to find the angle α:**

$$d_3^2 = z^2 + d_4^2 - 2zd_4\cos(\alpha) - - - (4)$$

- **Rearrange Eq.4 :**

$$\alpha = \cos^{-1}\left[\frac{d_3^2 - z^2 - d_4^2}{-2zd_4}\right]$$

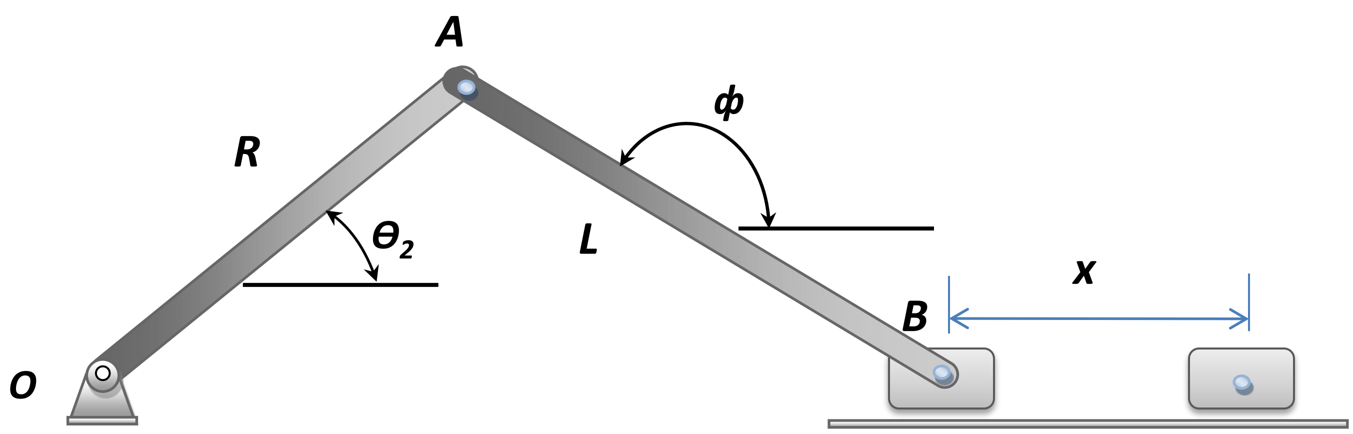


- **Finally: $\Theta_3 = 180 - \alpha - \gamma$**
- **$\Theta_4 = 180 - \alpha - \beta - \Theta_1$**

Position analysis

Slider crank mechanism

Assume the there is the following slider crank mechanism where L, R and Θ_1 are given data and Θ_2 is input and the required is to find both ϕ and x

