Ordinary Differential equations

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Ordinary Differential equations

General review

What is ODE

❖ is a equation that involves the notation of ordinary derivative *dy/dx* and the function *y* is only win terms of *x.*

❖Examples :

1. Linear equations which are differential equations have function (y) and its derivatives ($y^{(n)}$) to power one:

$$a_n(x)\frac{d^n y}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_1(x)\frac{dy}{dx}+a_0 y$$

2. Non linear equations are otherwise.
3. The numerical methods are used when the ODE can not be solved analytically.
4. The main method to solve ODE is Runge – Kutta methods

Ordinary Differential equations

Rung – Kutta Methods

Methodology

- **General formula** $y_{i+1} = y_i + \phi(x_i, y_i, h)h$

- **Where ϕ is incremental function and is given as:** $\phi = a_1k_1 + a_2k_2 + \ldots + a_nk_n$ **a's are constants and k's are given as:**

$$
\begin{gathered}
k_1 = f(x_i, y_i) \\
k_2 = f(x_i + p_1h, y_i + q_{11}k_1h) \\
k_3 = f(x_i + p_2h, y_i + q_{21}k_1h + q_{22}k_2h) \\
. \\
. \\
. \\
k_n = f(x_i + p_{n-1}h, y_i + q_{n-1,1}k_1h + q_{n-1,2}k_2h + \ldots + q_{n-1,n-1}k_{n-1}h)
\end{gathered}
$$

Where: q's and p's are constants

Remember:
What we need to do is to solve the ODE which means finding the function *y(x).*

Runge– Kutta Methods

First order

Euler's method

- **Euler method is the 1st order Runge – Kutta methods**

$$y_{i+1} = y_i + f(x_i, y_i)h$$

- **at this stage, we find the value of y in iterative technique.**

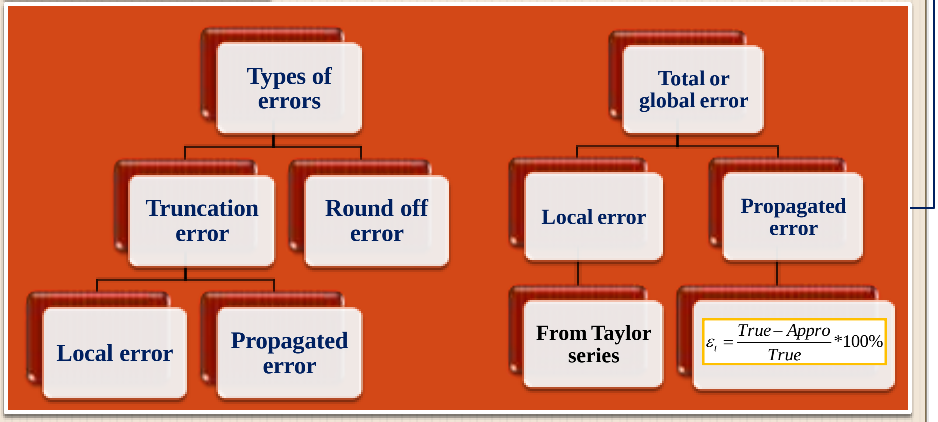
$$
\begin{gathered}
y_1 = y_0 + f(x_0, y_0)h \\
y_2 = y_1 + f(x_1, y_1)h \\
y_3 = y_2 + f(x_2, y_2)h \\
. \\
. \\
. \\
y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h
\end{gathered}
$$

As you can see, you need to know initial independent value (x0) and the dependent value at this point (y0). For that, this kind problems are called initial value problem because it needs initial values to solve it. In addition, higher order methods will show the need for more initial values (conditions)

