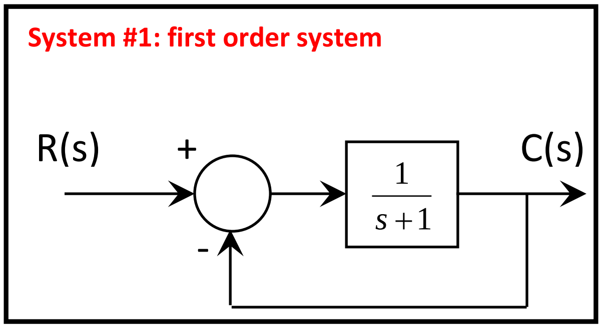
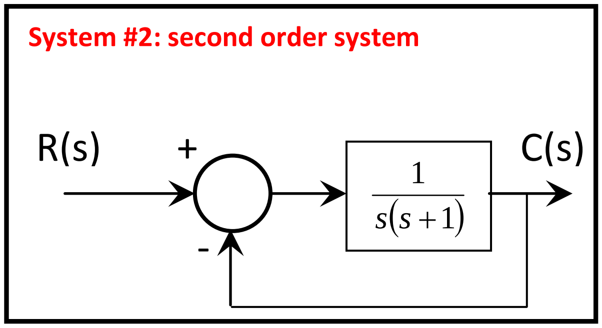
Example on PID controller selection

Problem statement: consider the following two unity feedback systems:





Requirements:

1. Find the steady state error (e_{ss}) for the following cases:
   a. r(t) = 1
   b. r(t) = t
   c. r(t) =t^2/2
2. add a PD or PI controller to in satisfy the following conditions:

For system #1:

Condition	Step	Ramp	Parabolic
Steady state error	<0.001	<0.001	<0.001
The maximum overshoot (%)	< 5	< 5	< 5
The settling time (s)	0.005	10	10
Rise time (s)	0.005	5	5

For system #2:

Condition	Step	Ramp	Parabolic
Steady state error	<0.001	<0.001	<0.001
The maximum overshoot (%)	< 5	< 5	< 5
The settling time (s)	10	10	10
Rise time (s)	5	5	5

Solution:-

For system #1: $G(s)=\dfrac{1}{s+1}$

For the step input r(t) = 1 or R(s) = 1/s, the steady state error can be calculated using Eq.1:

$$e_{ss}=\frac{R}{1+k_p};k_p=\lim_{s\to 0}G(s)=\lim_{s\to 0}\frac{1}{s+1}=1\Rightarrow e_{ss}=\frac{1}{1+1}=\frac{1}{2} \quad (1)$$

For the ramp input r(t) = t or R(s) = $1/s^2$, the steady state error can be calculated using Eq.2:

$$e_{ss}=\frac{R}{k_v};k_v=\lim_{s\to 0}sG(s)=\lim_{s\to 0}\frac{s}{s+1}=0\Rightarrow e_{ss}=\frac{1}{0}=\infty \quad (2)$$

For the parabolic input r(t) = $t^2/2$ or R(s) = $1/2s^3$, the steady state error can be calculated using Eq3:

$$e_{ss}=\frac{R}{k_a};k_a=\lim_{s\to 0}s^2G(s)=\lim_{s\to 0}\frac{s^2}{s+1}=0\Rightarrow e_{ss}=\frac{1}{0}=\infty \quad (3)$$

As seen from equations 1, 2 and 3, the steady state error or this system does not satisfy condition number 1. So, defiantly we need a controller for system 1

For system #2: $G(s)=\dfrac{1}{s(s+1)}$

For the step input r(t) = 1 or R(s) = 1/s, the steady state error can be calculated using Eq.4:

$$e_{ss}=\frac{R}{1+k_p};k_p=\lim_{s\to 0}G(s)=\lim_{s\to 0}\frac{1}{s(s+1)}=\infty\Rightarrow e_{ss}=\frac{1}{1+\infty}=0 \quad (4)$$

For the ramp input r(t) = t or R(s) = $1/s^2$, the steady state error can be calculated using Eq.5:

$$e_{ss}=\frac{R}{k_v};k_v=\lim_{s\to 0}sG(s)=\lim_{s\to 0}\frac{s}{s(s+1)}=1\Rightarrow e_{ss}=\frac{1}{1}=1 \quad (5)$$

For the parabolic input r(t) = $t^2/2$ or R(s) = $1/2s^3$, the steady state error can be calculated using Eq.6:

$$e_{ss} = \frac{R}{k_a}; k_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{s^2}{s(s+1)} = 0 \Rightarrow e_{ss} = \frac{1}{0} = \infty \qquad (6)$$

As seen from Eq.4, the steady state error is zero but this does not mean that we do not need a controller because we need to satisfy the other conditions: max. Overshot, rise time and settling time. On the other hand, the ramp and parabolic inputs need a controller.