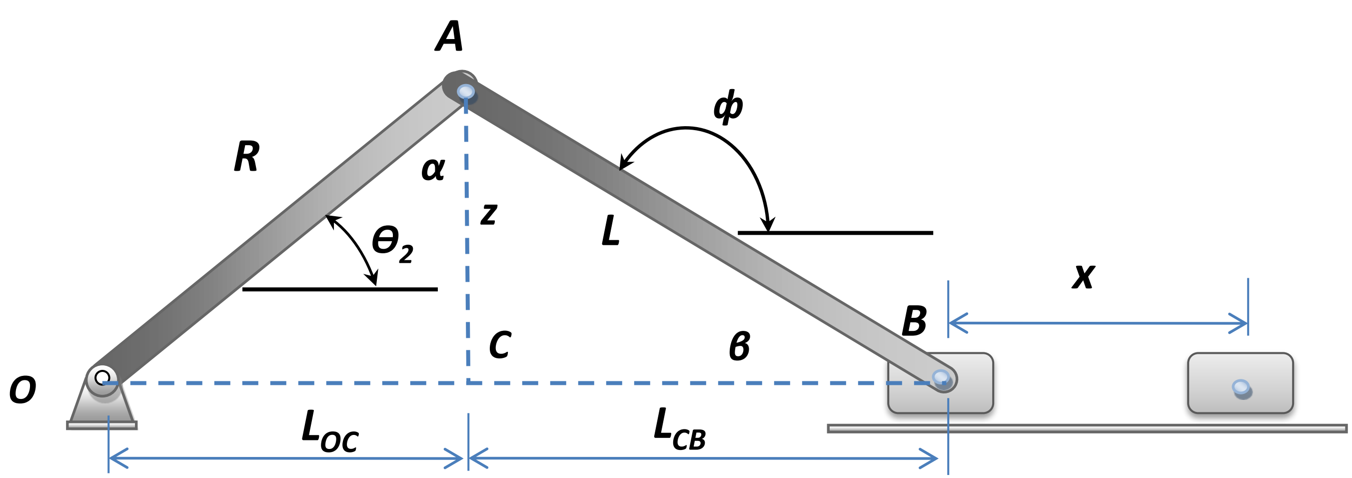


Position analysis

Slider crank mechanism

Solution

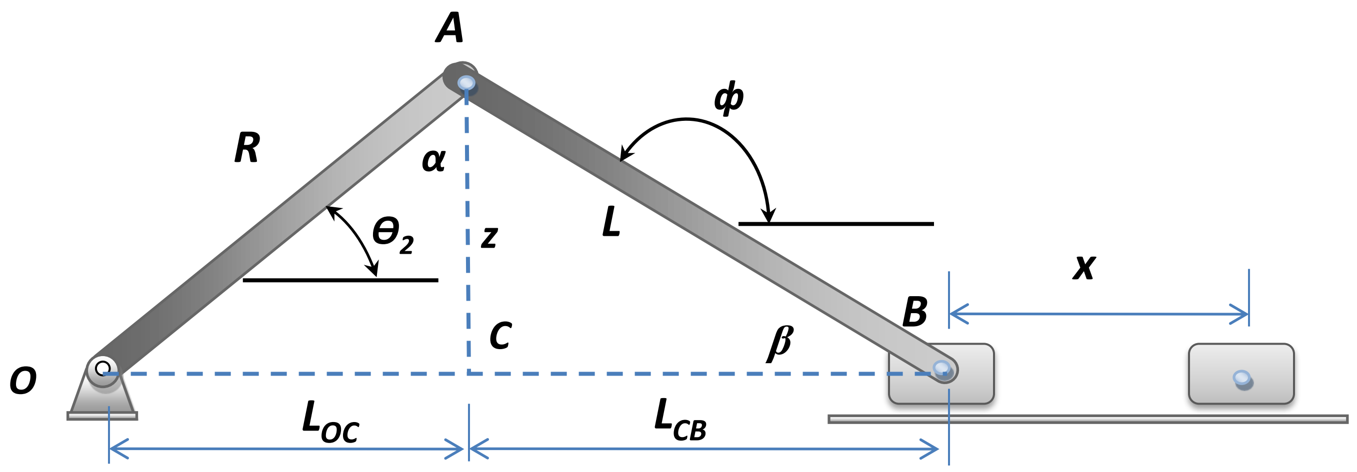
- Drop a perpendicular line from point A to line OB as shown. The length of this line is z.
- As shown: $\boldsymbol{L_{OC} + L_{BC} + x = R+L}$. *this is the 1st equitation.*
- Take the triangle OAC.
- Using sine law to find z: $\boldsymbol{z = R \sin(\theta_2)}$
- You can find that $\boldsymbol{\alpha = 90 - \theta_2}$ **.** After knowing all the angels of triangle OAC, we can find the distance L_{OC} using cosine law



Position analysis

Slider crank mechanism

- Or L_{OC} can be found easily by : $L_{OC} = R\cos(\theta_2)$**.** As you can see, you can find the distance L_{OC} in many ways
- Let us go to the triangle ABC: the angle β can be found as : $\beta = \sin^{-1}(z/R)$ directly. In direct way: $\phi = 180 - \beta$.
- $L_{CB} = L\cos(\beta)$**.**
- Back to the first equation: $x = R+L - (L_{OC} + L_{CB})$



Position analysis

Using vector algebra

The most common and the easiest way to perform position analysis is by using vector algebra. In this method, the links are assumed as vectors. So, it is convenient at this stage to review some of the vector algebra principles. Because we are dealing with planner mechanisms, the vectors that represent the links are in two dimensional forms:

$$\vec{V} = L\cos(\theta)i + L\sin(\theta)j = LU_\theta$$

$\vec{V}$ is a vector

- L is the vector length
- Ө is the polar position of the vector(or the angle between the vector and the x-axis)
- i and j are unit vectors in x and y dimensions respectively.
- U_θ is a unit vector in the direction of Ө: $U_\theta = \cos(\theta)i + \sin(\theta)j$

Theory of machinery

Position analysis

Using vector algebra

If we assume that V_1 and V_2 are vectors represented as:

$$V_1 = L_1\cos(\theta_1)i + L_1\sin(\theta_1)j = L_1U_{\theta 1} \text{ and } V_2 = L_2\cos(\theta_2)i + L_2\sin(\theta_2)j = L_2U_{\theta 2}$$

Then

$$V_1 \pm V_2 = [L_1\cos(\theta_1) \pm L_2\cos(\theta_2)]i + [L_1\sin(\theta_1) \pm L_2\sin(\theta_2)]j$$