Runge– Kutta Methods

Euler's method

Error analysis



Types of errors

Truncation error

Round off error

Local error

Propagated error

Total or global error

Local error

Propagated error

From Taylor series

$\varepsilon_t = \frac{True - Appro}{True} *100\%$

Runge– Kutta Methods

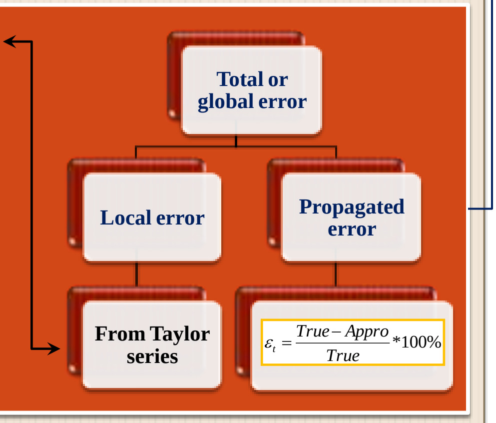
Euler's method

Error analysis

$$\boxed{y_{i+1} = y_i + y_i^{'} h} + \frac{y_i^{''}}{2!} h^2 + \ldots + \frac{y_i^{(n)}}{n!} h^n$$

$$E_t = \frac{y_i^{''}}{2!} h^2 + \frac{y_i^{''}}{3!} h^3 + \ldots + \frac{y_i^{(n)}}{n!} h^n$$

Note that if the function is a polynomial, the calculations of local error returned with finite number of terms (equal n-1). For this case, error is found exactly. However, for other case, the error depends on number of terms involved in the calculations.



Total or global error

Local error

Propagated error

From Taylor series

$\varepsilon_t = \frac{True - Appro}{True} *100\%$

Runge– Kutta Methods

Euler’s method

Example

❖**Integrate the following equation**

$$\frac{dy}{dx} = y\left(x^2 - 1.1\right)$$

over the interval from 0.0 to 2.0 to find the function y over this interval. The initial condition is y(0)=1 and take the step size (h) = 0.5 .

Runge– Kutta Methods

Euler’s method

Example

❖**Solution:**

❖**Exact solution:**

$$\frac{dy}{y} = \left(x^2 - 1.1\right)dx \Rightarrow \int \frac{1}{y}.dy = \int \left(x^2 - 1.1\right)dx$$

$$\Rightarrow \ln(y) = \frac{x^3}{3} - 1.1x \Rightarrow y = e^{\left(\frac{x^3}{3} - 1.1x\right)}$$

i	x_i	y_t
0	0.0	1.00000
1	0.5	0.60149
2	1.0	0.46455
3	1.5	0.59155
4	2.0	1.59466

Runge– Kutta Methods

Euler's method

Example

❖Solution:

❖Euler method:

❖***f(x$_i$,y$_i$):*** $f(x_i, y_i) = y_i(x_i^2 - 1.1)$

❖**1st step size : 0.0 → 0.5:**

$$y_1 = y_0 + f(x_0, y_0)h$$
$$y(0.5) = y(0.0) + f(0.0, 1.0)0.5$$
$$f(0,1) = 1(0^2 - 1.1) = -1.1$$
$$\Rightarrow y(0.5) = 1 + (-1.1)0.5 = 0.45$$

❖ **error analysis: 1st , global error**

ε_t = (true-approximate) / true*100%

= (0.60149 -0.45)/0.45 = 25.1

This is the local relative error

$$E_t = \frac{y_i''}{2!}h^2 = \frac{d^2y_i/dx_i^2}{2!}h^2$$
$$\frac{d^2y_i}{dx_i^2} = 2x_iy_i + (x_i^2 - 1.1)\frac{dy_i}{dx_i} = 2x_iy_i + (x_i^2 - 1.1)f(x_i, y_i)$$
$$\Rightarrow \frac{d^2y_0}{dx_0^2} = 2x_0y_0 + (x_0^2 - 1.1)f(x_0, y_0)$$
$$\Rightarrow \frac{d^2y_0}{dx_0^2} = 2(0)(1) + ((0)^2 - 1.1)(-1.1) = 1.21$$
$$\Rightarrow E_t = \frac{1.21}{2!}0.5^2 = 0.15125$$
$$\Rightarrow \varepsilon_t = \frac{0.15125}{0.60149} * 100\% = 25.1\%$$

Runge– Kutta Methods

Euler's method

Example

❖Summary of calculations

i	x_i	y_t	y_{Euler}	*Global error*	*Local error*
0	0.0	1.00000	1.0000	----	----
1	0.5	0.60149	0.45	25.1	25.1
2	1.0	0.46455	0.2587	44.3	
3	1.5	0.59155	0.24581	58.4	
4	2.0	1.59466	0.38715	75.7	