A MATLAB code was used to draw both systems for the three inputs and the results are shown in Fig.1.

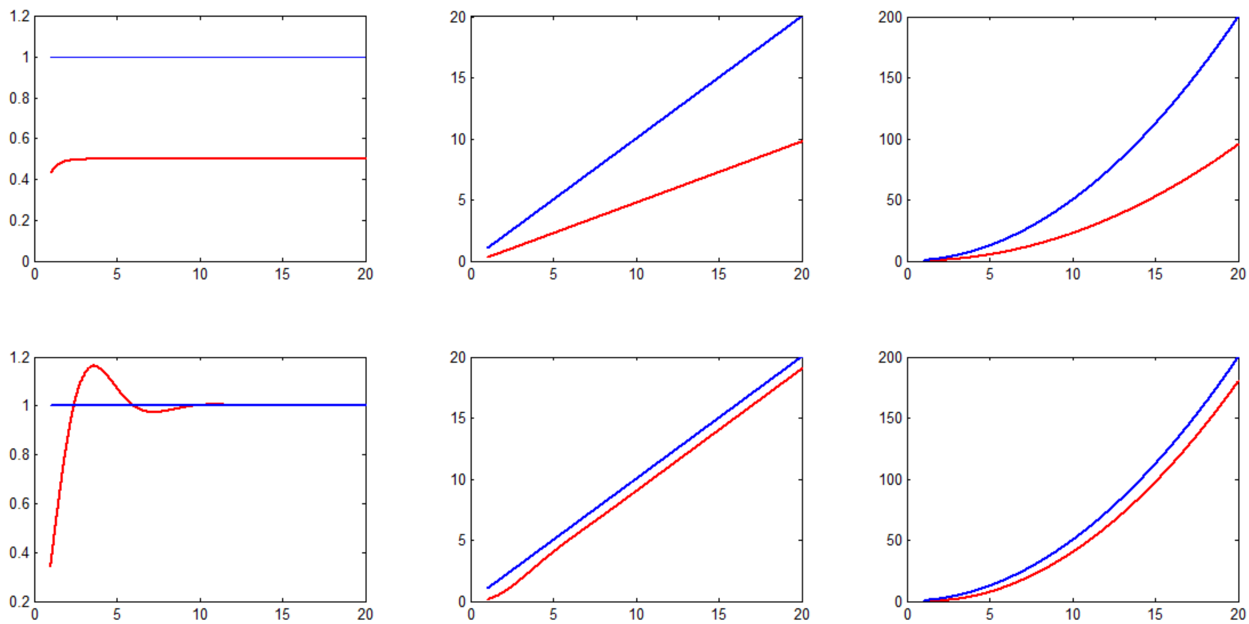


Fig.1. time response for system 1 (the upper raw) and system 2 (the lower raw). The red line is system response and the blue line is the reference input.

As seen from Fig.1, when unity step input is applied to the first system, the output reaches a value equal ½ which is half of the original input. This result is proved by Eq.1 where the error is ½. On the other hand, the second system shows that this system will reach the reference input eventually which is proved by Eq.4 where the error is zero.

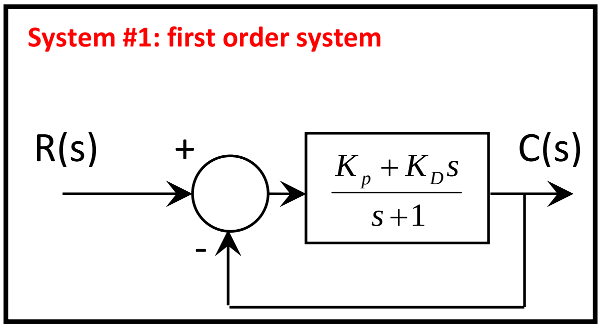
Also it's obvious that when the input of the first system (first order system) is ramp or parabolic, the outputs diverge from the input which is proved by Eqs 2 and 3.

