Mechanics of materials

Chapter seven + nine

Shear + stress transformation

By

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M e c h a n i c s o f m a t e r i a l s

Shear

7.1. shear formula

Transverse shear stress always has its associated longitudinal shear stress acting along longitudinal planes of the beam.

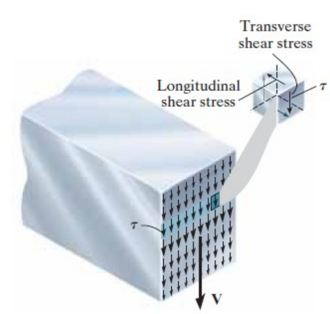


Fig. 7–1

Shear

7.1. shear formula

Effects of Shear Stresses:

Fig. 7–2

Warping of cross section

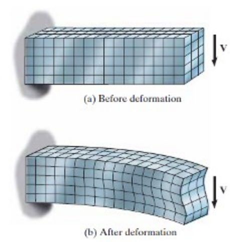
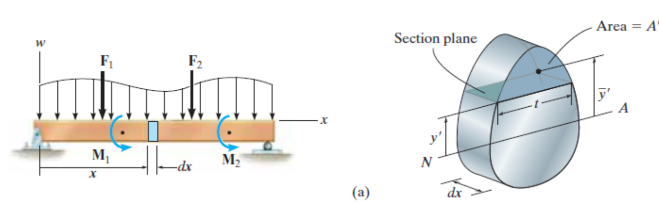


Fig. 7–3

Shear

7.1. shear formula

SHEAR FORMULA

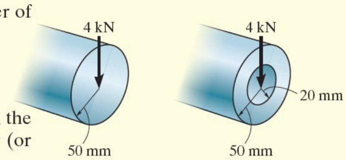


$$\tau = \frac{VQ}{It}$$

$$\text{where } Q = \int_{A'} y\,dA = \bar{y}'A'$$

EXAMPLE 7.1

The solid shaft and tube shown in Fig. 7–9*a* are subjected to the shear force of 4 kN. Determine the shear stress acting over the diameter of each cross section.



(a)

SOLUTION

Section Properties. Using the table on the inside front cover, the moment of inertia of each section, calculated about its diameter (or neutral axis), is

$$I_{\text{solid}} = \frac{1}{4}\pi c^4 = \frac{1}{4}\pi(0.05\text{ m})^4 = 4.909(10^{-6})\text{ m}^4$$

$$I_{\text{tube}} = \frac{1}{4}\pi(c_o^4 - c_i^4) = \frac{1}{4}\pi[(0.05\text{ m})^4 - (0.02\text{ m})^4] = 4.783(10^{-6})\text{ m}^4$$

The semicircular area shown shaded in Fig. 7–9*b*, above (or below) each diameter, represents Q, because this area is "held onto the member" by the longitudinal shear stress along the diameter.

$$Q_{\text{solid}} = \bar{y}'A' = \frac{4c}{3\pi}\left(\frac{\pi c^2}{2}\right) = \frac{4(0.05\text{ m})}{3\pi}\left(\frac{\pi(0.05\text{ m})^2}{2}\right) = 83.33\,(10^{-6})\text{ m}^3$$

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EXAMPLE 7.1 CONTINUED

$$Q_{\text{tube}} = \Sigma\bar{y}'A' = \frac{4c_o}{3\pi}\left(\frac{\pi c_o^2}{2}\right) - \frac{4c_i}{3\pi}\left(\frac{\pi c_i^2}{2}\right)$$

$$= \frac{4(0.05\text{ m})}{3\pi}\left(\frac{\pi(0.05\text{ m})^2}{2}\right) - \frac{4(0.02\text{ m})}{3\pi}\left(\frac{\pi(0.02\text{ m})^2}{2}\right)$$

$$= 78.0(10^{-6})\text{ m}^3$$

Shear Stress. Applying the shear formula where $t = 0.1$ m for the solid section, and $t = 2(0.03\text{ m}) = 0.06$ m for the tube, we have

$$\tau_{\text{solid}} = \frac{VQ}{It} = \frac{4(10^3)\text{ N}(83.33(10^{-6})\text{ m}^3)}{4.909(10^{-6})\text{ m}^4(0.1\text{ m})} = 679\text{ kPa} \qquad \textit{Ans.}$$

$$\tau_{\text{tube}} = \frac{VQ}{It} = \frac{4(10^3)\text{ N}(78.0(10^{-6})\text{ m}^3)}{4.783(10^{-6})\text{ m}^4(0.06\text{ m})} = 1.09\text{ MPa} \qquad \textit{Ans.}$$

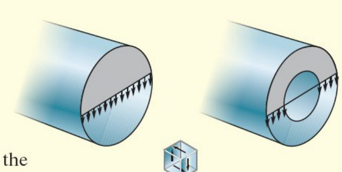
(b)

Fig. 7–9

NOTE: As discussed in the limitations for the shear formula, the calculations performed here are valid since the shear stress along the diameter is vertical and therefore tangent to the boundary of the cross section. An element of material on the diameter is subjected to "pure shear" as shown in Fig. 7–9*b*.

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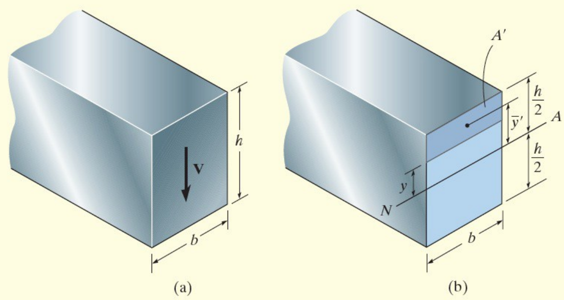
EXAMPLE 7.2

Determine the distribution of the shear stress over the cross section of the beam shown in Fig. 7–10*a*.



