Linear Algebraic Equations

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Linear algebraic equations

Matrices

- **matrix notation**

$$A=\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & . & . & . & a_{mn} \end{bmatrix}$$

Linear algebraic equations

Special cases of Matrices

- **Vector matrix**

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ a_{m1} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \end{bmatrix}$$

- **Square matrix (n=m)**

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & a_{2m} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & . & . & . & a_{mm} \end{bmatrix}$$

Special cases of Matrices

Square matrix

- **Symmetric matrix** ($a_{ij} = a_{ji}$)
  - $a_{12} = a_{21}$
  - $a_{13} = a_{31}$
  - $a_{23} = a_{32}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- **diagonal matrix**
  - $a_{ij} = 0\ ;\ i \neq j$
  - $a_{ij} \neq 0\ ;\ i = j$

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

Special cases of Matrices

Square matrix

- **Identity matrix**
  - $a_{ij} = 0 \; ; \; i \neq j$
  - $a_{ij} \neq 1 \; ; \; i = j$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Upper and lower triangular matrices**

Upper matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Lower matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Special cases of Matrices

Square matrix

- **Bounded matrix**

This example has a bandwidth equal 3

Bandwidth

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

Linear algebraic equations

Matrices operations

Addition

[C] = [A]+[B] then $c_{ij} = a_{ij} + b_{ij}$
[A]+[B] = [B] +[A]
([A]+[B]) +[C] = [A] +([B] + [C])

Multiplication

([A] [B]) [C] = [A] ([B] [C])
[A] ([B]+[C])=[A][B]+[A][C]
[A][B]≠ [B][A] unless
$[A][A]^{-1}=[A]^{-1}[A]=[I]$

Multiplication

[D] = g[A]+[B]

$$g[A]=\begin{bmatrix} ga_{11} & ga_{12} & ga_{13} \\ ga_{21} & ga_{22} & ga_{23} \\ ga_{31} & ga_{32} & ga_{33} \end{bmatrix}$$

[C] = [A] [B] ; $c_{ih} = \sum_{k=1}^{n} a_{ik} b_{kj}$

$$c_{12} = (a_{11}b_{12}) + (a_{12}b_{22}) + (a_{13}b_{32})$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

Linear algebraic equations

Matrices operations

Matrix Determent |A|

$$\begin{matrix} + & - & + \end{matrix}$$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Linear algebraic equations

Matrices operations

Matrix transpose and inverse

- **Transpose :** $[A]_{ij} = [A]_{ji}$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow [A]^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{32} & a_{33} \end{bmatrix}$$

- **Inverse** $[A]^{-1}$ =

$$[A]^{-1} = \frac{1}{det(A)}[cofactor\ matrix\ of\ A]^T$$

Linear algebraic equations

Matrices operations

Matrix inverse

- **Adjoint method**

$$[A]^{-1} = \frac{1}{det(A)}[cofactor\ matrix\ of\ A]^T$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow [cofactor\ matrix\ of\ A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, A_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\ and\ so\ on$$

$$[cofactor\ matrix\ of\ A] = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Linear algebraic equations

Terminology

Transformation to matrix notation

- the given equations are in form of and transferred to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\;\;\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$

$$[A]\{x\} = \{B\}$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{bmatrix} \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{Bmatrix} \quad \{B\} = \begin{Bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{Bmatrix}$$

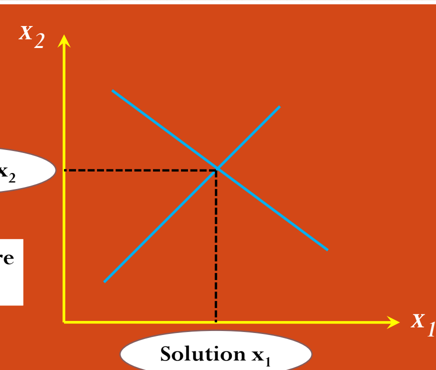
Linear algebraic equations

Solution methods

Graphical method

- example :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$



Can not be applied for more than three equations

Linear algebraic equations

Solution methods

Cramer's rule

$$[A]=\begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & . & . & . & a_{nn} \end{bmatrix} \quad \{x\}=\begin{Bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{Bmatrix} \quad \{B\}=\begin{Bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{Bmatrix}$$

$$x_1=\frac{\begin{vmatrix} b_{11} & a_{12} & . & . & a_{1n} \\ b_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{n1} & . & . & . & a_{nn} \end{vmatrix}}{|A|} \quad x_2=\frac{\begin{vmatrix} a_{11} & b_{12} & . & . & a_{1n} \\ a_{21} & b_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & b_{n2} & . & . & a_{nn} \end{vmatrix}}{|A|} \quad x_n=\frac{\begin{vmatrix} a_{11} & a_{12} & . & . & b_{1n} \\ a_{21} & a_{22} & . & . & b_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & b_{nn} \end{vmatrix}}{|A|}$$