$$V_1 \bullet V_2 = L_1L_2\cos(\theta_2 - \theta_1)$$

$$R = V_1 x V_2 \Rightarrow |R| = L_1L_2\sin(\theta_2 - \theta_1)$$

T h e o r y o f m a c h i n e r y

Position analysis

Performing position analysis using vectors

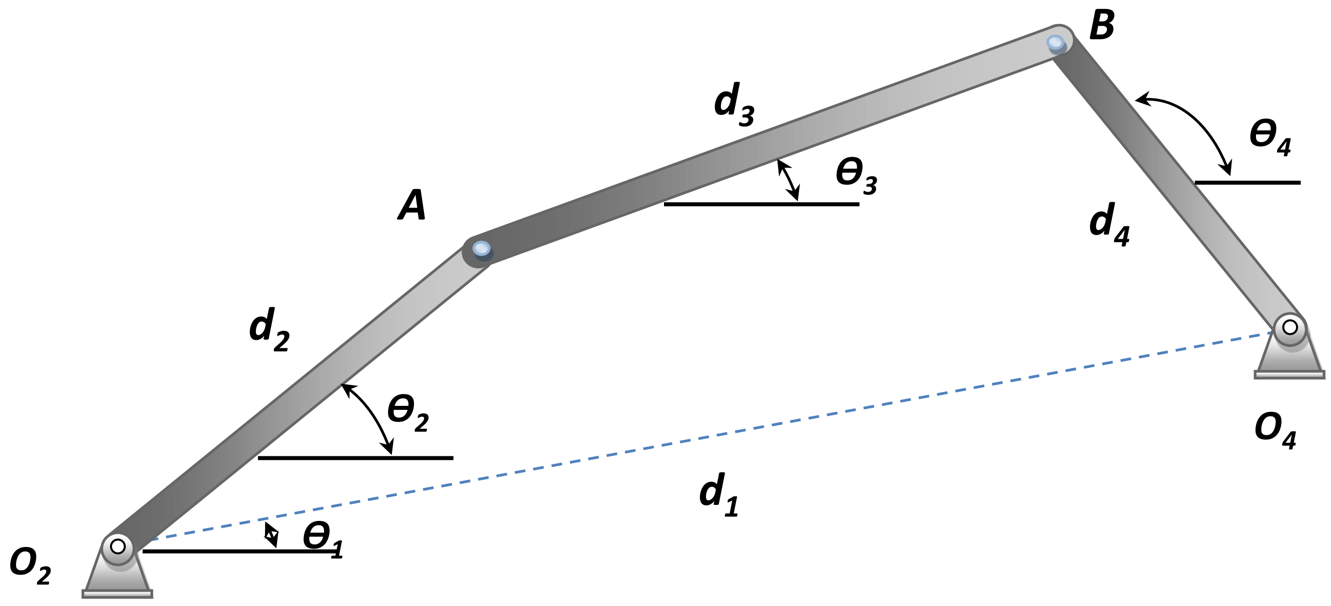
To perform position analysis using vector algebra, follow the following steps

- Connect between kinematic pairs using vector (i.e. by lengths and angels)
- Using vector algebra to find vector equations that satisfy the mechanism connectivity.
- Solve these equations in term of vector parameters relating known quantities to unknown quantities

Position analysis

4-bar mechanism: vector algebra approach

As in the previous example, d_1, d_2, d_3, d_4 and Θ_1 are given data and Θ_2 is input and the required is to find both Θ_3 and Θ_4



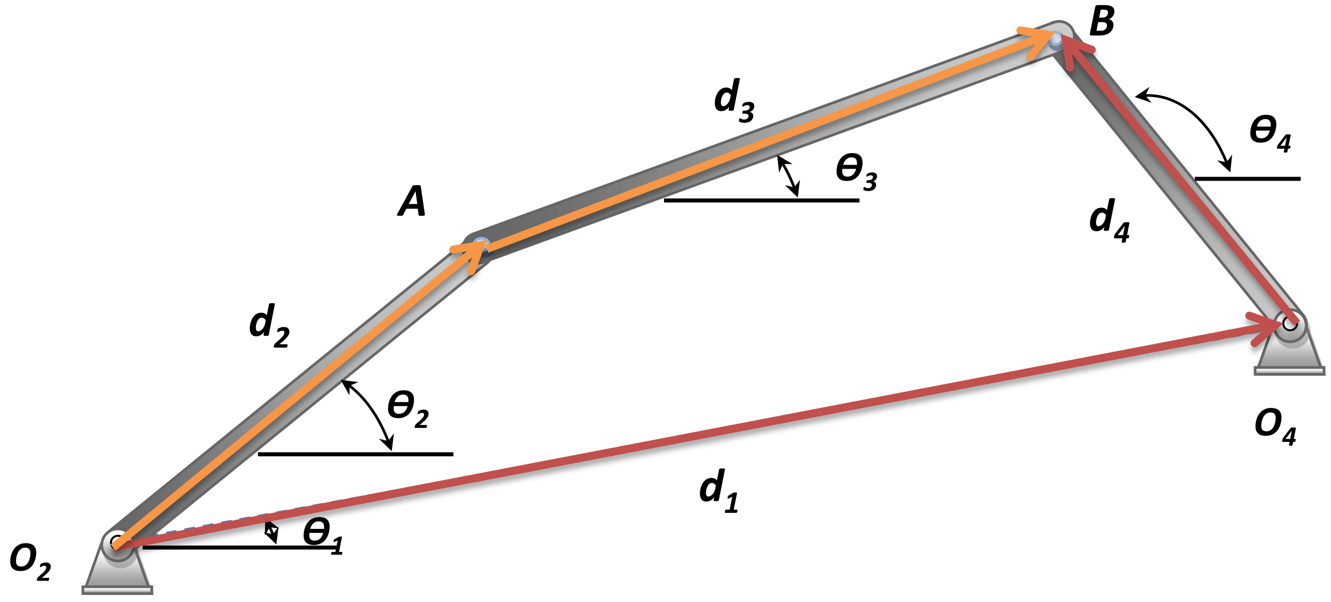
Position analysis

4-bar mechanism: vector algebra approach

Vector equation

The vector equation can be derived as shown in Eq.1

$$\overrightarrow{d_2} + \overrightarrow{d_3} = \overrightarrow{d_1} + \overrightarrow{d_4} --- (1)$$



T h e o r y o f m a c h i n e r y

Position analysis

4-bar mechanism: vector algebra approach

Vector equation

All the vectors in Eq.1 can be represent as:-

$$\begin{aligned} \overrightarrow{d_1} &= d_1 U_{\theta_1} \\ \overrightarrow{d_2} &= d_2 U_{\theta_2} \\ \overrightarrow{d_3} &= d_3 U_{\theta_3} \\ \overrightarrow{d_4} &= d_4 U_{\theta_4} \end{aligned} \quad ---(2)$$

Where: d_1, d_2, d_3 and d_4 are the lengths of the links 1, 2, 3 and 4 respectively and $U_{\theta 1}$, $U_{\theta 2}$, $U_{\theta 3}$, and $U_{\theta 4}$ are unit vectors in the direction for the links 1, 2, 3 and 4 respectively.