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EXAMPLE 7.2 CONTINUED

SOLUTION

The distribution can be determined by finding the shear stress at an *arbitrary height y* from the neutral axis, Fig. 7–10*b*, and then plotting this function. Here, the dark colored area A' will be used for Q.* Hence

$$Q = \bar{y}'A' = \left[y + \frac{1}{2}\left(\frac{h}{2} - y\right)\right]\left(\frac{h}{2} - y\right)b = \frac{1}{2}\left(\frac{h^2}{4} - y^2\right)b$$

Applying the shear formula, we have

$$\tau = \frac{VQ}{It} = \frac{V\left(\frac{1}{2}\right)[(h^2/4) - y^2]b}{\left(\frac{1}{12}bh^3\right)b} = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) \qquad (1)$$

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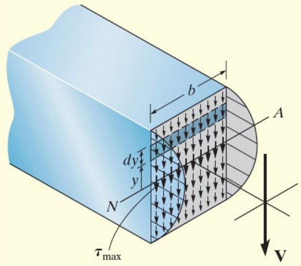
EXAMPLE 7.2 CONTINUED



Shear–stress distribution

(c)

Fig. 7–10

Applying the shear formula, we have

$$\tau = \frac{VQ}{It} = \frac{V\left(\frac{1}{2}\right)[(h^2/4) - y^2]b}{\left(\frac{1}{12}bh^3\right)b} = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) \qquad (1)$$

This result indicates that the shear-stress distribution over the cross section is ***parabolic***. As shown in Fig. 7–10*c*, the intensity varies from zero at the top and bottom, $y = \pm h/2$, to a maximum value at the neutral axis, $y = 0$. Specifically, since the area of the cross section is $A = bh$, then at $y = 0$ we have

$$\tau_{max} = 1.5\frac{V}{A} \qquad (2)$$

*The area below y can also be used $[A' = b(h/2 + y)]$, but doing so involves a bit more algebraic manipulation.

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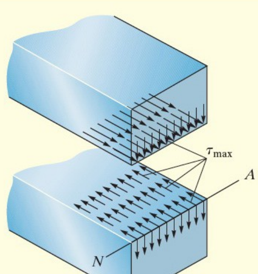
EXAMPLE 7.2 CONTINUED



(d)

Fig. 7–10 (cont.)

Typical shear failure of this wooden beam occurred at the support and through the approximate center of its cross section.

This same value for τ_{max} can be obtained directly from the shear formula, $\tau = VQ/It$, by realizing that τ_{max} occurs where Q is *largest*, since V, I, and t are *constant*. By inspection, Q will be a maximum when the entire area above (or below) the neutral axis is considered; that is, $A' = bh/2$ and $\bar{y}' = h/4$. Thus,

$$\tau_{max} = \frac{VQ}{It} = \frac{V(h/4)(bh/2)}{\left[\frac{1}{12}bh^3\right]b} = 1.5\frac{V}{A}$$

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EXAMPLE 7.2 CONTINUED

By comparison, τ_{max} is 50% greater than the *average* shear stress determined from Eq. 1–7; that is, $\tau_{avg} = V/A$.

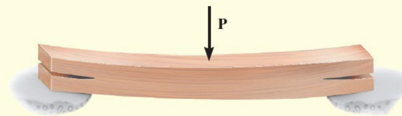
It is important to realize that τ_{max} also acts in the longitudinal direction of the beam, Fig. 7–10*d*. It is this stress that can cause a timber beam to fail as shown Fig. 7–10*e*. Here horizontal splitting of the wood starts to occur through the neutral axis at the beam's ends, since there the vertical reactions subject the beam to large shear stress and wood has a low resistance to shear along its grains, which are oriented in the longitudinal direction.

It is instructive to show that when the shear-stress distribution, Eq. 1, is integrated over the cross section it yields the resultant shear V. To do this, a differential strip of area $dA = b\,dy$ is chosen, Fig. 7–10*c*, and since τ acts uniformly over this strip, we have

$$\int_A \tau\, dA = \int_{-h/2}^{h/2} \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right) b\, dy$$

$$= \frac{6V}{h^3}\left[\frac{h^2}{4}y - \frac{1}{3}y^3\right]_{-h/2}^{h/2}$$

$$= \frac{6V}{h^3}\left[\frac{h^2}{4}\left(\frac{h}{2} + \frac{h}{2}\right) - \frac{1}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right)\right] = V$$



(e)

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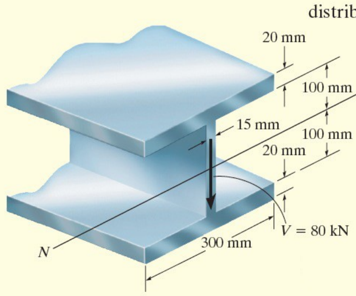
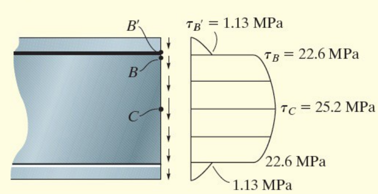
EXAMPLE 7.3

A steel wide-flange beam has the dimensions shown in Fig. 7–11*a*. If it is subjected to a shear of $V = 80$ kN, plot the shear-stress distribution acting over the beam's cross-sectional area.



(a)



(b)

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EXAMPLE 7.3 CONTINUED

SOLUTION

Since the flange and web are rectangular elements, then like the previous example, the shear-stress distribution will be parabolic and in this case it will vary in the manner shown in Fig. 7–11*b*. Due to symmetry, only the shear stresses at points B', B, and C have to be determined. To show how these values are obtained, we must first determine the moment of inertia of the cross-sectional area about the neutral axis. Working in meters, we have

$$I = \left[\frac{1}{12}(0.015\text{ m})(0.200\text{ m})^3\right] + 2\left[\frac{1}{12}(0.300\text{ m})(0.02\text{ m})^3 + (0.300\text{ m})(0.02\text{ m})(0.110\text{ m})^2\right] = 155.6(10^{-6})\text{ m}^4$$

