Numerical integration

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Numerical integration

Newton – Cotes formulas

When to use

❖We use the numerical methods instead of the well known analytical methods when :

1. **The function is none integrable**
2. **When the independent values are found as discrete points (eg. Experimentally).**

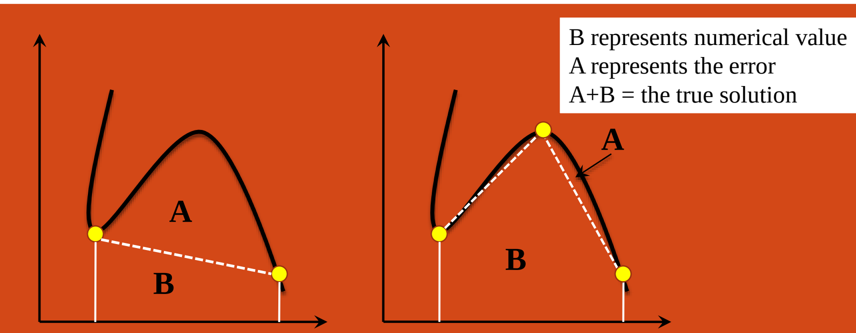
❖ General formula:

$$I = \int_a^b f(x).dx \cong \int_a^b f_n(x).dx \;\; ; \; f_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Numerical integration

Newton – Cotes formulas

Graphical representation



Numerical integration

Newton – Cotes formulas

Closed and open forms

- **Closed form: the form where the beginning and end of limits of integration are known**
- **Open forms: the beginning and end of limits of integration are unknown**

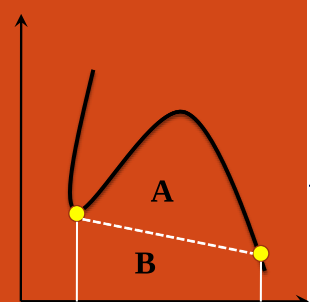
Numerical integration

Trapezoidal rule

Methodology and formula

$$I \cong (b-a)\frac{f(a)+f(b)}{2}$$

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$



Numerical integration

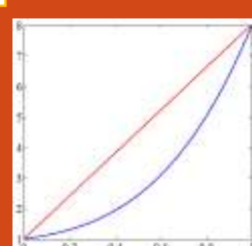
Trapezoidal rule

Example

$$f(x) = x^4 + 3x^3 + 2x^2 + x + 1 \;;\; a = 0 \;\; b = 1$$

- **Solution:**
- **Exact solution :** $\int_0^1 x^4 + 3x^3 + 2x^2 + x + 1.dx = 187/60 = 3.1166667$
- **Trapezoidal :**

$$f(0) = 1$$
$$f(1) = 8$$
$$I = (1-0)\frac{f(1)+f(0)}{2} = \frac{9}{2} = 4.5$$



Numerical integration

Trapezoidal rule

***Example** cont*

- error

$$f''(x) = 12x^2 + 18x + 4$$

$$f''(\xi) = \frac{\int_a^b f''(x).dx}{b-a} = \frac{\int_0^1 12x^2 + 18x + 4.dx}{1-0} = 30$$

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3 = -\frac{1}{12}(30)(1)^3 = -2.5$$

Numerical integration

Trapezoidal rule

***Example** cont*

- alternative method to find the error

$$f''(x) = 12x^2 + 18x + 4$$

$$\xi = \frac{1-0}{2} = 0.5 \Rightarrow mid\ point$$

$$f''(0.5) = 12(0.5)^2 + 18(0.5) + 4 = 16$$

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3 = -\frac{1}{12}(16)(1)^3 = -1.3333$$