Position analysis

4-bar mechanism: vector algebra approach

Vector equation

Substitute Eq.1 in Eq.2:-

$$d_2U_{\theta 2} + d_3U_{\theta 3} = d_1U_{\theta 1} + d_4U_{\theta 4} \; --- (3)$$

Rearrange

$$d_3U_{\theta 3} = d_1U_{\theta 1} + d_4U_{\theta 4} - d_2U_{\theta 2} \; --- (4)$$

loop closure equation

Position analysis

4-bar mechanism: vector algebra approach

Vector equation

- Dot product each side by itself to eliminate U_θ (e.g. $d_3 U_{\theta 3} \bullet d_3 U_{\theta 3} = d_3^2$).
- The dot product for the right side is calculated as:-

$$\left(d_1 U_{\theta 1} + d_4 U_{\theta 4} - d_2 U_{\theta 2}\right)\left(d_1 U_{\theta 1} + d_4 U_{\theta 4} - d_2 U_{\theta 2}\right) \quad ---(5)$$

$$= d_1^2 + 2 d_1 d_4 \cos(\theta_1 - \theta_4) - 2 d_1 d_2 \cos(\theta_1 - \theta_2) + d_4^2 - 2 d_4 d_2 \cos(\theta_4 - \theta_2) + d_2^2$$

Remember: $V_1 \bullet V_2 = L_1 L_2 \cos(\theta_2 - \theta_1)$

•Eq.5 equals the dot product of the left side of Eq. 4 which is simply d_3^2

Now, we have a single scalar equation with one unknown: Θ_4

Theory of machinery

Position analysis

4-bar mechanism: vector algebra approach

Vector equation

$$d_1^2 + 2d_1 d_4 \cos(\theta_1 - \theta_4) - 2d_1 d_2 \cos(\theta_1 - \theta_2) + d_4^2 - 2d_4 d_2 \cos(\theta_4 - \theta_2) + d_2^2 = d_3^2 - - - (6)$$

Note that: cos(a+b) = cos(a) cos(b) – sin(a) sin(b); cos(a-b) = cos(a) cos(b)+sin(a) sin(b)

Apply the previous identity to Eq.6:

$$d_1^2 + d_2^2 + d_4^2 - d_3^2 - 2d_1 d_2 \cos(\theta_1 - \theta_2) + 2d_1 d_4 [\cos(\theta_1)\cos(\theta_4) + \sin(\theta_1)\sin(\theta_4)]$$
$$- 2d_4 d_2 [\cos(\theta_4)\cos(\theta_2) + \sin(\theta_4)\sin(\theta_2)] = 0$$

Eq.7

Theory of machinery

Position analysis

4-bar mechanism: vector algebra approach

Vector equation

- To simplify Eq.7, make the following assumptions

$$a = 2d_1 d_4 \cos(\theta_1) - 2d_2 d_4 \cos(\theta_2)$$

$$b = 2d_1 d_4 \sin(\theta_1) - 2d_2 d_4 \sin(\theta_2)$$

$$c = d_1^2 + d_2^2 + d_4^2 - d_3^2 - 2d_1 d_2 \cos(\theta_1 - \theta_2)$$

$$a\cos(\theta_4) + b\sin(\theta_4) + c = 0 --- (8)$$

•To find a solution for Eq.8, use the following identity

$$\text{If } \Phi = \tan\left(\frac{a}{2}\right) \text{ then } \sin(a) = \frac{2\Phi}{1+\Phi^2} \text{ and } \cos(a) = \frac{1-\Phi^2}{1+\Phi^2} \quad --- (9)$$

T h e o r y o f m a c h i n e r y

Position analysis

4-bar mechanism: vector algebra approach

Vector equation

Assume $\Phi = \tan\left(\frac{\theta_4}{2}\right)$ and substitute in Eq.9

$$a\frac{1-\Phi^2}{1+\Phi^2} + b\frac{2\Phi}{1+\Phi^2} + c = 0 - - - (10)$$

Rearrange Eq.10

$$a\left(1-\Phi^2\right) + 2\Phi b + c\left(1+\Phi^2\right) = \left(c-a\right)\Phi^2 + 2b\Phi + \left(a+c\right) = 0 - - - (11)$$

Eq.11 is quadratic equation in Φ and the solution is found as:

$$\Phi_{1,2} = \frac{-b \pm \sqrt{b^2 - c^2 + a^2}}{c-a} \Rightarrow \theta_{4-1,2} = 2\tan^{-1}\left(\Phi_{1,2}\right)$$

Position analysis

4-bar mechanism: vector algebra approach

Vector equation

go back to Eq.4 to find Θ_3. separate the sine ($\sin(\Theta)$) and the cosine ($\cos(\Theta)$) terms and by equalizing the sine terms with each other or the cosine terms to each other, the following equations will be produced:

$$d_3 \sin(\theta_3) = d_1 \sin(\theta_1) + d_4 \sin(\theta_4) - d_2 \sin(\theta_2) \qquad \text{(a)}$$

$$d_3 \cos(\theta_3) = d_1 \cos(\theta_1) + d_4 \cos(\theta_4) - d_2 \cos(\theta_2) \qquad \text{(b)}$$

---(12)

