Roots of Equation

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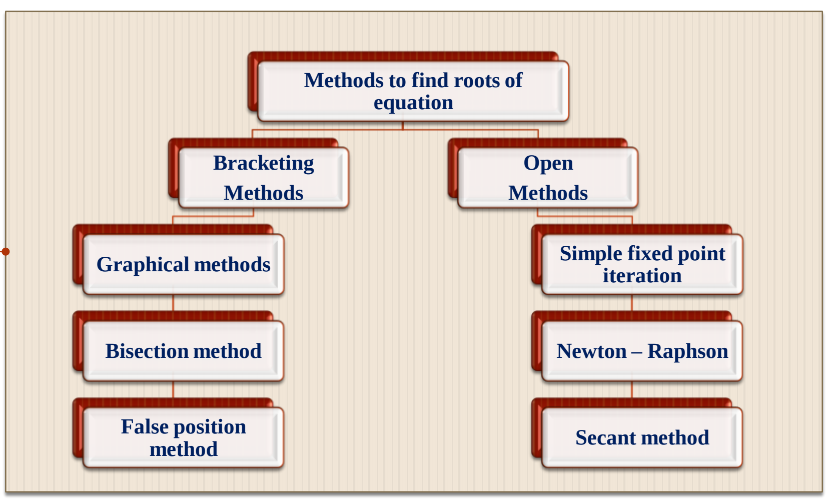
Roots of equation



Roots of equation

Bracketing Methods:

- Uses the fact that a function typically changes sign in the vicinity of a root
- Two initial guesses for the root are required

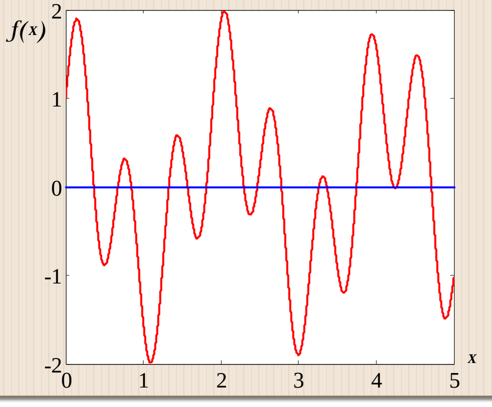
Open Methods:

- Single initial guesses for the root are required

Bracketing Methods

Graphical Methods

❖ **Example :** $f(x) = \sin(10x) + \cos(3x)$



Bracketing Methods

Graphical Methods

- **Advantages**
  - easy to conduct
- **Disadvantages**
  - Not precise enough
  - In multi – roots case such as triangular functions , it may mislead you when the chosen range of data used to draw the function passes some roots.

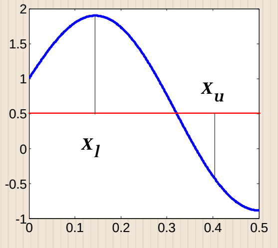
Bracketing Methods

Bisection method

- **Philosophy**

function changes its sign when it passes from one side of the root to the other side.

- **Graphical representation**



- **Mathematical translation**

$f(x_l)\,f(x_u)<0$

Bisection method

Procedures

❖ The following procedures are used to find root of equation by Bisection method :

Step 1: Chose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. You can check this out by insuring that $f(x_l)\, f(x_u)<0$

Step 2: An estimate of root x_r is determined by:

$$x_r = \frac{x_l + x_u}{2}$$

Step3: make the following evaluations to determine the next iteration:

- If $f(x_l)\, f(x_r)<0$ Set $x_u = x_r$ and return to step 2
- If $f(x_l)\, f(x_r)>0$ Set $x_l = x_r$ and return to step 2
- If $f(x_l)\, f(x_r)=0$ stop the iteration. You find the true root

Bisection method

Termination criteria

❖ To terminate the iteration, we have to define a termination criteria. Because bisection method is a numerical method and the true value is not found in typical ways, the use of approximation error is convenient to this case:

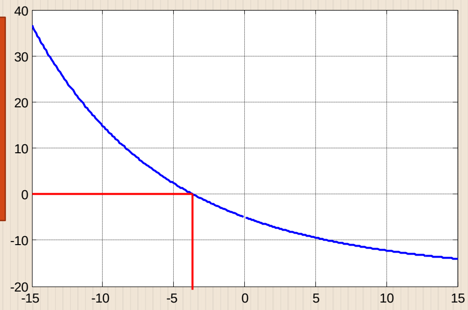
$$\varepsilon_a = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| 100\%$$

Bisection method

Example #1

- **Find the root of the following function :** $f(x)=\frac{100}{x}\left(1-e^{-0.15x}\right)-20$
- **Solution:**
- **Graphical representation**

As you can see, the exact root of the given equation is hard to find so numerical method must be used



Bisection method

Example #1 cont

- **Solution: 1st iteration**

Step 1: **assume** $x_l = -5$ **and** $x_u = 1$ $\quad (f(-5)\ f(1) < 0)$

Step2: $x_r = \frac{x_l + x_u}{2} = \frac{1+-5}{2} = -2 \Rightarrow f(-5) = 2.34,\ f(-2) = -2.51$

Step3: $f(-5)\ f(-2) < 0 \ \rightarrow x_u = x_r \rightarrow x_u = -2$

Bisection method

Example #1 cont

- **Solution: 2nd iteration**

Step 1: assume x_l = -5 and x_u =-2

Step2: $x_r = \frac{x_l + x_u}{2} = \frac{-2 + -5}{2} = -3.5 \Rightarrow f(-5) = 2.34,\ f(-3.5) = -0.273$

Step3: $f(-5)\ f(-3.5) < 0 \rightarrow x_u = x_r \rightarrow x_u = -3.5$

Bisection method

Example #1 cont

- **Solution: 3rd iteration**

Step 1: assume x_l = -5 and x_u =-3.5

Step2: $x_r = \frac{x_l + x_u}{2} = \frac{-3.5 + -5}{2} = -4.25 \Rightarrow f(-5) = 2.34,\ f(-4.25) = 0.982$

Step3: $f(-5)\ f(-4.25) > 0 \rightarrow x_l = x_r \rightarrow x_l = -4.25$

Bisection method

Example #1 cont

Solution: 4th iteration

Step 1: assume x_l = -4.25 and x_u = -3.5

Step2: $x_r = \frac{x_l + x_u}{2} = \frac{-3.5 + -4.25}{2} = -3.875 \Rightarrow f(-4.25) = 0.982,\ f(-3.875) = 0.353$

Step3: *f(-5) f(-4.25) >0* $\rightarrow x_l = x_r \rightarrow x_l$ = -3.875

Bisection method

Example #1 cont

Solution: 5th iteration

Step 1: assume x_l = -3.875 and x_u = -3.5

Step2: $x_r = \frac{x_l + x_u}{2} = \frac{-3.5 + -3.875}{2} = -3.6875 \Rightarrow f(-3.875) = 0.3425,\ f(-3.6875) = -8.4786$

Step3: *f(-5) f(-4.25) <0* $\rightarrow x_l = x_r \rightarrow x_u$ = -4.25

Bisection method

Example #1 cont

Solution: iteration summary

Iteration	x_l	x_u	x_r	ε_a
1	-5	1	-2	------
2	-5	-2	-3.5	42.86
3	-5	-3.5	-4.25	17.41
4	-4.25	-3.5	-3.875	9.68
5	-3.875	-3.5	-3.6875	5.08

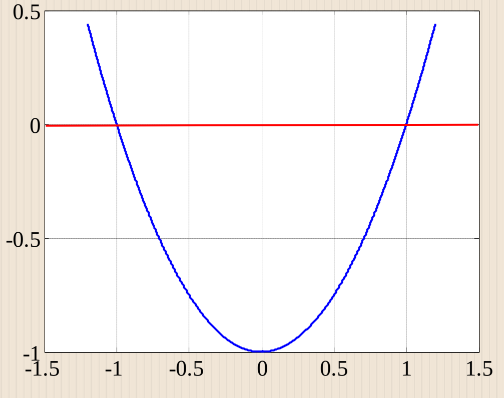
Bisection method

Example #2

Find the root of the following function: $f(x) = x^2 - 1$

- **Solution:**
- **Graphical representation**

As you can see, the roots of the given equation are -1 and +1. we will aim to find the second root and you can try to find the other root.