Runge– Kutta Methods

First order

Heun's method

❖This method improves the Euler method by estimate average of the slope from two estimations one ath the beginning of the interval and the other at the end.

$$y^0_{i+1} = y_i + f(x_i, y_i)h$$

Predictor

$$y'_{i+1} = f\left(x_{i+1}, y^0_{i+1}\right)$$

$$\bar{y}' = \frac{y'_i + y'_{i+1}}{2} = \frac{f(x_i, y_i) + f\left(x_{i+1}, y^0_{i+1}\right)}{2}$$

$$\Rightarrow y_{i+1} = y_i + \frac{f(x_i, y_i) + f\left(x_{i+1}, y^0_{i+1}\right)}{2}h$$

Corrector

$$|\varepsilon_a| = \left|\frac{y^j_{i+1} - y^{j-1}_{i+1}}{y^j_{i+1}}\right| *100\%$$

Ordinary Differential equations

Rung – Kutta Methods

2nd order

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2)h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h)$$

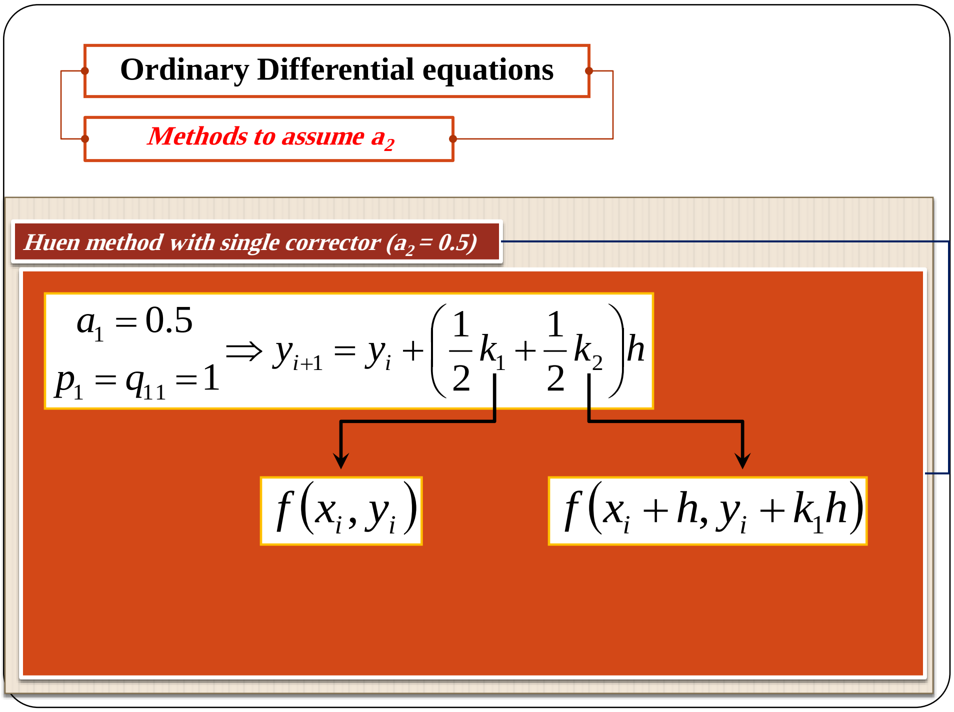
Setting these equations to Taylor series will results the following equations

$$a_1 + a_2 = 1$$
$$a_1p_1 = 1/2$$
$$a_2q_{11} = 1/2$$

Three equations with 4 unknowns so assume a_2 for example

$$a_1 = 1 - a_2$$
$$p_1q_{11} = \frac{1}{2a_2}$$

Because we can assume infinite number of a2 values, there are infinite number or R-K methods