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EXAMPLE 7.3 CONTINUED

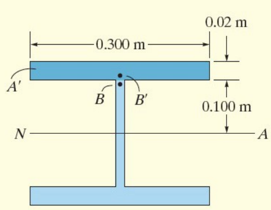

Fig. 7–11

For point B', $t_{B'} = 0.300$ m, and A' is the dark shaded area shown in Fig. 7–11*c*. Thus,

$$Q_{B'} = \bar{y}'A' = [0.110\text{ m}](0.300\text{ m})(0.02\text{ m}) = 0.660(10^{-3})\text{ m}^3$$

so that

$$\tau_{B'} = \frac{VQ_{B'}}{It_{B'}} = \frac{80(10^3)\text{ N}(0.660(10^{-3})\text{ m}^3)}{155.6(10^{-6})\text{ m}^4(0.300\text{ m})} = 1.13\text{ MPa}$$

For point B, $t_B = 0.015$ m and $Q_B = Q_{B'}$, Fig. 7–11*c*. Hence

$$\tau_B = \frac{VQ_B}{It_B} = \frac{80(10^3)\text{ N}(0.660(10^{-3})\text{ m}^3)}{155.6(10^{-6})\text{ m}^4(0.015\text{ m})} = 22.6\text{ MPa}$$

Note from the discussion of "Limitations on the Use of the Shear Formula" that the calculated value for both $\tau_{B'}$ and τ_B will actually be very misleading. Why?

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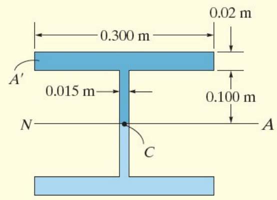
EXAMPLE 7.3 CONTINUED



(d)

Fig. 7–11 (cont.)

For point C, $t_C = 0.015$ m and A' is the dark shaded area shown in Fig. 7–11*d*. Considering this area to be composed of two rectangles, we have

$$\begin{aligned} Q_C = \Sigma \bar{y}' A' &= [0.110 \text{ m}](0.300 \text{ m})(0.02 \text{ m}) \\ &+ [0.05 \text{ m}](0.015 \text{ m})(0.100 \text{ m}) \\ &= 0.735(10^{-3}) \text{ m}^3 \end{aligned}$$

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EXAMPLE 7.3 CONTINUED

Thus,

$$\tau_C = \tau_{\max} = \frac{VQ_C}{It_C} = \frac{80(10^3) \text{ N}[0.735(10^{-3}) \text{ m}^3]}{155.6(10^{-6}) \text{ m}^4(0.015 \text{ m})} = 25.2 \text{ MPa}$$

NOTE: From Fig. 7–11*b*, note that most of the shear stress occurs in the web and is almost uniform throughout its depth, varying from 22.6 MPa to 25.2 MPa. It is for this reason that for design, some codes permit the use of calculating the average shear stress on the cross section of the web rather than using the shear formula. This will be discussed further in Chapter 11.

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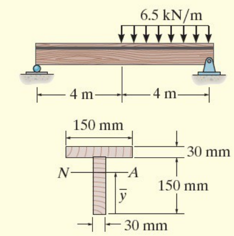
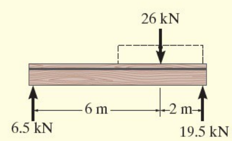
EXAMPLE 7.4



(a)



The beam shown in Fig. 7–12*a* is made from two boards. Determine the maximum shear stress in the glue necessary to hold the boards together along the seam where they are joined.

SOLUTION

Internal Shear. The support reactions and the shear diagram for the beam are shown in Fig. 7–12*b*. It is seen that the maximum shear in the beam is 19.5 kN.

Section Properties. The centroid and therefore the neutral axis will be determined from the reference axis placed at the bottom of the cross-sectional area, Fig. 7–12*a*. Working in units of meters, we have

$$\bar{y} = \frac{\Sigma \tilde{y} A}{\Sigma A}$$

$$= \frac{[0.075 \text{ m}](0.150 \text{ m})(0.030 \text{ m}) + [0.165 \text{ m}](0.030 \text{ m})(0.150 \text{ m})}{(0.150 \text{ m})(0.030 \text{ m}) + (0.030 \text{ m})(0.150 \text{ m})} = 0.120 \text{ m}$$

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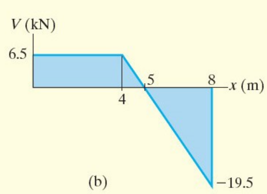
EXAMPLE 7.4 CONTINUED



(b)

The moment of inertia, about the neutral axis, Fig. 7–12*a*, is therefore

$$I = \left[\frac{1}{12}(0.030 \text{ m})(0.150 \text{ m})^3 + (0.150 \text{ m})(0.030 \text{ m})(0.120 \text{ m} - 0.075 \text{ m})^2\right]$$

$$+ \left[\frac{1}{12}(0.150 \text{ m})(0.030 \text{ m})^3 + (0.030 \text{ m})(0.150 \text{ m})(0.165 \text{ m} - 0.120 \text{ m})^2\right]$$

$$= 27.0(10^{-6}) \text{ m}^4$$

The top board (flange) is being held onto the bottom board (web) by the glue, which is applied over the thickness $t = 0.03$ m. Consequently A' is defined as the area of the top board, Fig. 7–12*a*. We have

$$Q = \bar{y}'A' = [0.180 \text{ m} - 0.015 \text{ m} - 0.120 \text{ m}](0.03 \text{ m})(0.150 \text{ m})$$

$$= 0.2025(10^{-3}) \text{ m}^3$$

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EXAMPLE 7.4 CONTINUED

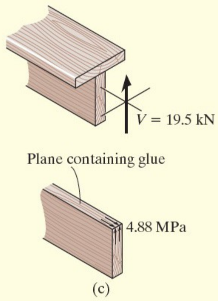

Fig. 7–12

Shear Stress. Using the above data and applying the shear formula yields

$$\tau_{\max} = \frac{VQ}{It} = \frac{19.5(10^3)\text{ N}(0.2025(10^{-3})\text{ m}^3)}{27.0(10^{-6})\text{ m}^4(0.030\text{ m})} = 4.88\text{ MPa} \qquad \textit{Ans.}$$

The shear stress acting at the top of the bottom board is shown in Fig. 7–12*c*.