Divide (a) on (b)

$$\theta_{3-1,2} = \tan^{-1}\left[\frac{d_1 \sin(\theta_1) + d_4 \sin(\theta_{4-1,2}) - d_2 \sin(\theta_2)}{d_1 \cos(\theta_1) + d_4 \cos(\theta_{4-1,2}) - d_2 \cos(\theta_2)}\right]$$

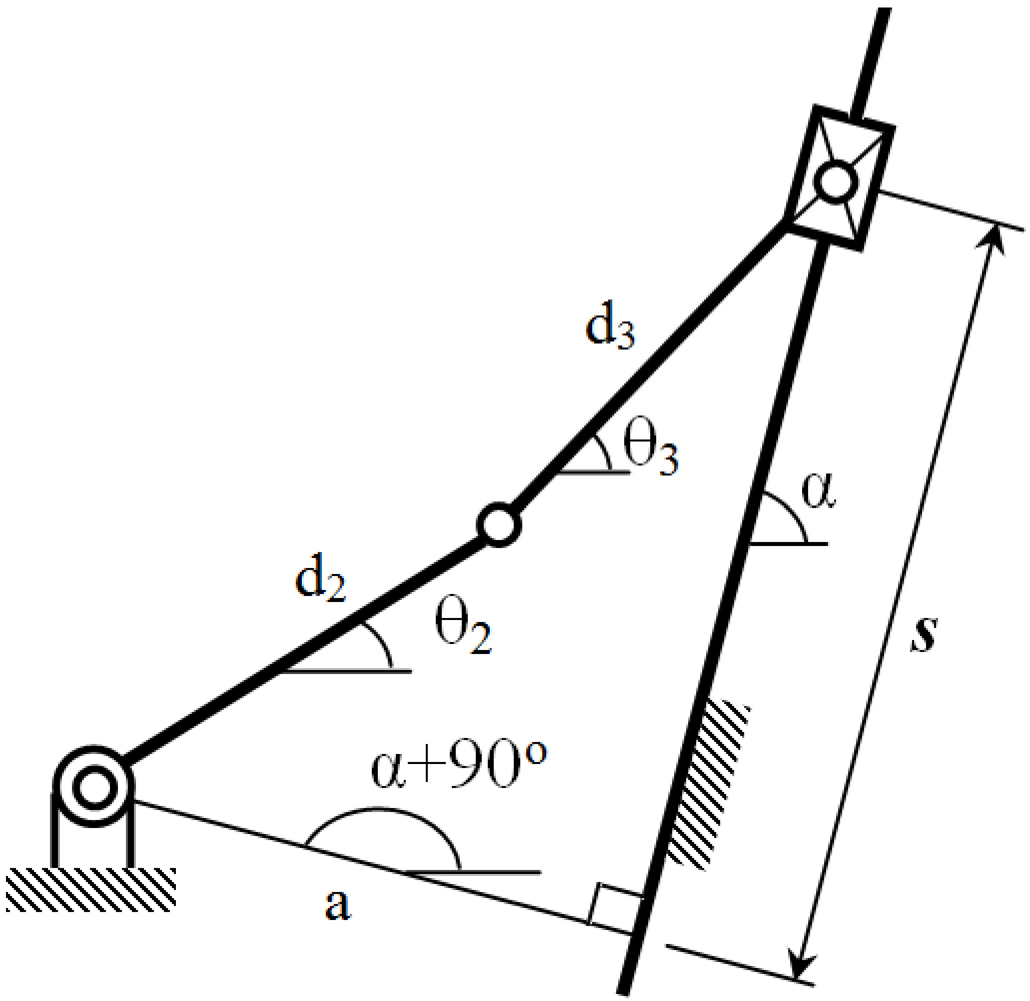
Position analysis

Slider crank mechansim

Problem statement

Find θ_3 and S for the crank - slider mechanism shown in figure.