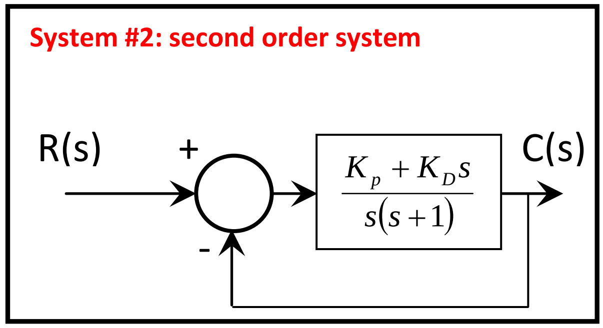
For the second system (second order system), the ramp input show a convergence to the input but with error equal 1. For example, the steady state response is shown in Fig.1. for ramp input shows that at time equal 20 sec – which can be considered enough to steady state condition – the output equal 19 while the input equal 20 and so the error is 1 which is proved by Eq.5.

From previous analysis, we show that for both first and second order systems a controller is needed to satisfy the conditions required previously.

Add PD controller for both systems

The transfer function of PD controller (Gc) is given as: $Gc(s)= K_p + K_D s$. when this controller is added to the systems they become:





Now, the error calculation must be performed again but this time the PD controller is added

For system #1:

Error condition:-

For the step input r(t) = 1 or R(s) = 1/s, the steady state error for the modified system can be calculated using Eq.7:

$$e_{ss} = \frac{R}{1+k_p}; k_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K_p + K_D s}{s+1} = K_p \Rightarrow e_{ss} = \frac{1}{1+K_p} < 0.001 \quad (7)$$

From Eq.7. we can find our first condition which is $K_p>999$. This condition can be satisfied by choosing $K_p=1000$.

Stability conditions:-

After adding Gc(s) to the system, the transfer function M(s) = C(s) / R(s) is calculated as:

$$M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\frac{K_p+K_D s}{s+1}}{\frac{s+1}{s+1}+\frac{K_p+K_D s}{s+1}}=\frac{K_p+K_D s}{s(K_D+1)+K_p+1} \qquad (8)$$

The characteristic equation of the system is:

$$s(K_D+1)+K_p+1=0 \Rightarrow s=\frac{-K_p-1}{(K_D+1)} \qquad (9)$$

Eq.9 gives the second condition (stability condition). Because K_p and K_D are real numbers, s must be negative real number to insure stable system and this can be satisfied when (K_D+1)>0 or $K_D>-1$ because the value of K_p is greater than +999 (i.e. the nominator is always negative therefore the dominator must be positive).

To draw the time response, a MATLAB simulink is used with values of K_p = 1000 and K_D = -0.5. Fig2 shows the block diagram and Fig.3 shows the time response for this system.

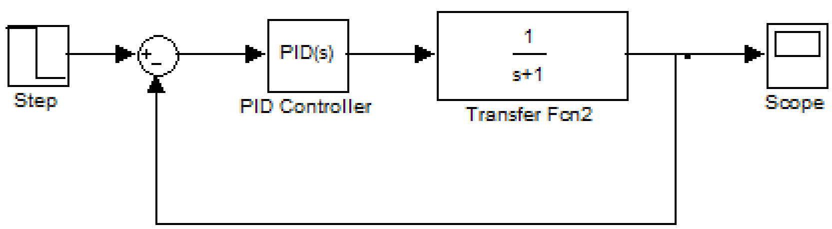


Fig.2. simulink block diagram for the first system when the input is unit step, K_p=1000 and K_D = -0.5

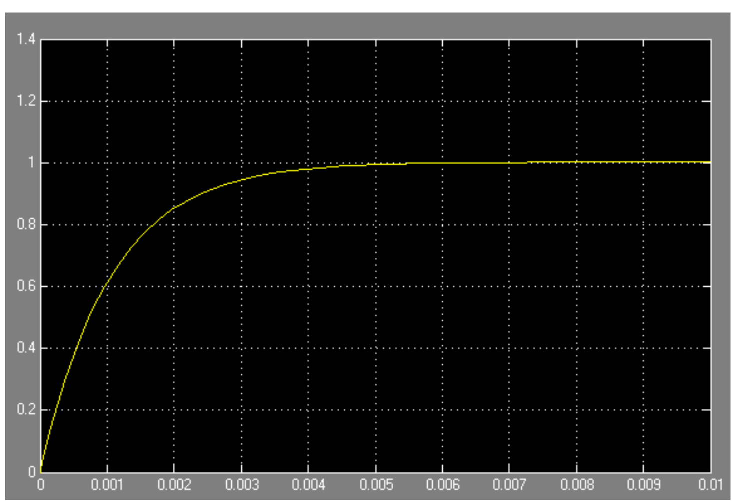


Fig.3. time response for the first system with PD controller when the input is unit step, K_p=1000 and K_D = -0.5

However, if the time is increased to 1sec instead of 0.01sec and the steady state response is magnified as shown in Fig.4, the response will oscillate in behavior show instability. This behavior can be ignored if the steady state error is less than 0.001 and we can assume that the system reaches steady state after 0.005 sec as shown in Fig.3.