Bisection method

Example #2

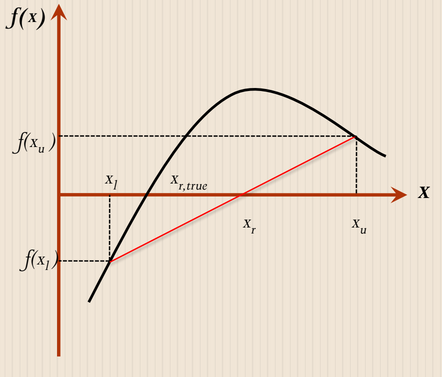
Solution: iteration summary

Iteration	x_l	x_u	x_r	ε_a	ε_t
1	0	1.2	0.6	----	40
2	0.6	1.2	0.9	33.3	10
3	0.9	1.2	1.05	14.3	5
4	0.9	1.05	0.975	7.7	2.5
5	0.975	1.05	1.0125	3.7	1.25

Bracketing Methods

False position method

- This method is based on a graphical insight
- Graphical presentation



Bracketing Methods

False position method

- This method is based on a graphical insight
- Mathematical formula

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

- Implementation :

Step 1: Chose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. You can check this out by insuring that $f(x_l)\, f(x_u)<0$

Step 2: An estimate of root x_r is determined by:

Step3: make the following evaluations to determine the next iteration:

- If $f(x_l)\, f(x_r)<0$ Set $x_u = x_r$ and return to step 2
- If $f(x_l)\, f(x_r)>0$ Set $x_l = x_r$ and return to step 2
- If $f(x_l)\, f(x_r)=0$ stop the iteration. You find the true root

False position method

Example #3

- Find the root of the following function : $f(x) = \frac{100}{x}\left(1 - e^{-0.15x}\right) - 20$
- Solution: 1st iteration

Step 1: assume x_l = -5 and x_u =1 $(f(-5)\ f(1) < 0)$

Step2: $x_r = 1 - \frac{f(1)(1 - -5)}{f(-5) - f(1)}$ = -3.4797

Step3: $f(-5)\ f(-3.4797) < 0 \rightarrow x_u = x_r \rightarrow x_u$ = -3.4797

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False position method

Example #3 cont

Solution: iteration summary

Iteration	x_l	x_u	x_r	ε_a
1	-5	1	-3.3307	------
2	-5	-3.3307	-3.6449	8.5
3	-5	-3.6449	-3.6664	0.58
4	-5	-3.6664	-3.6679	0.106
5	-5	-3.6679	-3.668	0.00272

Open Methods

Simple fixed – point iteration

- *It is also called the one-point iteration or successive iteration .*
- *It depends on change the function to be in the form :* $x = g(x)$.
- *This transformation could be done by:*
  - *Algebraic manipulation*
  - *Simply add x to both sides of equation*
- *Examples*

$$x^2 - 2x + 3 = 0 \Rightarrow x = \frac{x^2 + 3}{2} \qquad \sin(x) = 0 \Rightarrow x = \sin(x) + x$$

- *you can use any method but its preferred to try the first one and then if failed you can go to the second method*

Open Methods

Simple fixed – point iteration

❖ Methodology and error

❖ methodology :

$$x_{i+1} = g(x_i)$$

❖ Error

$$\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$$

Open Methods

Simple fixed – point iteration

❖ Example#4 :

Find the roots of the following function: $f(x) = e^{-x} - x$

❖ Solution:

step1: $e^{-x} - x = 0 \Rightarrow x = e^{-x}$ then $g(x) = e^{-x}$

step 2: define the iterative formula : $x_{i+1} = e^{-x_i}$

step 3: start iteration : assume x_i =0 then x_{i+1} = 1

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False position method

Example #4 cont

Solution: iteration summary

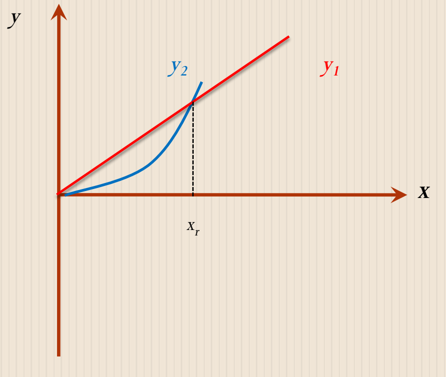
i	x_i	ε_a (%)
0	0	----
1	1.000000	100
2	0.367879	171.8
3	0.692201	46.9
4	0.500473	38.3
5	0.606244	17.4