Linear algebraic equations

Cramer's rule

Example

$$x_1+2x_2+x_3=13............Eq.1$$
$$2x_1+2x_2+4x_3=30.........Eq.2$$
$$x_1+3x_2+2x_3=21...........Eq.3$$

$$[A]=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \\ 1 & 3 & 2 \end{bmatrix} \quad \{x\}=\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \quad \{B\}=\begin{Bmatrix} 13 \\ 30 \\ 21 \end{Bmatrix}$$

$$x_1=\frac{\begin{vmatrix} 13 & 2 & 1 \\ 30 & 2 & 4 \\ 21 & 3 & 2 \end{vmatrix}}{|A|} \quad x_2=\frac{\begin{vmatrix} 1 & 13 & 1 \\ 2 & 30 & 4 \\ 1 & 21 & 2 \end{vmatrix}}{|A|} \quad x_3=\frac{\begin{vmatrix} 1 & 2 & 13 \\ 2 & 2 & 30 \\ 1 & 3 & 21 \end{vmatrix}}{|A|}$$

Linear algebraic equations

Solution methods

Elimination of unknowns

❖**Forward elimination**

$$(a_{11}x_1 + a_{12}x_2 = b_1) * a_{21} \qquad a_{11}a_{21}x_1 + a_{12}a_{21}x_2 = a_{21}b_1$$

$$(a_{21}x_1 + a_{22}x_2 = b_2) * a_{11} \qquad a_{21}a_{11}x_1 + a_{22}a_{11}x_2 = a_{11}b_2$$

Subtract (2) from (1): $a_{22}a_{11}x_2 - a_{12}a_{21}x_2 = a_{11}b_2 - a_{21}b_1$

And so: $x_2 = \dfrac{a_{11}b_2 - a_{21}b_1}{a_{22}a_{11} - a_{12}a_{21}}$

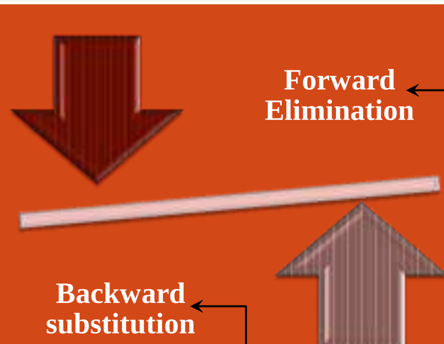
❖**Backward substitution:** $x_1 = \dfrac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}}$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad and \quad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Linear algebraic equations

Solution methods

Naive Gauss Elimination



Forward Elimination

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Backward substitution

$$x_3 = b''_3 / a''_{33}$$
$$x_2 = (b'_2 - a'_{23}x_3) / a'_{22}$$
$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3) / a_{11}$$

Linear algebraic equations

Naive Gauss Elimination

Example

$$\begin{aligned} x_1+2x_2+x_3&=13\text{.......... Eq.1}\\ 2x_1+2x_2+4x_3&=30\text{........ Eq.2}\\ x_1+3x_2+2x_3&=21\text{.......... Eq.3}\end{aligned} \Rightarrow \left[\begin{array}{ccc|c}1&2&1&13\\2&2&4&30\\1&3&2&21\end{array}\right]$$

Multiply Eq.1 by(2) and subtract Eq.2 from it

$$\begin{aligned} x_1+2x_2+x_3&=13\\ 0x_1+2x_2-2x_3&=-4\\ x_1+3x_2+2x_3&=21\end{aligned} \Rightarrow \left[\begin{array}{ccc|c}1&2&1&13\\0&2&-2&-4\\1&3&2&21\end{array}\right]$$

Subtract Eq.3 from Eq.1

$$\begin{aligned} x_1+2x_2+x_3&=13\\ 0x_1+2x_2-2x_3&=-4\\ 0x_1+3x_2+2x_3&=21\end{aligned} \Rightarrow \left[\begin{array}{ccc|c}1&2&1&13\\0&2&-2&-4\\0&1&1&8\end{array}\right]$$

Multiply Eq.3 by(2) and subtract it from Eq.1

$$\begin{aligned} x_1+2x_2+x_3&=13\\ 0x_1+2x_2-2x_3&=-4\\ 0x_1+0x_2+2x_3&=21\end{aligned} \Rightarrow \left[\begin{array}{ccc|c}1&2&1&13\\0&2&-2&-4\\0&0&-4&-20\end{array}\right]$$