NOTE: It is the glue's resistance to this longitudinal *shear* stress that holds the boards from slipping at the right-hand support.

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Stress transformation

Mechanics of materials

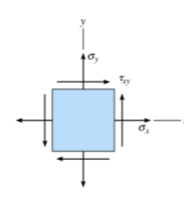
GENERAL EQUATIONS OF PLANE-STRESS TRANSFORMATION

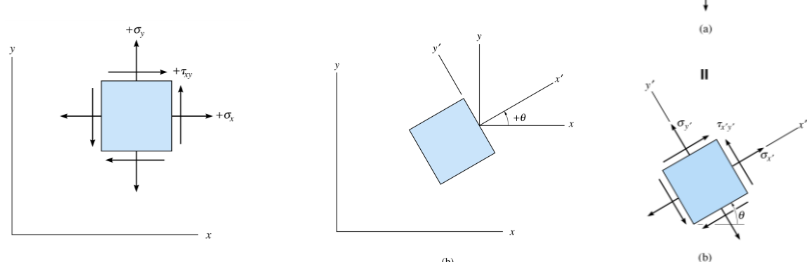
The state of plane stress at a point is uniquely represented by three components acting on an element that has a specific orientation at the point.

Sign Convention:

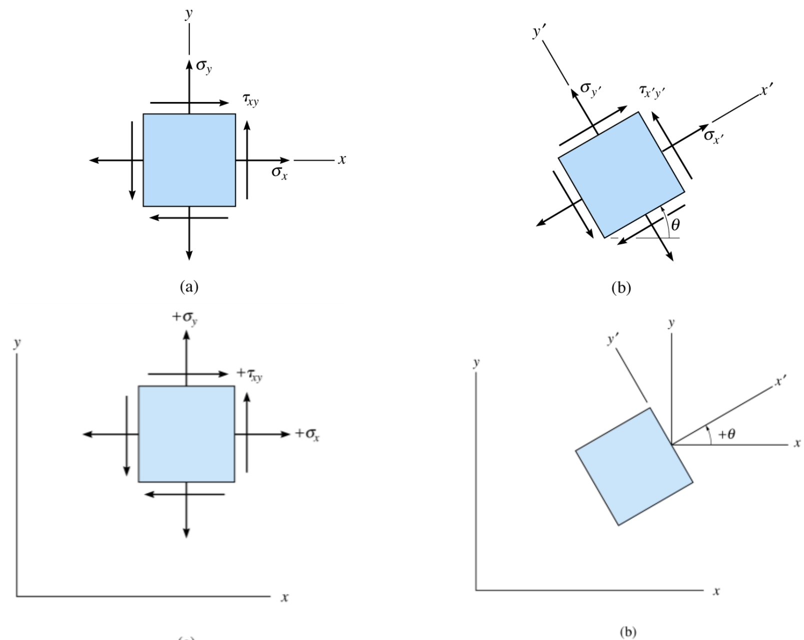
Positive normal stress acts outward from all faces

Positive shear stress acts upwards on the right-hand face of the element





Positive Sign Convention



Positive Sign Convention

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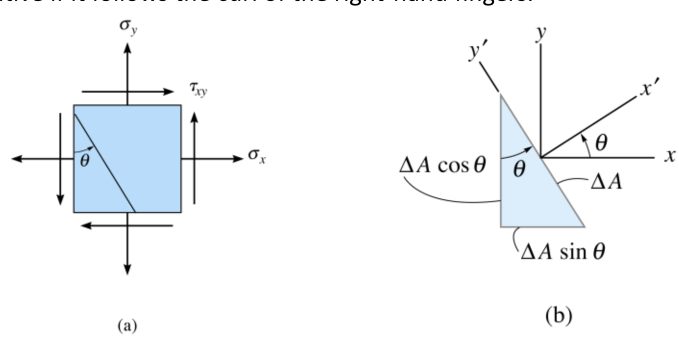
GENERAL EQUATIONS OF PLANE-STRESS TRANSFORMATION (cont)

- Sign convention (continued)
- Both the x-y and x'-y' system follow the right-hand rule
- The orientation of an inclined plane (on which the normal and shear stress components are to be determined) will be defined using the angle θ. The angle θ is measured from the positive x to the positive x'-axis. It is positive if it follows the curl of the right-hand fingers.



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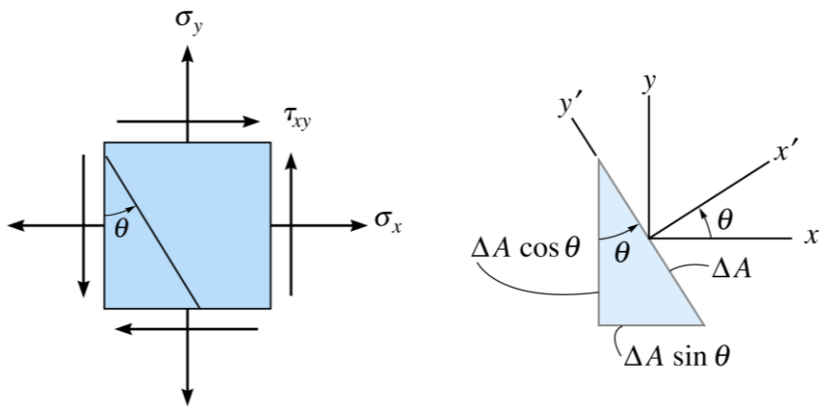

(a)

(b)