Error increased

False position method

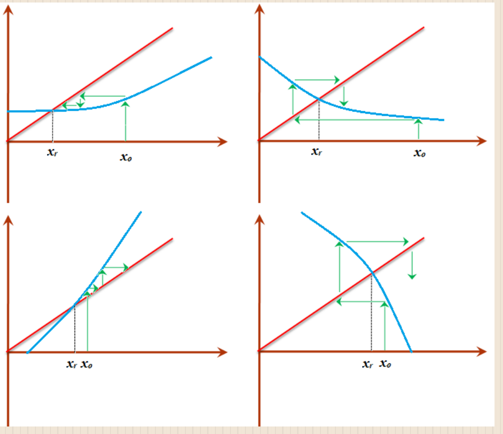
Convergence

- **Separate the iteration formula into two functions :** $y_1 = x, \ y_2 = g(x)$
- **Draw both functions and the intersect point between them will be the root:**



False position method

Convergence examples



Open Methods

Newton – Raphson Method

- Methodology and error
- methodology :

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

- Error

$$\varepsilon_a = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| 100\%$$

Open Methods

Newton – Raphson Method

❖ **Example#5 :**

Find the roots of the following function: $f(x) = e^{-x} - x$

❖ **Solution:**

step1: $f'(x) = -e^{-x} - 1$

step 2: define the iterative formula : $x_{i+1} = x_i - \dfrac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$

step 3: start iteration : assume x_i =0 then x_{i+1} = 1

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Newton – Raphson Method

Example #5 cont

Solution: iteration summary

i	x_i	ε_a (%)
0	0	----
1	0.500000000	100
2	0.566311003	11.8
3	0.567143165	0.0000220
4	0.567143290	$<10^{-8}$

Newton – Raphson Method

Pitfalls

❖ In some cases, the Newton – Raphson method shows slow rate of convergence. The following example Illustrates that:

❖ Given data $f(x) = x^{10} - 1$ $\quad x_0 = 0.5$

Solution $x_{i+1} = x_i - \dfrac{x_i^{10} - 1}{10x_i^9}$

i	x_i
0	0.5
1	51.65
2	46.486
3	41.8362
4	37.65285
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∞	1.0000000

Open Methods

Secant method

❖ Methodology and error

❖ Methodology :

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

❖ Notes:

- ❖ Although two initial guesses are required but it is still open method because it dose not need change in sign in the function like the bracketing methods
- ❖ Its derived by substitution of the first derivative evaluated from the backward divided difference into the Newton – Raphson formula

❖ Error $\quad \varepsilon_a = \left| \dfrac{x_{i+1} - x_i}{x_{i+1}} \right| 100\%$

Open Methods

Secant method

❖ Example#6 :

Find the roots of the following function: $f(x) = e^{-x} - x$

❖ Solution:

step1: define the initial iterations x_{i-1} and x_i.

$x_{-1} = 0 \rightarrow f(x_{-1}) = 1.000000$

$x_0 = 1 \rightarrow f(x_0) = -0.63212$

step2: $$x_1 = 1 - \frac{-0.63212(0-1)}{1-(-0.63212)} = 0.61270$$

Open Methods

Secant method

❖ Example#6 cont :

❖ 2nd iteration :

step3: define the 2nd iterations x_{i-1} and x_i.

$x_0 = 1 \rightarrow f(x_0) = -0.63212$

$x_1 = 0.61270 \rightarrow f(x_1) = -0.07081$

step4: $$x_2 = 0.61270 - \frac{-0.0708(1-0.61270)}{-0.63212-(-0.0708)} = 0.56384$$

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