Assume d_2, d_3, α and a are given data and θ_2 is input



Position analysis

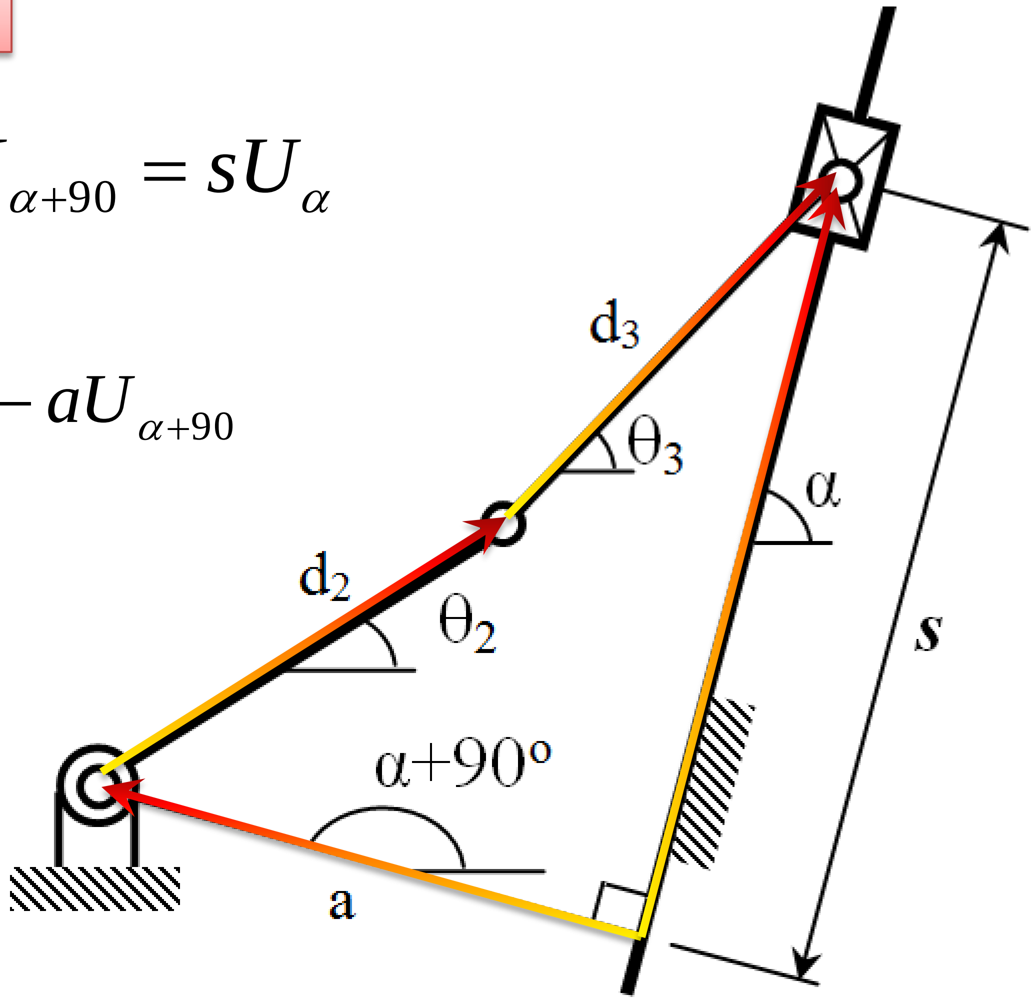
Slider crank mechansim

Loop closure equation

$$d_2 U_{\theta 2} + d_3 U_{\theta 3} + a U_{\alpha+90} = s U_{\alpha}$$

Rearrange

$$d_3 U_{\theta 3} = s U_{\alpha} - d_2 U_{\theta 2} - a U_{\alpha+90}$$



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Position analysis

Slider crank mechansim

Loop closure equation

Dot product each side by itself to eliminate $U_{\theta 3}$

$$d_3^2 = s^2 - 2sd_2\cos(\alpha - \theta_2) + a^2 + 2ad_2\sin(\alpha - \theta_2) + d_2^2$$

$$\Rightarrow s^2 - 2sd_2\cos(\alpha - \theta_2) + a^2 + 2ad_2\sin(\alpha - \theta_2) + d_2^2 - d_3^2 = 0$$

Remember: $U_{\alpha+90} . U_\alpha = 0$ and $\cos(\alpha+90-\theta_2)=\sin(\alpha-\theta_2)$

To simplify the previous equation make the following assumptions

$$b = -2d_2\cos(\alpha - \theta_2) \qquad c = a^2 + 2ad_2\sin(\alpha - \theta_2) + d_2^2 - d_3^2$$

and substitute them in loop equation: $s^2 + bs + c = 0$

T h e o r y o f m a c h i n e r y

Position analysis

Slider crank mechansim

Loop closure equation

This equation $s^2 + bs + c = 0$ is quadratic in S and it has solution equal

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

go back to loop equation to find Θ_3 as in the previous example:-

$$\theta_{3-1,2} = \tan^{-1}\left[\frac{s_{1,2}\sin(\alpha) + a\cos(\alpha) - d_2\sin(\theta_2)}{s_{1,2}\cos(\alpha) + a\cos(\alpha) - d_2\cos(\theta_2)}\right]$$