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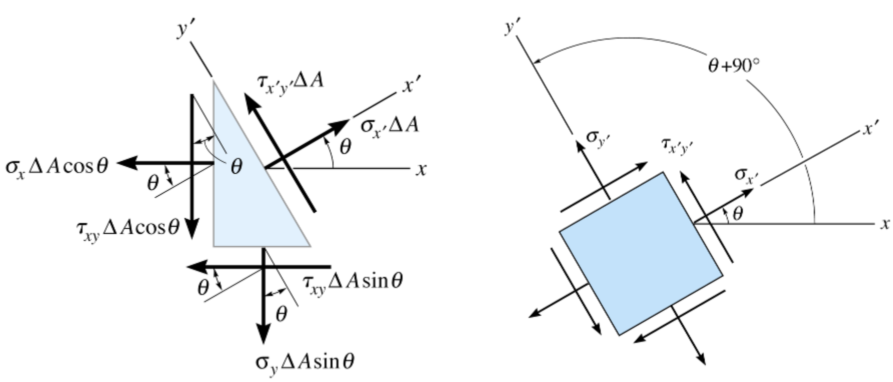
GENERAL EQUATIONS OF PLANE-STRESS TRANSFORMATION (cont)

- Normal and shear stress components:
  - Consider the free-body diagram of the segment



(c)

(d)

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GENERAL EQUATIONS OF PLANE-STRESS TRANSFORMATION (cont)

$$+\nearrow\Sigma F_{x'} = 0; \quad \sigma_{x'}\,\Delta A - (\tau_{xy}\,\Delta A\ \sin\theta)\cos\theta - (\sigma_y\,\Delta A \sin\theta)\sin\theta$$
$$- (\tau_{xy}\,\Delta A\ \cos\theta)\sin\theta - (\sigma_x\,\Delta A \cos\theta)\cos\theta = 0$$
$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\,(2\sin\theta\cos\theta)$$

$$+\nwarrow\Sigma F_{y'} = 0; \quad \tau_{x'y'}\,\Delta A + (\tau_{xy}\,\Delta A\ \sin\theta)\sin\theta - (\sigma_y\,\Delta A \sin\theta)\cos\theta$$
$$- (\tau_{xy}\,\Delta A\ \cos\theta)\cos\theta + (\sigma_x\,\Delta A \cos\theta)\sin\theta = 0$$
$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}\,(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\,\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x + \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\,\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\,\theta$$

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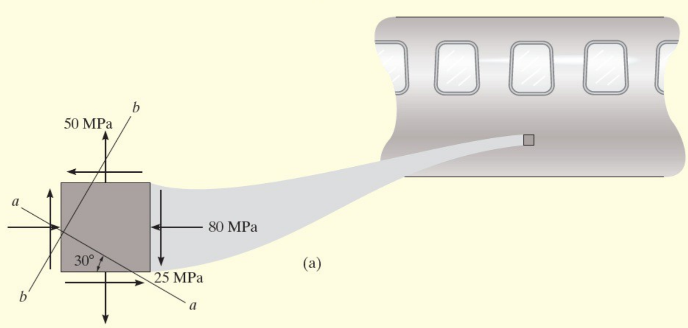
EXAMPLE 9.1

The state of plane stress at a point on the surface of the airplane fuselage is represented on the element oriented as shown in Fig. 9–4*a*. Represent the state of stress at the point on an element that is oriented 30° clockwise from the position shown.



(a)

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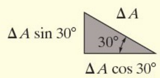
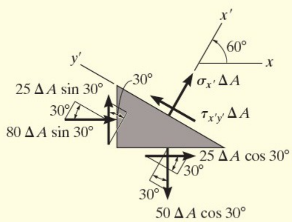
EXAMPLE 9.1 CONTINUED



(b)



(c)

Fig. 9–4

SOLUTION

The rotated element is shown in Fig. 9–4*d*. To obtain the stress component on this element we will first section the element in Fig. 9–4*a* by the line *a–a*. The bottom segment is removed, and assuming the sectioned (inclined) plane has an area ΔA, the horizontal and vertical planes have the areas shown in Fig. 9–4*b*. The free-body diagram of this segment is shown in Fig. 9–4*c*. Applying the equations of force equilibrium in the x' and y' directions to avoid a simultaneous solution for the two unknowns $\sigma_{x'}$ and $\tau_{x'y'}$, we have

$$+\nearrow\Sigma F_{x'} = 0; \quad \sigma_{x'}\,\Delta A - (50\,\Delta A \cos 30°)\cos 30° + (25\,\Delta A \cos 30°)\sin 30° + (80\,\Delta A \sin 30°)\sin 30° + (25\,\Delta A \sin 30°)\cos 30° = 0$$

$$\sigma_{x'} = -4.15 \text{ MPa} \qquad \textit{Ans.}$$

$$+\nwarrow\Sigma F_{y'} = 0; \quad \tau_{x'y'}\,\Delta A - (50\,\Delta A \cos 30°)\sin 30° - (25\,\Delta A \cos 30°)\cos 30° - (80\,\Delta A \sin 30°)\cos 30° + (25\,\Delta A \sin 30°)\sin 30° = 0$$

$$\tau_{x'y'} = 68.8 \text{ MPa} \qquad \textit{Ans.}$$

Since $\sigma_{x'}$ is negative, it acts in the opposite direction of that shown in Fig. 9–4*c*. The results are shown on the *top* of the element in Fig. 9–4*d*, since this surface is the one considered in Fig. 9–4*c*.