Numerical integration

Trapezoidal rule

Multiple application

$$I = (b-a)\frac{f(x_0) + \sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}$$

$$E_a = -\frac{(b-a)^3}{12n^2}\bar{f}''$$

Numerical integration

Trapezoidal rule

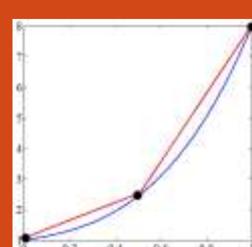
Example: Multiple application

$f(x) = x^4 + 3x^3 + 2x^2 + x + 1\ ;\ a = 0\ \ b = 1$

- ❖ **Solution:**
- ❖ **Assume n=2 → h = 0.5**

$$f(0) = 1$$
$$f(0.5) = 2.4375$$
$$f(1) = 8$$
$$I = (1-0)\frac{f(0) + 2f(0.5) + f(1)}{(2)(2)} = 3.4688$$

$$\varepsilon_t = \left|\frac{3.1166667 - 3.4688}{3.1166667}\right| 100\% = 11.2968$$



Numerical integration

Trapezoidal rule

Example: Multiple application

$f(x) = x^4 + 3x^3 + 2x^2 + x + 1\ ;\ a = 0\ \ b = 1$

❖ **Solution:**

$f(0) = 1$

$f(0.25) = 1.4258$

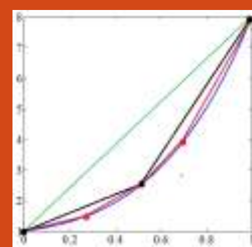
$f(0.5) = 2.4375$

$f(0.75) = 4.4570$

$f(1) = 8$

$I = (1-0)\dfrac{f(0) + 2(f(0.25) + f(0.5) + f(0.75)) + f(1)}{(2)(4)} = 3.2050$

$\varepsilon_t = \left|\dfrac{3.1166667 - 3.2050}{3.1166667}\right| 100\% = 2.836$



Numerical integration

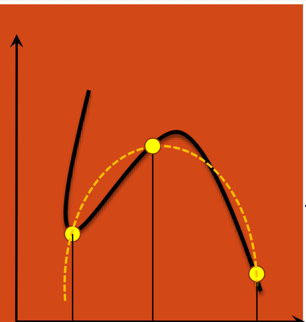
Simpson's rule s

1/3 Simpsons rule

❖ **Second order polynomial**

$$I \cong (b-a)\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$



Numerical integration

Simpson's rule s

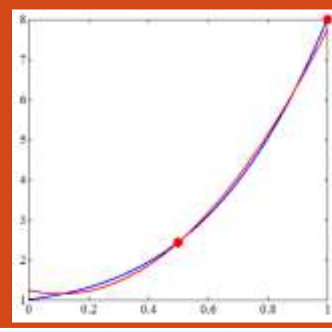
1/3 Simpsons rule

$$f(x) = x^4 + 3x^3 + 2x^2 + x + 1 \; ; \; a = 0 \;\; b = 1$$

❖ **1/3 Simpsons rule :**

$$f(0) = 1$$
$$f(0.5) = 2.4375$$
$$f(1) = 8$$
$$I = (1-0)\frac{f(0) + 4f(0.5) + f(1)}{6} = 3.125$$

$$\varepsilon_t = \left|\frac{3.1166667 - 3.125}{3.1166667}\right| 100\% = 0.2673$$



Numerical integration

1/3 Simpsons rule

Multiple application

$$I = (b-a)\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}$$

$$E_a = -\frac{(b-a)^5}{180n^4}\,\bar{f}^{(4)}$$

Numerical integration

1/3 Simpsons rule

Example: Multiple application

$f(x) = x^4 + 3x^3 + 2x^2 + x + 1\ ;\ a = 0\ \ b = 1$

- **Solution:**
- **Assume n=4 → h = 0.25**

$f(0) = 1$

$f(0.25) = 1.4258$

$f(0.5) = 2.4375$

$f(0.75) = 4.4570$

$f(1) = 8$

$$I \cong (1-0)\frac{f(0)+4(f(0.25)+f(0.75))+(2)f(0.5)+f(1)}{(3)(4)} = 3.1171833$$

$$\varepsilon_t = \left|\frac{3.1166667-3.1171833}{3.1166667}\right|100\% = 0.0165$$