Position analysis

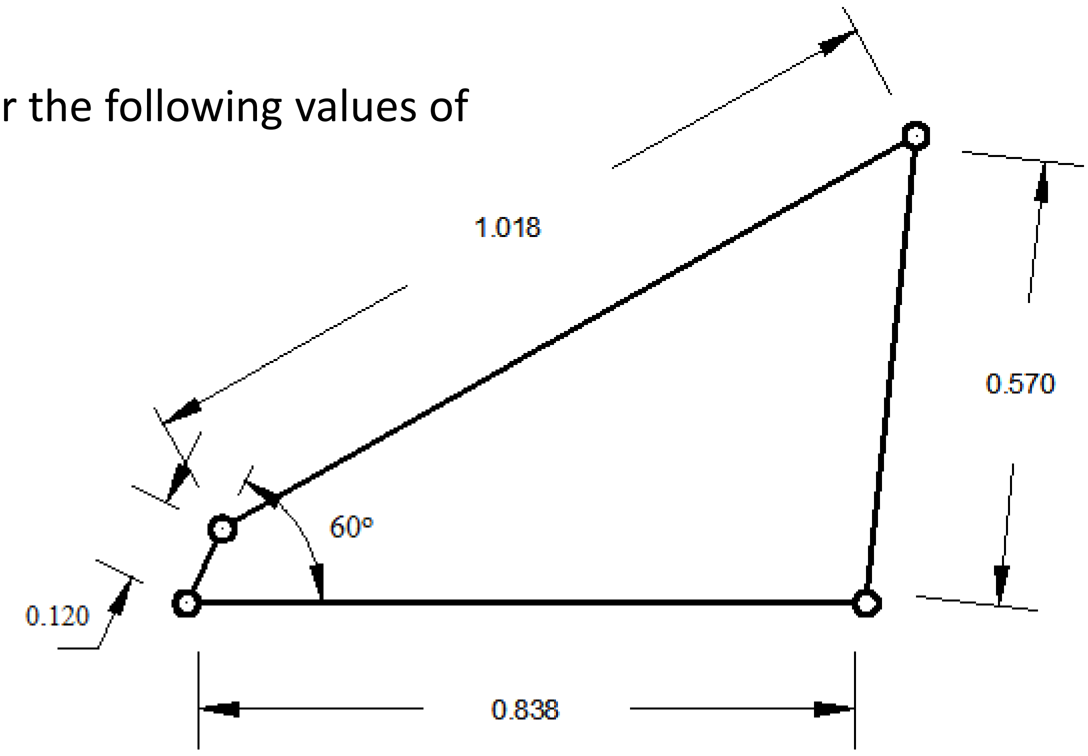
Example: 4-bar mechanism

Find Θ_3 and Θ_4 for the given 4-bar mechanism

Solution

Using the previous analysis for the following values of

- $d_1 = 0.868$
- $d_2 = 0.12$
- $d_3 = 1.018$
- $d_4 = 0.570$
- $\theta_1 = 0.0^o$
- $\theta_2 = 60.0^o$



Position analysis

Theory of machinery

Example: 4-bar mechanism

Solution

$$a = 2d_1 d_4 \cos(\theta_1) - 2d_2 d_4 \cos(\theta_2) = 0.9696$$

$$b = 2d_1 d_4 \sin(\theta_1) - 2d_2 d_4 \sin(\theta_2) = -0.1247$$

$$c = d_1^2 + d_2^2 + d_4^2 - d_3^2 - 2d_1 d_2 \cos(\theta_1 - \theta_2) = -0.01266$$

$$\Phi_{1,2} = \frac{-b \pm \sqrt{b^2 - c^2 + a^2}}{c - a} \Rightarrow \theta_{4-1,2} = 2\tan^{-1}(\Phi_{1,2}) = -96.58\,, 81.93$$

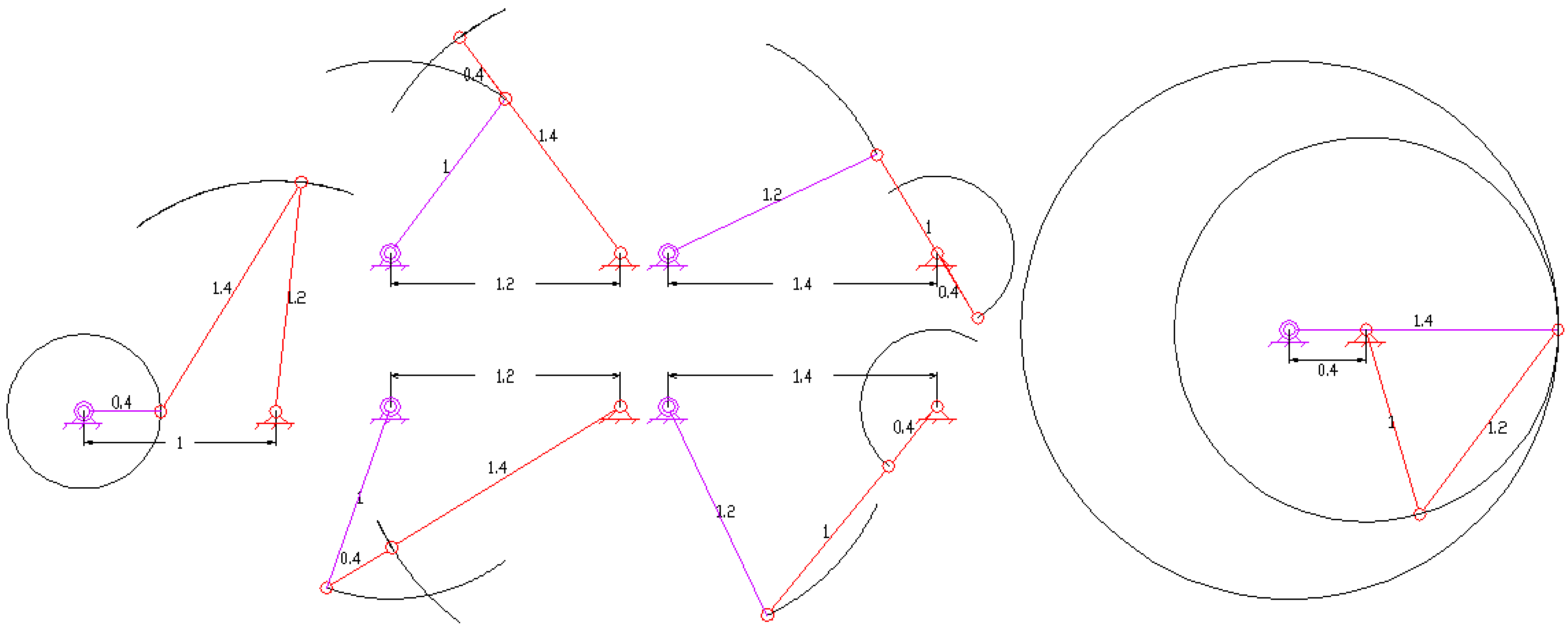
To find θ_3:

$$\theta_{3-1,2} = \tan^{-1}\left[\frac{d_1 \sin(\theta_1) + d_4 \sin(\theta_{4-1,2}) - d_2 \sin(\theta_2)}{d_1 \cos(\theta_1) + d_4 \cos(\theta_{4-1,2}) - d_2 \cos(\theta_2)}\right] = -43.43, 28.78$$

Position analysis

4-bar mechanism Grashof condition

Statement: the sum of the shortest and longest link of a planar 4-bar linkage is less than or equal to the sum of the remaining two links, then the shortest link can rotate fully with respect to a neighboring link.