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EXAMPLE 9.1 CONTINUED

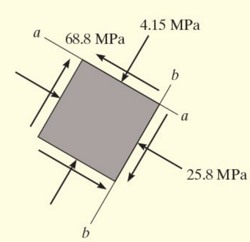
We must now repeat the procedure to obtain the stress on the *perpendicular* plane *b–b*. Sectioning the element in Fig. 9–4*a* along *b–b* results in a segment having sides with areas shown in Fig. 9–4*e*. Orienting the $+x'$ axis outward, perpendicular to the sectioned face, the associated free-body diagram is shown in Fig. 9–4*f*. Thus,

$$+\searrow\Sigma F_{x'} = 0; \quad \sigma_{x'}\,\Delta A - (25\,\Delta A \cos 30°)\sin 30° + (80\,\Delta A \cos 30°)\cos 30° - (25\,\Delta A \sin 30°)\cos 30° - (50\,\Delta A \sin 30°)\sin 30° = 0$$

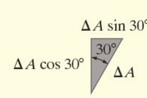
$$\sigma_{x'} = -25.8 \text{ MPa} \qquad \textit{Ans.}$$

$$+\nearrow\Sigma F_{y'} = 0; \quad -\tau_{x'y'}\,\Delta A + (25\,\Delta A \cos 30°)\cos 30° + (80\,\Delta A \cos 30°)\sin 30° - (25\,\Delta A \sin 30°)\sin 30° + (50\,\Delta A \sin 30°)\cos 30° = 0$$

$$\tau_{x'y'} = 68.8 \text{ MPa} \qquad \textit{Ans.}$$



(d)



(e)

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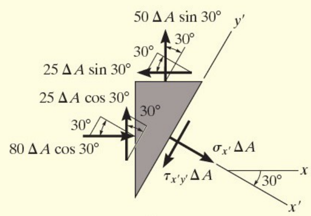
EXAMPLE 9.1 CONTINUED

Since $\sigma_{x'}$ is a negative quantity, it acts opposite to its direction shown in Fig. 9–4*f*. The stress components are shown acting on the *right side* of the element in Fig. 9–4*d*.

From this analysis we may therefore conclude that the state of stress at the point can be represented by choosing an element oriented as shown in Fig. 9–4*a*, or by choosing one oriented as shown in Fig. 9–4*d*. In other words, these states of stress are equivalent.



(f)

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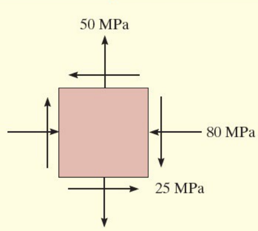
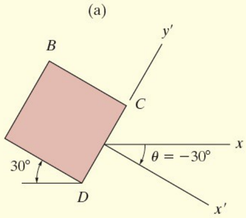
EXAMPLE 9.2



(a)



(b)

The state of plane stress at a point is represented by the element shown in Fig. 9–7*a*. Determine the state of stress at the point on another element oriented 30° clockwise from the position shown.

SOLUTION

This problem was solved in Example 9.1 using basic principles. Here we will apply Eqs. 9–1 and 9–2. From the established sign convention, Fig. 9–5, it is seen that

$$\sigma_x = -80\text{ MPa} \qquad \sigma_y = 50\text{ MPa} \qquad \tau_{xy} = -25\text{ MPa}$$

Plane *CD*. To obtain the stress components on plane *CD*, Fig. 9–7*b*, the positive x' axis is directed outward, perpendicular to *CD*, and the associated y' axis is directed along *CD*. The angle measured from the x to the x' axis is $\theta = -30°$ (clockwise). Applying Eqs. 9–1 and 9–2 yields

$$\begin{aligned}\sigma_{x'} &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ &= \frac{-80 + 50}{2} + \frac{-80 - 50}{2}\cos 2(-30°) + (-25)\sin 2(-30°) \\ &= -25.8\text{ MPa}\end{aligned}$$

Ans.

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EXAMPLE 9.2 CONTINUED

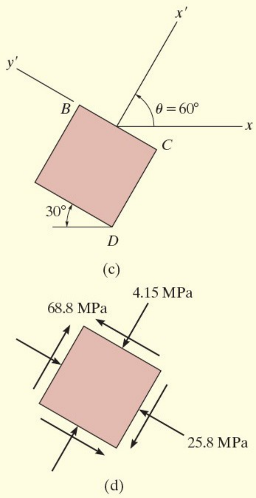

Fig. 9–7

The negative signs indicate that $\sigma_{x'}$ and $\tau_{x'y'}$ act in the negative x' and y' directions, respectively. The results are shown acting on the element in Fig. 9–7*d*.

Plane *BC*. In a similar manner, the stress components acting on face *BC*, Fig. 9–7*c*, are obtained using $\theta = 60°$. Applying Eqs. 9–1 and 9–2,* we get

$$\sigma_{x'} = \frac{-80 + 50}{2} + \frac{-80 - 50}{2}\cos 2(60°) + (-25)\sin 2(60°)$$
$$= -4.15 \text{ MPa} \qquad \textit{Ans.}$$

$$\tau_{x'y'} = -\frac{-80 - 50}{2}\sin 2(60°) + (-25)\cos 2(60°)$$
$$= 68.8 \text{ MPa} \qquad \textit{Ans.}$$

Here $\tau_{x'y'}$ has been calculated twice in order to provide a check. The negative sign for $\sigma_{x'}$ indicates that this stress acts in the negative x' direction, Fig. 9–7*c*. The results are shown on the element in Fig. 9–7*d*.

*Alternatively, we could apply Eq. 9–3 with $\theta = -30°$ rather than Eq. 9–1.

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IN-PLANE PRINCIPAL STRESS

- The principal stresses represent the maximum and minimum normal stress at the point.
- When the state of stress is represented by the principal stresses, no shear stress will act on the element.

$$\frac{d\sigma_{x'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}(2\sin 2\theta) + 2\tau_{xy}\cos 2\theta$$

Solving this equation leads to θ = θ_p

$$\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

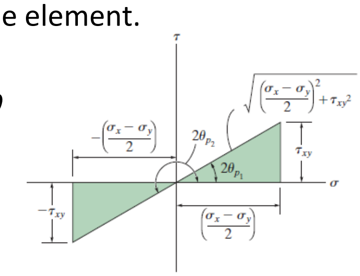

Fig. 9–8

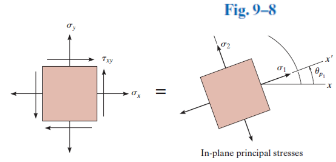

Fig. 9–9

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IN-PLANE PRINCIPAL STRESS (cont)

$$\frac{d\sigma_{x'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}(2\sin 2\theta) + 2\tau_{xy}\cos 2\theta$$