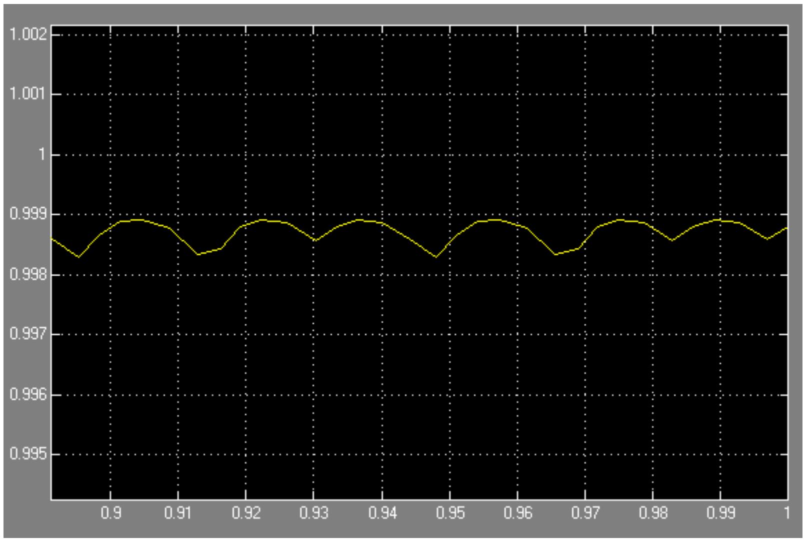


Fig.4. magnified time response for the first system with PD controller when the input is unit step, K_p=1000 and K_D = -0.5

In separate experiment, higher values of K_D were tested and the results show that as K_D increases, the oscillation frequency increases and both settling and rise time decreases.

For the ramp input r(t) = t or R(s) = $1/s^2$, the steady state error can be calculated using Eq.11:

$$e_{ss} = \frac{R}{k_v}; k_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{s(K_p + K_D s)}{s+1} = 0 \Rightarrow e_{ss} = \frac{1}{0} = \infty \qquad (11)$$

It is obvious here that adding a PD controller do not solve the problem. This can be concluded for the parabolic input too as shown in Eq.12

$$e_{ss} = \frac{R}{k_a}; k_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} \frac{s^2(K_p + K_D s)}{s+1} = 0 \Rightarrow e_{ss} = \frac{1}{0} = \infty \qquad (12)$$

As a conclusion, PD controller is not suitable for first order system when the input is either ramp or parabolic.

For system #2: $G(s)=\frac{1}{s(s+1)}$

For the step input r(t) = 1 or R(s) = 1/s, the steady state error can be calculated using Eq.13:

$$e_{ss}=\frac{R}{1+k_p};k_p=\lim_{s\to 0}G(s)=\lim_{s\to 0}\frac{K_p+K_D s}{s(s+1)}=\infty\Rightarrow e_{ss}=\frac{1}{1+\infty}=0 \quad (13)$$

As seen we have to give no extra care about the steady state error. However, we need to test system stability. The transfer function M(s) = C(s) / R(s) is calculated as:

$$M(s)=\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\frac{K_p+K_D s}{s(s+1)}}{\frac{s(s+1)}{s(s+1)}+\frac{K_p+K_D s}{s(s+1)}}=\frac{K_p+K_D s}{s^2+s(K_D+1)+K_p} \quad (14)$$

The characteristic equation of the system is:

$$s^2+s(K_D+1)+K_p=0 \quad (15)$$

This is a square equation and so two roots for s may be found or the stability can be examined using Routh method as illustrated in the following table:

s^2	1	K_p
s^1	K_D+1	0
s^0	$\frac{(K_D+1)K_p}{(K_D+1)}=K_p$	0

To insure stable system, the value of K_p and (K_D+1) must be positive and so: Kp>0 and K_D > -1. The value of K_p is chosen as 0.5 and the values of K_D were iterated and the results are shown in table below:

K_p	**K_D**	**Max. overshot (%)**	**tr (s)**	**ts(s)**
0.5	**-0.90**	**230**	**4**	**70**
0.5	**-0.50**	**35**	**3.8**	**15**
0.5	**-0.40**	**25**	**3.5**	**11**
0.5	**-0.10**	**10**	**3.8**	**9**
0.5	**-0.05**	**4.5**	**4**	**8**
0.5	**0.00**	**4**	**4.5**	**7**
0.5	**0.05**	**2**	**4.8**	**6**

As seen from table, all the shaded rows satisfy all the conditions: max. Overshot <5%, t_r <5sec and t_s<10 sec. Fig.5 shows the time response when K_p =0.5 and K_D =0.05.