Numerical integration

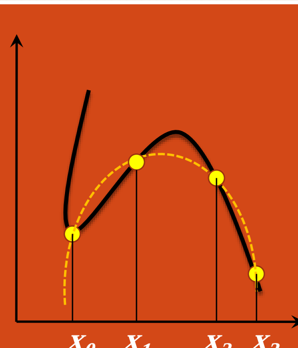
Simpson's rule s

3/8 Simpsons rule

- **Second order polynomial**

$$I \cong (b-a)\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$$

$$E_t = -\frac{(b-a)^5}{6480}f^{(4)}(\xi)$$



Numerical integration

Simpson's rule s

3/8 Simpsons rule

$$f(x) = x^4 + 3x^3 + 2x^2 + x + 1 \; ; \; a = 0 \;\; b = 1$$

❖**1/3 Simpsons rule :**

$$f(0) = 1$$
$$f(0.33333) = 1.679$$
$$f(0.66667) = 3.642$$
$$f(1) = 8$$
$$I \cong (1-0)\frac{1 + (3)(1.679) + (3)(3.642) + 8}{8} = 3.1204$$

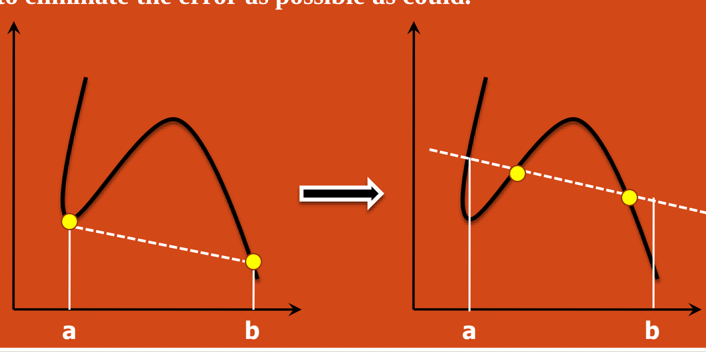
$$\varepsilon_t = \left|\frac{3.1166667 - 3.1204}{3.1166667}\right| 100\% = 0.119$$

Numerical integration

Gauss Quadrature

What is Gauss Quadrature

❖ **it is a method used to locate the negative and positive errors in balancing manner to eliminate the error as possible as could.**



Numerical integration

Gauss Quadrature

Methodology

❖Method of undetermined coefficients:

Trapezoidal rule implies that $I \cong (b-a)\frac{f(a)+f(b)}{2}$ which can be represented as:

Where: c_0 and c_1 are constants

$$I \cong c_0 f(a) + c_1 f(b)$$

Trapoziodal rule results an exact solution when the function needed to be integrated is constant ($y = 1$) or linear ($y = x$). So:

$$c_0 + c_1 = \int_{-(b-a)/2}^{(b-a)/2} 1.dx$$

$$-c_0\frac{(b-a)}{2} + c_1\frac{(b-a)}{2} = \int_{-(b-a)/2}^{(b-a)/2} x.dx$$

Numerical integration

Gauss Quadrature

Methodology

❖Method of undetermined coefficients cont:

Evaluate these integrals to have:

$$c_0 + c_1 = b - a$$

$$-c_0\frac{(b-a)}{2} + c_1\frac{(b-a)}{2} = 0$$

These are two equations with two unknowns : c_0 and c_1 and can be solved:

$$c_0 = c_1 = \frac{b-a}{2}$$

Numerical integration

Gauss Quadrature

Methodology

❖**Method of undetermined coefficients** [cont]**:**

❖ **the *Gauss Quadrature* aims to find the undetermined coefficients of the following equation:**

$$I \cong c_0 f(x_0) + c_1 f(x_1)$$

❖**In contrast with trapezoidal rule, this method has four unknowns: c_0, c_1, x_0 and x_1. this integral is fitted to parabolic function ($y=x^2$) and cubic function ($y=x^3$)**

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} 1.dx = 2$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x.dx = 0$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^2.dx = \frac{2}{3}$$

$$c_0 f(x_0) + c_1 f(x_1) = \int_{-1}^{1} x^3.dx = 0$$

$$c_0 = c_1 = 1$$

$$x_0 = -1/\sqrt{3}$$

$$x_1 = 1/\sqrt{3}$$

$$I \cong f(1/\sqrt{3}) + f(-1/\sqrt{3})$$