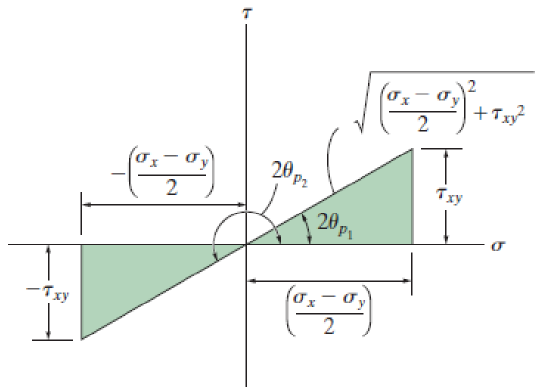

Fig. 9–8

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IN-PLANE PRINCIPAL STRESS (cont)

Solving this equation leads to $\theta = \theta_p$; i.e $\tan 2\theta_p = \dfrac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$

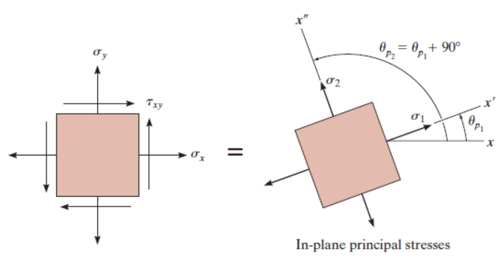

Fig. 9–9

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

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MAXIMUM IN-PLANE PRINCIPAL STRESS

- The state of stress can also be represented in terms of the *maximum in-plane shear stress*. In this case, an *average stress* will also act on the element.

$$\frac{d\tau_{x'y'}}{d\theta} = -\frac{\sigma_x - \sigma_y}{2}(2\cos\theta) - \tau_{xy}(2\sin 2\theta) = 0$$

- Solving this equation leads to θ = θs; i.e $\tan 2\theta_s = \dfrac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}$

$$\tau_{\text{max in-plane}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

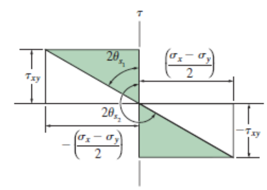

Fig. 9–10

- And there is a normal stress on the plane of maximum in-plane shear stress

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

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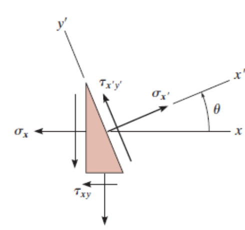
MOHR'S CIRCLE OF PLANE STRESS

- A geometrical representation of equations 9.1 and 9.2; i.e.

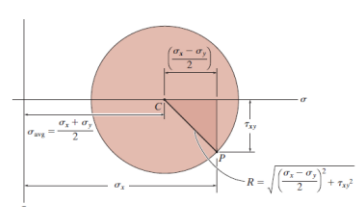
$$\sigma_{x'} - \left(\frac{\sigma_x + \sigma_y}{2}\right) = \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\sin 2\theta$$



- Sign Convention:

  σ is positive to the right, and τ is positive downward.



(b)

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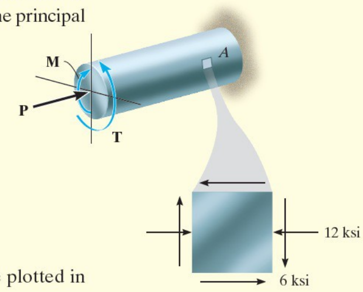
EXAMPLE 9.7

Due to the applied loading, the element at point A on the solid shaft in Fig. 9–18*a* is subjected to the state of stress shown. Determine the principal stresses acting at this point.



SOLUTION

Construction of the Circle. From Fig. 9–18*a*,

$$\sigma_x = -12 \text{ ksi} \qquad \sigma_y = 0 \qquad \tau_{xy} = -6 \text{ ksi}$$

The center of the circle is at

$$\sigma_{\text{avg}} = \frac{-12 + 0}{2} = -6 \text{ ksi}$$

The reference point $A(-12, -6)$ and the center $C(-6, 0)$ are plotted in Fig. 9–18*b*. The circle is constructed having a radius of

$$R = \sqrt{(12 - 6)^2 + (6)^2} = 8.49 \text{ ksi}$$

Principal Stress. The principal stresses are indicated by the coordinates of points B and D. We have, for $\sigma_1 > \sigma_2$,

$$\sigma_1 = 8.49 - 6 = 2.49 \text{ ksi} \qquad \textit{Ans.}$$

$$\sigma_2 = -6 - 8.49 = -14.5 \text{ ksi} \qquad \textit{Ans.}$$

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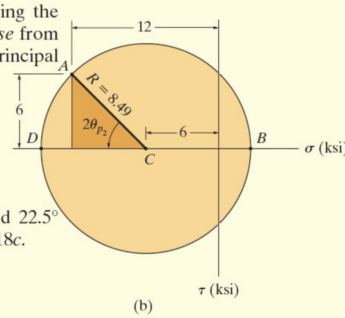
EXAMPLE 9.7 CONTINUED

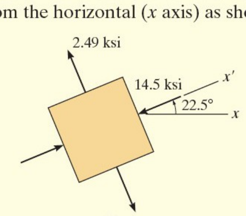
The orientation of the element can be determined by calculating the angle $2\theta_{p_2}$ in Fig. 9–18*b*, which here is measured *counterclockwise* from CA to CD. It defines the direction θ_{p_2} of σ_2 and its associated principal plane. We have



$$2\theta_{p_2} = \tan^{-1}\frac{6}{12 - 6} = 45.0^\circ$$

$$\theta_{p_2} = 22.5^\circ$$

The element is oriented such that the x' axis or σ_2 is directed 22.5° *counterclockwise* from the horizontal (x axis) as shown in Fig. 9–18*c*.



(c)

Fig. 9–18

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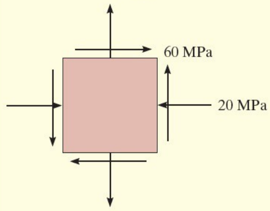
EXAMPLE 9.8



(a)

The state of plane stress at a point is shown on the element in Fig. 9–19*a*. Determine the maximum in-plane shear stress at this point.