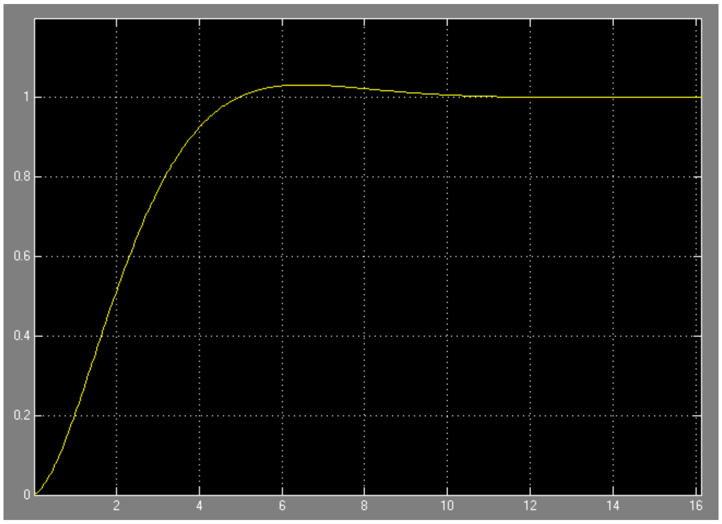


Fig.5. time response for the second system with PD controller when the input is unit step, K_p=0.5 and K_D = 0.05

For the ramp input r(t) = t or R(s) = 1/s², the steady state error can be calculated using Eq.16:

$$e_{ss} = \frac{R}{k_v}; k_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \frac{s(K_p + K_D s)}{s(s+1)} = K_p \Rightarrow e_{ss} = \frac{1}{K_p} < 0.001 \qquad (16)$$

So Kp >1000 to satisfy the steady state condition.

For stability condition, the characteristic equation does not been affected by the type of input and so the previous stability conditions: Kp>0 and K_D > -1 or Kp>1000 and K_D > -1 to satisfy both error and stability conditions are valid. Choosing Kp = 1000 and KD = 0.05 will satisfy all conditions as shown in Fig.6 where the red line in the magnified picture shows the reference input.

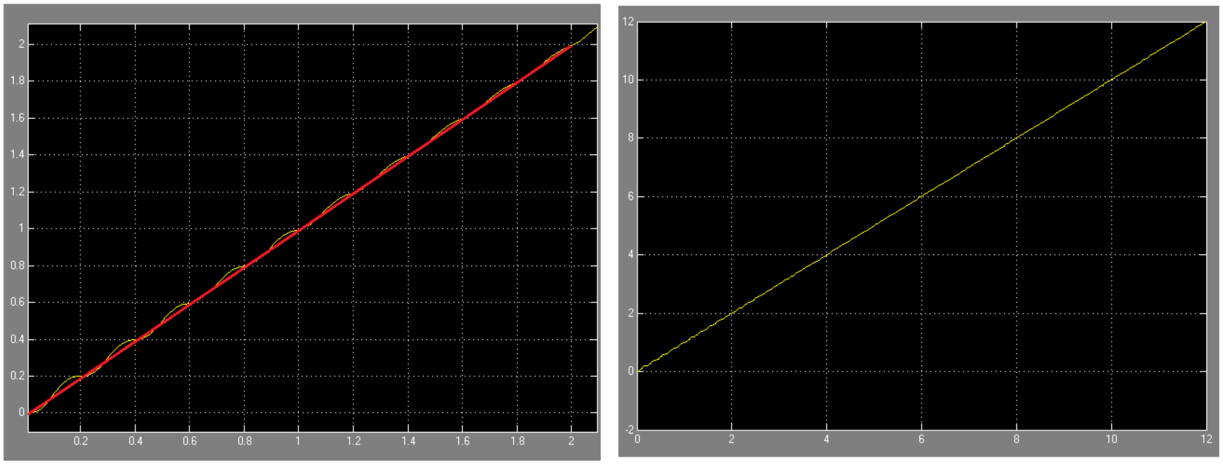


Fig.6. time response for the second system with PD controller when the input is unit ramp, K_p=1000 and K_D = 0.05.