Back substitution

$$x_3=-20/-4=5$$
$$x_2=\left(-4-(-2)(5)\right)/2=3$$
$$x_1=\left(13-(2)(3)-(1)(5)\right)/1=2$$

Verification

Linear algebraic equations

Gauss – Jordan Method

Methodology

$$\left[\begin{array}{ccc|c}a_{11}&a_{12}&a_{13}&b_1\\a_{21}&a_{22}&a_{23}&b_2\\a_{31}&a_{32}&a_{33}&b_3\end{array}\right] \quad \left[\begin{array}{ccc|c}1&0&0&B_1\\0&1&0&B_2\\0&0&1&B_3\end{array}\right] \quad \begin{aligned}x_1&=B_1\\x_2&=B_2\\x_3&=B_3\end{aligned}$$

❖ this is done by :

1. Eliminating x_1 from Eqs 2 & 3
2. Eliminating x_2 from Eqs 1& 3
3. Eliminating x_3 from Eqs 1 & 2

Linear algebraic equations

Gauss – Jordan Method

Example

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 13 \text{............ } Eq.1 \\ 2x_1 + 2x_2 + 4x_3 &= 30 \text{......... } Eq.2 \Rightarrow \\ x_1 + 3x_2 + 2x_3 &= 21 \text{.......... } Eq.3 \end{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 13 \\ 2 & 2 & 4 & 30 \\ 1 & 3 & 2 & 21 \end{array}\right]$$

❖**Solution**

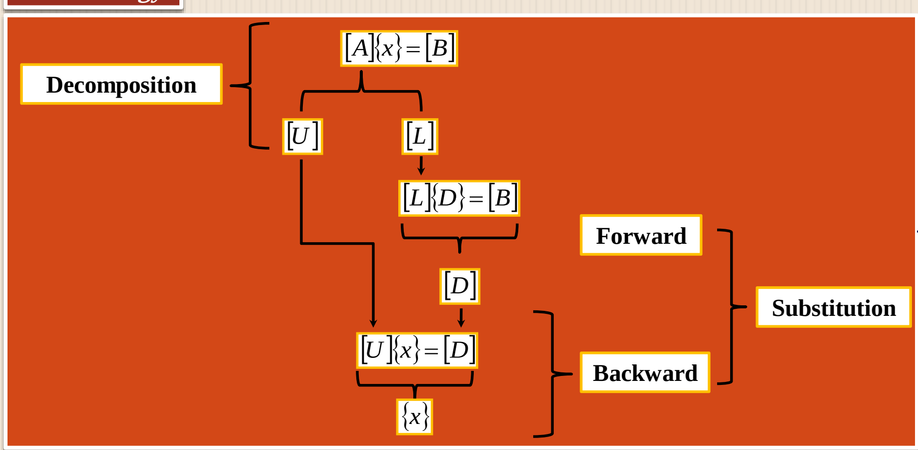
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 13 \\ 2 & 2 & 4 & 30 \\ 1 & 3 & 2 & 21 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 13 \\ 0 & 2 & -2 & -4 \\ 0 & 1 & 1 & 8 \end{array}\right] 2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -17 \\ 0 & 2 & -2 & -4 \\ 0 & 1 & 1 & 8 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -17 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 8 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -17 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -2 & -10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -17 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -17 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Linear algebraic equations

LU- Decomposition method

Methodology



Linear algebraic equations

LU- Decomposition

Decomposition phase

Decomposition

$$[A]\{x\}=[B]$$

$[U]$ and $[L]$

1

$$[U]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

2

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ f_{21} & 1 & 0 \\ f_{31} & f_{32} & 1 \end{bmatrix}$$

$$f_{21}=\frac{a_{21}}{a_{11}} \qquad f_{31}=\frac{a_{31}}{a_{11}} \qquad f_{32}=\frac{a'_{32}}{a'_{22}}$$

Linear algebraic equations

LU- Decomposition

Substitution phase

$$[L]\{D\}=[B]$$

$$[D]$$

$$[U]\{x\}=[D]$$

$$\{x\}$$

Linear algebraic equations

LU- Decomposition

Detailed procedures

❖ decomposition phase

- ❖ to find [U]:
  - ❖ multiply Eq1 by the factor (a_{21}/a_{11})
  - ❖ Subtract Eq2 from Eq1 to eliminate x_1 from Eq2
  - ❖ multiply Eq1 by the factor (a_{31}/a_{11})
  - ❖ Subtract Eq3 from Eq1 to eliminate x_1 from Eq3
  - ❖ At this point, you eliminate x_1 from Eqs 1 and 2 and the matrix will be in the following form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix}$$