Position analysis

4-bar mechanism Grashof condition

Consider the previous 4-bar mechanism and the values T_1, T_2 and T_3:

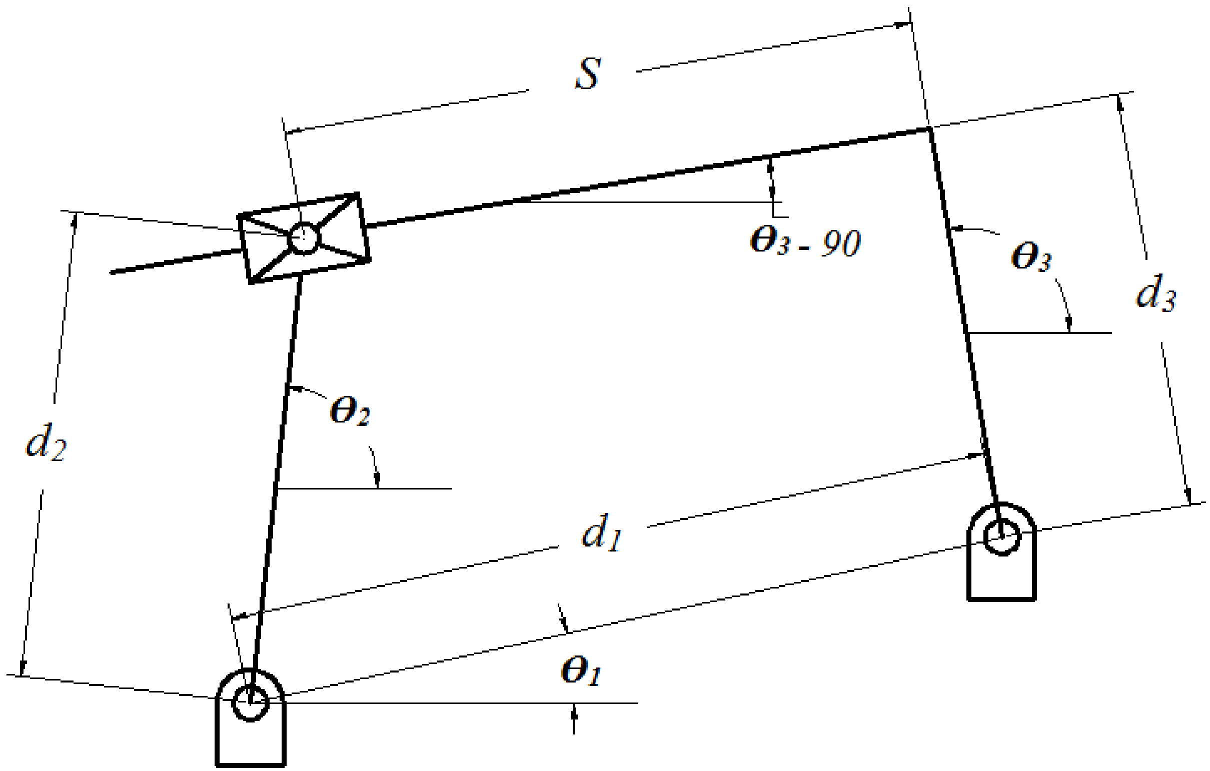
- ❑ $T_1 = d_1 + d_3 - d_2 - d_4$
- ❑ $T_2 = d_4 + d_1 - d_2 - d_3$
- ❑ $T_3 = d_4 + d_3 - d_2 - d_1$

T_1	T_2	T_3	Grashof condition	Input link	Output link
−	−	+	Grashof	Crank	Crank
+	+	+	Grashof	Crank	Rocker
+	−	−	Grashof	Rocker	Crank
−	+	−	Grashof	Rocker	Rocker
−	−	−	Non-Grashof	0-Rocker	0-Rocker
−	+	+	Non-Grashof	π-Rocker	π-Rocker
+	−	+	Non-Grashof	π-Rocker	0-Rocker
+	+	−	Non-Grashof	0-Rocker	π-Rocker

Position analysis

Exercise #1 : inverted slider mechanism

Find θ_3 and *s* for the inverted slider mechanism shown in figure. Assume d_1, d_2, d_3 and θ_1 are given data and θ_2 is input



Position analysis

Exercise #1 : 6 bar mechanism

Analyze the 6-bar mechanism shown in figure If

- θ_1, θ_7, α, d_1, d_2, d_3, d_4, d_5, d_6, d_7 and h are known
- θ_2 is input
- θ_3, θ_4, θ_5 and θ_6 are unknowns

Hint: loop closure equations are:

- $d_2 U_{\theta 2} + d_3 U_{\theta 3} = d_7 U_{\theta 7} + d_4 U_{\theta 4}$
- $d_2 U_{\theta 2} + h U_{\alpha+\theta 3} = d_5 U_{\theta 5} + d_6 U_{\theta 6} + d_1 U_{\theta 1}$