SOLUTION

Construction of the Circle. From the problem data,

$$\sigma_x = -20 \text{ MPa} \qquad \sigma_y = 90 \text{ MPa} \qquad \tau_{xy} = 60 \text{ MPa}$$

The σ, τ axes are established in Fig. 9–19*b*. The center of the circle C is located on the σ axis, at the point

$$\sigma_{avg} = \frac{-20 + 90}{2} = 35 \text{ MPa}$$

Point C and the reference point $A(-20, 60)$ are plotted. Applying the Pythagorean theorem to the shaded triangle to determine the circle's radius CA, we have

$$R = \sqrt{(60)^2 + (55)^2} = 81.4 \text{ MPa}$$

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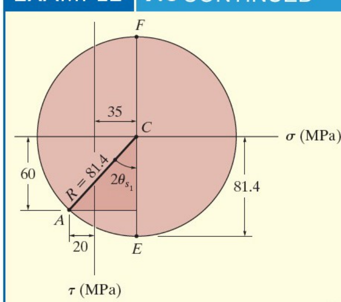
EXAMPLE 9.8 CONTINUED



(b)

Maximum In-Plane Shear Stress. The maximum in-plane shear stress and the average normal stress are identified by point E (or F) on the circle. The coordinates of point $E(35, 81.4)$ give

$$\tau_{\substack{max \\ in\text{-}plane}} = 81.4 \text{ MPa} \qquad \textit{Ans.}$$

$$\sigma_{avg} = 35 \text{ MPa} \qquad \textit{Ans.}$$

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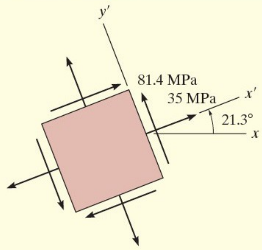
EXAMPLE 9.8 CONTINUED



(c)

Fig. 9–19

The angle θ_{s_1}, measured *counterclockwise* from *CA* to *CE*, can be found from the circle, identified as $2\theta_{s_1}$. We have

$$2\theta_{s_1} = \tan^{-1}\left(\frac{20 + 35}{60}\right) = 42.5^\circ$$

$$\theta_{s_1} = 21.3^\circ \qquad \textit{Ans.}$$

This *counterclockwise* angle defines the direction of the x' axis, Fig. 9–19*c*. Since point *E* has *positive* coordinates, then the average normal stress and the maximum in-plane shear stress both act in the *positive* x' and y' directions as shown.

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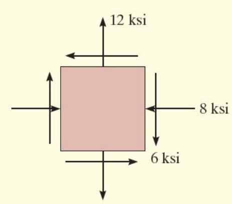
EXAMPLE 9.9

The state of plane stress at a point is shown on the element in Fig. 9–20*a*. Represent this state of stress on an element oriented 30° counterclockwise from the position shown.



(a)

SOLUTION

Construction of the Circle. From the problem data,

$$\sigma_x = -8\text{ ksi} \qquad \sigma_y = 12\text{ ksi} \qquad \tau_{xy} = -6\text{ ksi}$$

The σ and τ axes are established in Fig. 9–20*b*. The center of the circle *C* is on the σ axis at

$$\sigma_{avg} = \frac{-8 + 12}{2} = 2\text{ ksi}$$

The reference point for $\theta = 0^\circ$ has coordinates $A(-8, -6)$. Hence from the shaded triangle the radius *CA* is

$$R = \sqrt{(10)^2 + (6)^2} = 11.66$$

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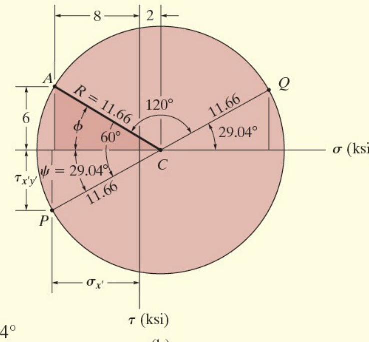
EXAMPLE 9.9 CONTINUED



(b)

Stresses on 30° Element. Since the element is to be rotated 30° *counterclockwise*, we must construct a radial line CP, $2(30°) = 60°$ *counterclockwise*, measured from $CA\,(\theta = 0°)$, Fig. 9–20*b*. The coordinates of point $P(\sigma_{x'}, \tau_{x'y'})$ must now be obtained. From the geometry of the circle,

$$\phi = \tan^{-1}\frac{6}{10} = 30.96° \qquad \psi = 60° - 30.96° = 29.04°$$

$$\sigma_{x'} = 2 - 11.66 \cos 29.04° = -8.20 \text{ ksi} \qquad \textit{Ans.}$$

$$\tau_{x'y'} = 11.66 \sin 29.04° = 5.66 \text{ ksi} \qquad \textit{Ans.}$$

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EXAMPLE 9.9 CONTINUED

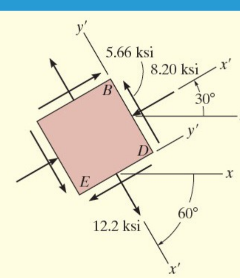
These two stress components act on face BD of the element shown in Fig. 9–20*c* since the x' axis for this face is oriented 30° *counterclockwise* from the x axis.

The stress components acting on the adjacent face DE of the element, which is 60° *clockwise* from the positive x axis, Fig. 9–20*c*, are represented by the coordinates of point Q on the circle. This point lies on the radial line CQ, which is 180° from CP. The coordinates of point Q are

$$\sigma_{x'} = 2 + 11.66 \cos 29.04° = 12.2 \text{ ksi} \qquad \textit{Ans.}$$

$$\tau_{x'y'} = -(11.66 \sin 29.04) = -5.66 \text{ ksi} \quad \text{(check)} \qquad \textit{Ans.}$$

NOTE: Here $\tau_{x'y'}$ acts in the $-y'$ direction.



(c)

Fig. 9–20

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