Ordinary Differential equations

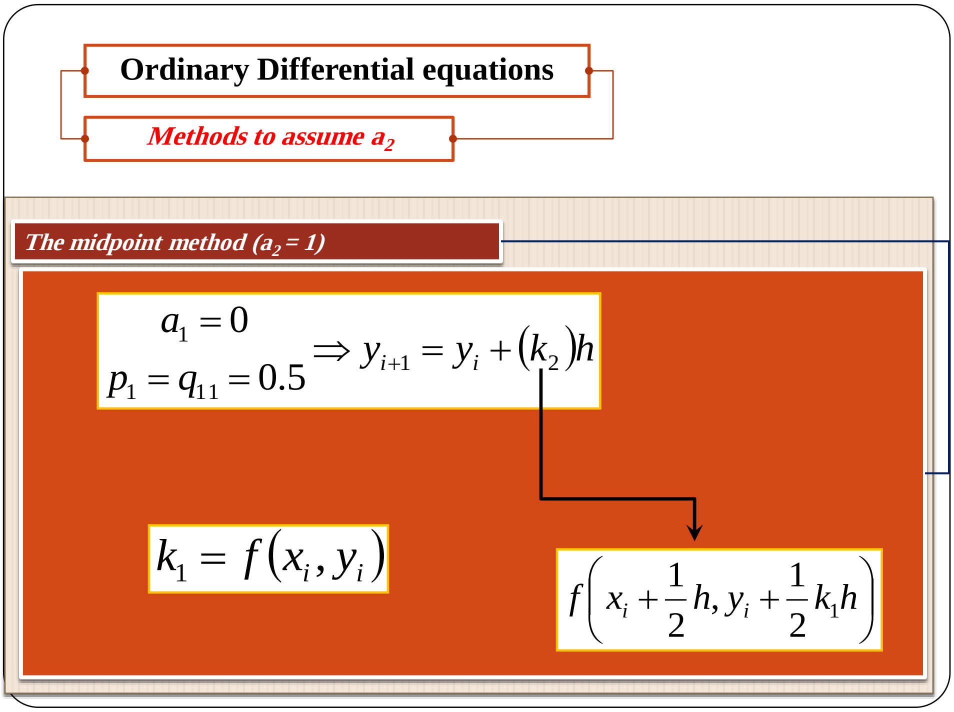
Methods to assume a_2

Huen method with single corrector ($a_2 = 0.5$)

$$\left.\begin{matrix} a_1 = 0.5 \\ p_1 = q_{11} = 1 \end{matrix}\right. \Rightarrow y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$f\left(x_i, y_i\right)$$

$$f\left(x_i + h, y_i + k_1 h\right)$$



Ordinary Differential equations

Methods to assume a_2

The midpoint method ($a_2 = 1$)

$$\left.\begin{matrix} a_1 = 0 \\ p_1 = q_{11} = 0.5 \end{matrix}\right. \Rightarrow y_{i+1} = y_i + \left(k_2\right)h$$

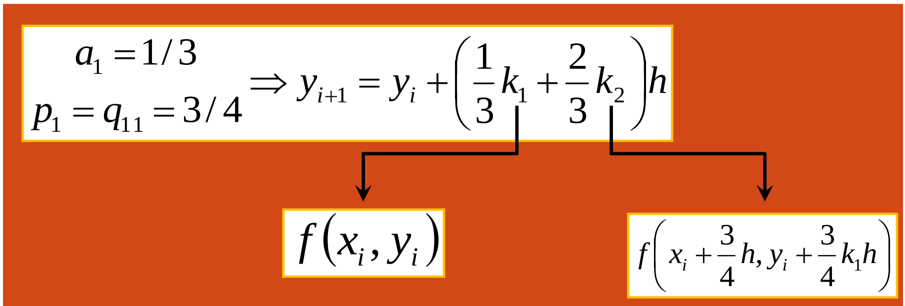
$$k_1 = f\left(x_i, y_i\right)$$

$$f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ordinary Differential equations

Methods to assume a_2

Ralston's method ($a_2 = 2/3$)



$$a_1 = 1/3$$
$$p_1 = q_{11} = 3/4 \Rightarrow y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

$$k_1 \rightarrow f(x_i, y_i)$$

$$k_2 \rightarrow f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

Ordinary Differential equations

Rung – Kutta Methods

3^{rd} and 4^{th} orders

- **3^{rd} order R-K methods**

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$$

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$
$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$$

- **4^{th} order R-K methods**

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$
$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$
$$k_4 = f(x_i + h, y_i + k_3 h)$$

Runge– Kutta Methods

Huen and higher order R-K

Example

❖Find the value of *y(0.5)* for the following function

$$y' = y\left(x^2 - 1.1\right)$$

Using:

1. Huen's method
2. 2nd order R-K method
3. 3rd order R-K method
4. 4th order R-K method

If the initial condition is y(0)=1 and h=0.5.