  - ❖ Save the values of a'_{22} and a'_{23} to use it later

Linear algebraic equations

LU- Decomposition

Detailed procedures

❖ decomposition phase

- ❖ continuo to find [U]:
  - ❖ multiply Eq2 by the factor (a'_{32}/a'_{22})
  - ❖ Subtract Eq3 from Eq2 to eliminate x_2 from Eq3
  - ❖ At this point, you eliminate x_2 from Eq 2 and the matrix will be in the following form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}$$

  - ❖ This matrix is called the upper matrix

Linear algebraic equations

LU- Decomposition

Detailed procedures

❖ decomposition phase

❖to find [L]:

$$[L]=\begin{bmatrix}1 & 0 & 0\\ f_{21} & 1 & 0\\ f_{31} & f_{32} & 1\end{bmatrix}$$

$$f_{21}=\frac{a_{21}}{a_{11}} \qquad f_{31}=\frac{a_{31}}{a_{11}} \qquad f_{32}=\frac{a'_{32}}{a'_{22}}$$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \qquad \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ 0 & a'_{22} & a'_{23}\\ 0 & a'_{32} & a'_{33}\end{bmatrix}$$

Linear algebraic equations

LU- Decomposition

Detailed procedures

❖ forward substitution phase

❖ arrange the matrices to be like: $[L]\{D\}=\{B\}$

$$\begin{bmatrix}1 & 0 & 0\\ f_{21} & 1 & 0\\ f_{31} & f_{32} & 1\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix}=\begin{Bmatrix}b_1\\ b_2\\ b_3\end{Bmatrix}\Rightarrow \begin{matrix}d_1=b_1\\ d_2=b_2-(f_{21}d_1)\\ d_3=b_3-(f_{31}d_1+f_{32}d_2)\end{matrix}$$

❖ backward substitution phase

❖ arrange the matrices to be like: $[U]\{x\}=[D]$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ 0 & a'_{22} & a'_{23}\\ 0 & 0 & a''_{33}\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix}=\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix}\Rightarrow \begin{matrix}x_2=(d_1-a_{12}x_2-a_{13}x_3)/a_{11}\\ x_2=(d_2-a'_{23}x_3)/a'_{22}\\ x_3=d_3/a''_{33}\end{matrix}$$

Linear algebraic equations

LU- Decomposition

Example

$$\begin{aligned}3x_1 - 0.1x_2 - 0.2x_3 &= 7.85\\ 0.1x_1 + 7x_2 - 0.3x_3 &= -19.3\\ 0.3x_1 - 0.2x_2 + 10x_3 &= 71.4\end{aligned} \Rightarrow [A] = \begin{bmatrix}3 & -0.1 & -0.2\\ 0.1 & 7 & 0.3\\ 0.3 & -0.2 & 10\end{bmatrix}, \{B\} = \begin{Bmatrix}7.85\\ -19.3\\ 71.4\end{Bmatrix}$$

Decomposition phase

$$[U] = \begin{bmatrix}3 & -0.1 & -0.2\\ 0 & 7.00333 & -0.293333\\ 0 & 0 & 10.0120\end{bmatrix} \quad [L] = \begin{bmatrix}1 & 0 & 0\\ 0.0333333 & 1 & 0\\ 0.100000 & -0.0271300 & 1\end{bmatrix}$$

$$f_{21} = \frac{0.1}{3} = 0.0333333$$

$$f_{31} = \frac{0.3}{3} = 0.1000000$$

$$f_{32} = \frac{-0.19}{7.00333} = -0.0271300$$

$$[L][U] = \begin{bmatrix}3 & -0.1 & -0.2\\ 0.1 & 7 & 0.3\\ 0.3 & -0.2 & 10\end{bmatrix}$$

Verification

Linear algebraic equations

LU- Decomposition

Example

Substitution phase

$$[L]\{D\} = \{B\}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0.0333333 & 1 & 0\\ 0.100000 & -0.0271300 & 1\end{bmatrix}\begin{Bmatrix}d_1\\ d_2\\ d_3\end{Bmatrix} = \begin{Bmatrix}7.85\\ -19.3\\ 71.4\end{Bmatrix} \Rightarrow \begin{aligned}d_1 &= 7.85\\ d_2 &= -19.5617\\ d_3 &= 70.0843\end{aligned}$$

Forward substitution

$$[U]\{x\} = \{D\}$$

$$\begin{bmatrix}3 & -0.1 & -0.2\\ 0 & 7.00333 & -0.293333\\ 0 & 0 & 10.0120\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix} = \begin{Bmatrix}7.85\\ -19.5617\\ 70.0843\end{Bmatrix} \Rightarrow \begin{aligned}x_1 &= 3\\ x_2 &= -2.5\\ x_3 &= 7\end{aligned}$$

Backward substitution