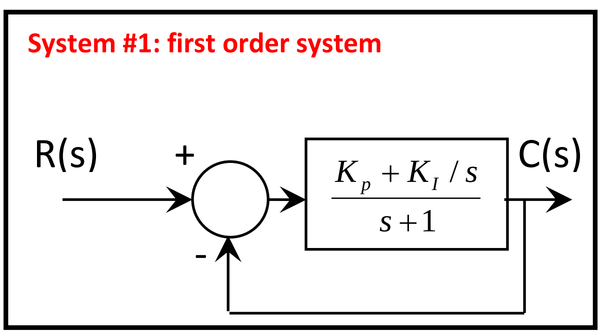
For the parabolic input, the steady state error can be calculated using Eq.17:

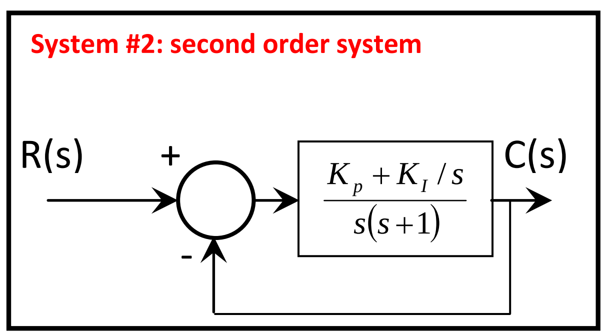
$$e_{ss} = \frac{R}{k_a}; k_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} \frac{s^2 (K_p + K_D s)}{s(s+1)} = 0 \Rightarrow e_{ss} = \frac{1}{0} = \infty \qquad (17)$$

From Eq.17 there is no benifet from adding PD controller when the input is parabolic because there will be s in the nomentor makes Ka = 0 and the steady state error goes to infinity.

Adding PI controller

The transfer function of PI controller (Gc) is given as: Gc(s)= K_p + K_I/s. when this controller is added to the systems they become:





Now, the error calculation must be performed again but this time the PI controller is added in similar way as we have done with PD controller. The next table summrize the new changes after adding the PI controller

System	G(s)	M(s) = C(s)/R(s)	Characteristic Eq.	Stability cond
1	$\frac{K_p s + K_I}{s(s+1)}$	$\frac{K_p s + K_I}{s^2 + s(K_p + 1) + K_I}$	$s^2 + s(K_p + 1) + K_I = 0$	**K_I>0 and K_p > -1**
2	$\frac{K_p s + K_I}{s^2(s+1)}$	$\frac{K_p s + K_I}{s^3 + s^2 + K_p s + K_I}$	$s^3 + s^2 + K_p s + K_I = 0$	**K_p>K_I and K_I > 0**

Error analysis:

Next table shows the error condition for the new proposed systems

System	Step	Ramp	Parabolic
1	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_I}$	$e_{ss} = \infty$
2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_I}$

According to previous tables we can conclude the followings:

1. PI controller can not help us to control the first system when the input is parabolic. May be addition to second integral controller can help.
2. For step input, both systems show exact steady state solutions (e_{ss}=0)
3. For ramp inputs, the first system needs to be controlled to satisfy the error condition while steady state error in the second system is canceled by the presence of the controller whatever the value of K_I and K_P.
4. Error of the second system when the input is parabolic can be controlled using PI controller

Now to satisfy the other requiermnet, we need to itterate for the values of K_p and K_I for each case individually.

Case 1: system 1 + step input + PI controller

An itteration values of KI and KP were tested and the results are shown in the next table

K_I	K_P	Max. overshot (%)	tr (s)	ts(s)
1	-0.5	48	2.3	13
1	0.5	4	2.2	8
1	1.0	----	2.1	6
1	10.0	----	0.3	3
1	100.0	----	0.025	0.037
1	500.0	----	0.004	0.006
1	750.0	----	0.003	0.045

As seen, the values of K_p = 750 and K_I =1 will satisfay all the conditions. Fig.7 show the time response for these values

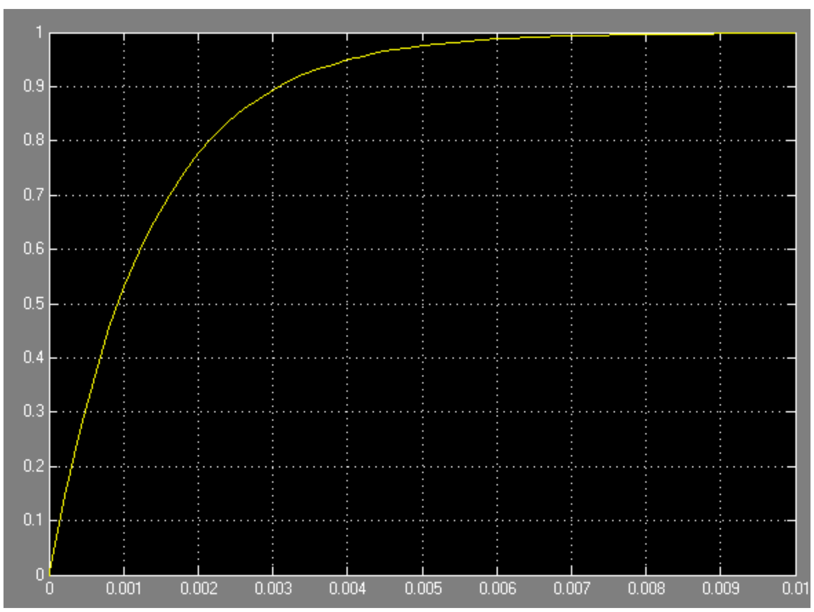


Fig.7. time response for the first system with PI controller when the input is unit step, K_p=1000 and K_I = 1.

Case 2: system 1 + ramp input + PI controller

To satisfy the stability condition, $K_I > 0$ and $Kp > -1$ or $K_I > 1000$ and $Kp > -1$ to satisfy both stability and error conditions. Arbittary values of $K_I = 1001$ and $K_P = 1$ will satisfy allconditions as shown in Fig.8.

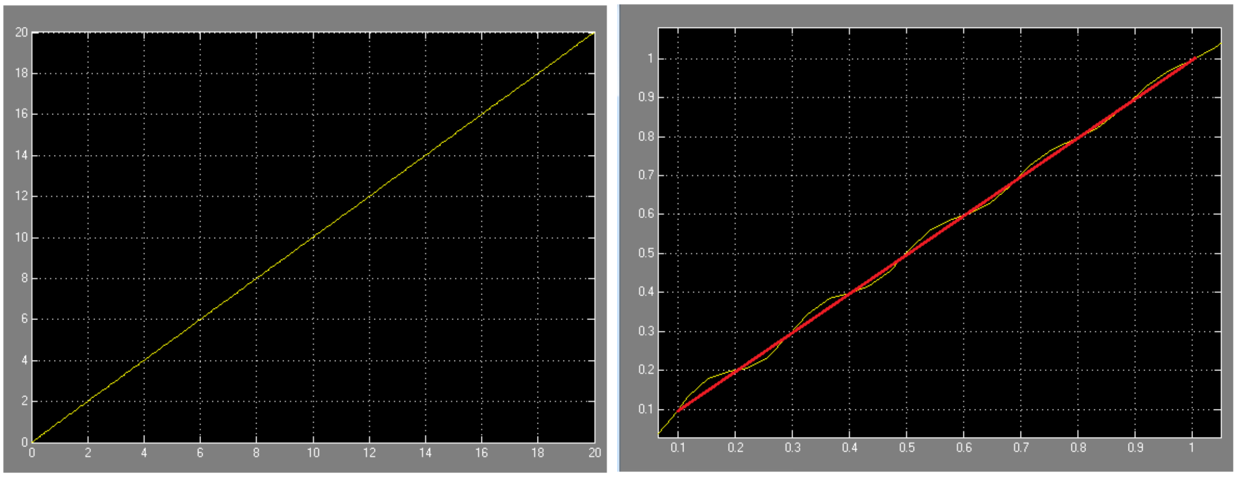


Fig.8. time response for the first system with PI controller when the input is unit ramp, $K_p=1$ and $K_I = 1001$.

Case 3: system 2 + step input + PI controller

After many itterations, the values of $K_P = 0.5$ and $K_I = 0.001$ satisfay all conditions and Fig.9 shows the time response under these values

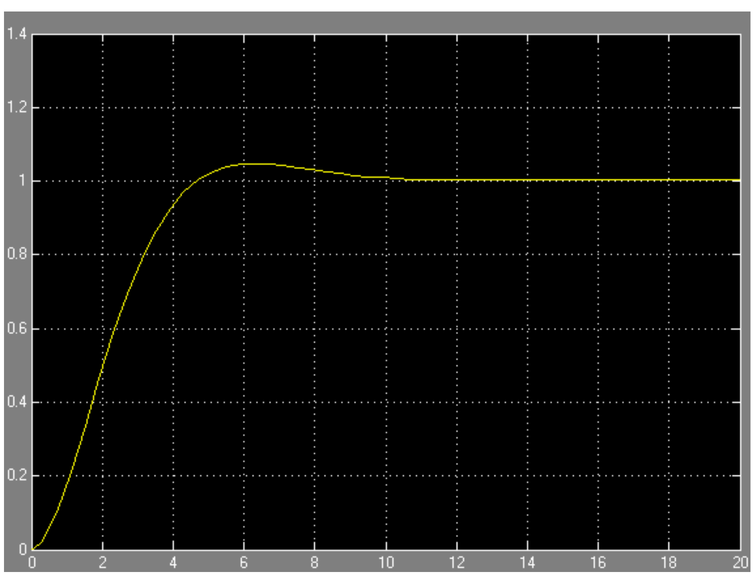


Fig.9. time response for the second system when the input is unit step, $K_p=0.5$ and $K_I = 0.001$.

Case 4: system 2 + ramp input + PI controller

After many itterations, the values of K_P = 10 and K_I = 0.001 satisfay all conditions and Fig.10 shows the time response under these values

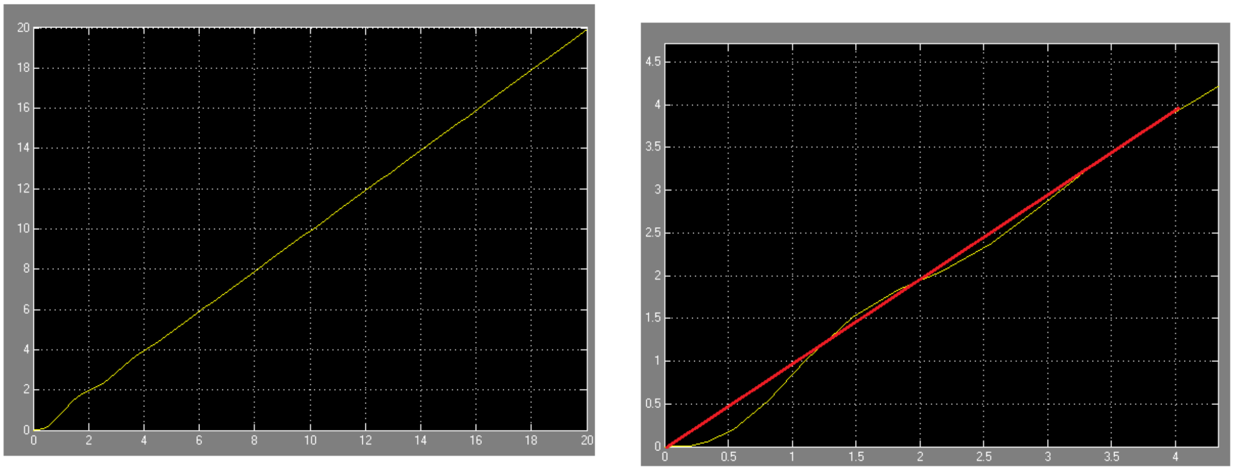


Fig.10. time response for the second system when the input is unit step, K_p=10 and K_I = 0.001.

Case 5: system 2 + parb input + PI controller

After many itterations, the values of K_P = 10000 and K_I = 1001 satisfay all conditions and Fig.11 shows the time response under these values

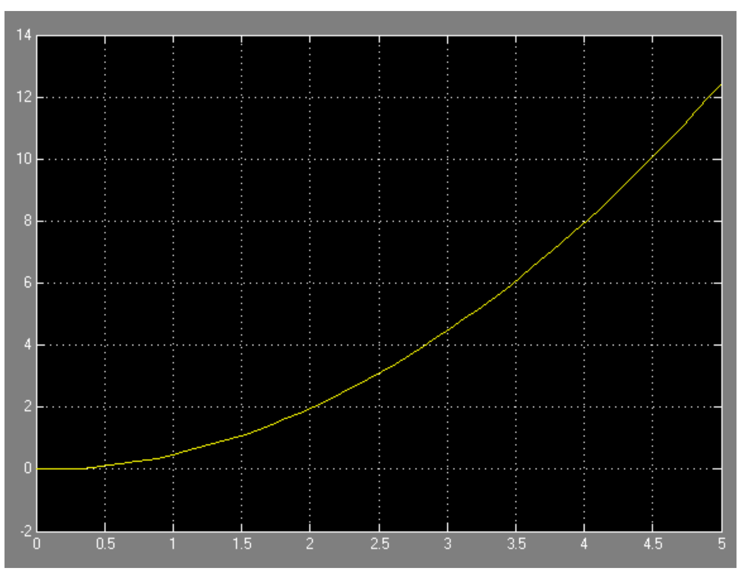


Fig.11. time response for the second system with PI controller when the input is ramp step, K_p=10000 and K_I = 1001.