Runge– Kutta Methods

Huen and higher order R-K

Hune method

$$y^0_{i+1} = y_i + f\left(x_i, y_i\right)h \Rightarrow \mathit{predictor}$$

$$y_{i+1} = y_i + \frac{f\left(x_i, y_i\right) + f\left(x_{i+1}, y^0_{i+1}\right)}{2}h \Rightarrow \mathit{corrector}$$

First, the predictor:

$$y^0_1 = y_0 + f\left(x_0, y_0\right)h$$

$$f\left(x_0, y_0\right) = y'\left(x_0, y_0\right) = 1(0^2 - 1.1) = -1.1$$

$$y^0_1 = 1 + (-1.1)(0.5) = 0.45$$

Runge– Kutta Methods

Huen and higher order R-K

Hune method

Second , the corrector:

$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2} h$$

$$y_1 = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^0)}{2} h$$

$$f(x_1,y_1^0) = y'(x_1,y_1^0) = y'(0.5,0.45)$$

$$f(x_1,y_1^0) = 0.45(0.5^2 - 1.1) = -0.4$$

$$y_1 = 1 + \frac{-1.1 - 0.4}{2}(0.5) = 0.625$$

Runge– Kutta Methods

Huen and higher order R-K

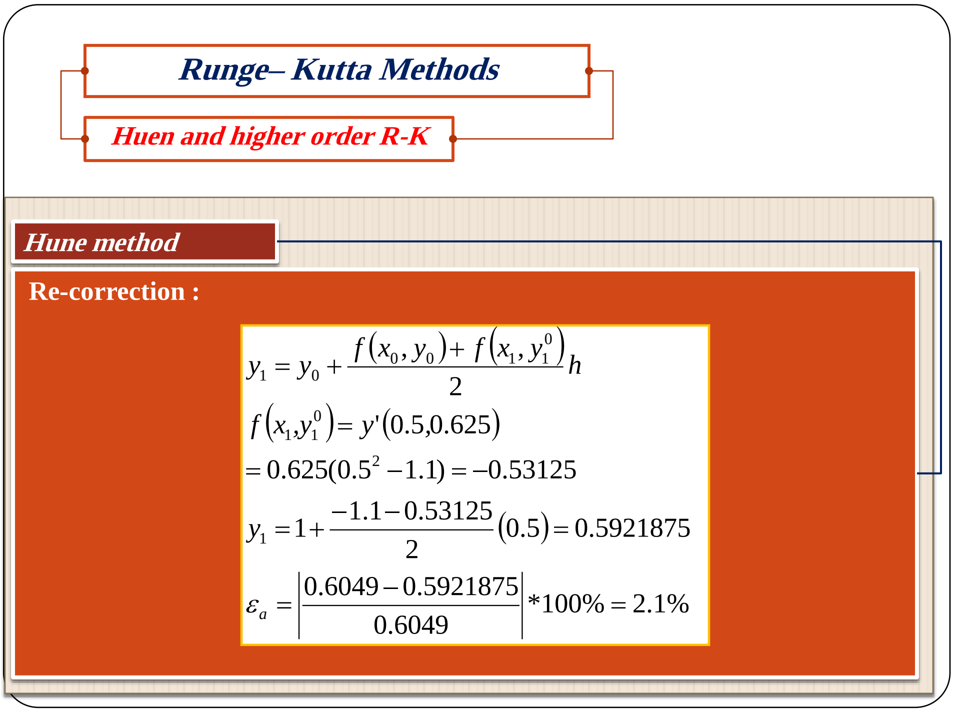
Hune method

Third , error comparison (global error) :

Remember that the global error in Euler method was 25.1 while the excat solution was 0.6049

$$\varepsilon_a = \left| \frac{0.6049 - 0.625}{0.6049} \right| * 100\% = 3.322\%$$

To reduce the error and increase the accuracy, the correction step may be done again and again until we reach the accuracy we desire.



Runge– Kutta Methods

Huen and higher order R-K

Hune method

Re-correction :

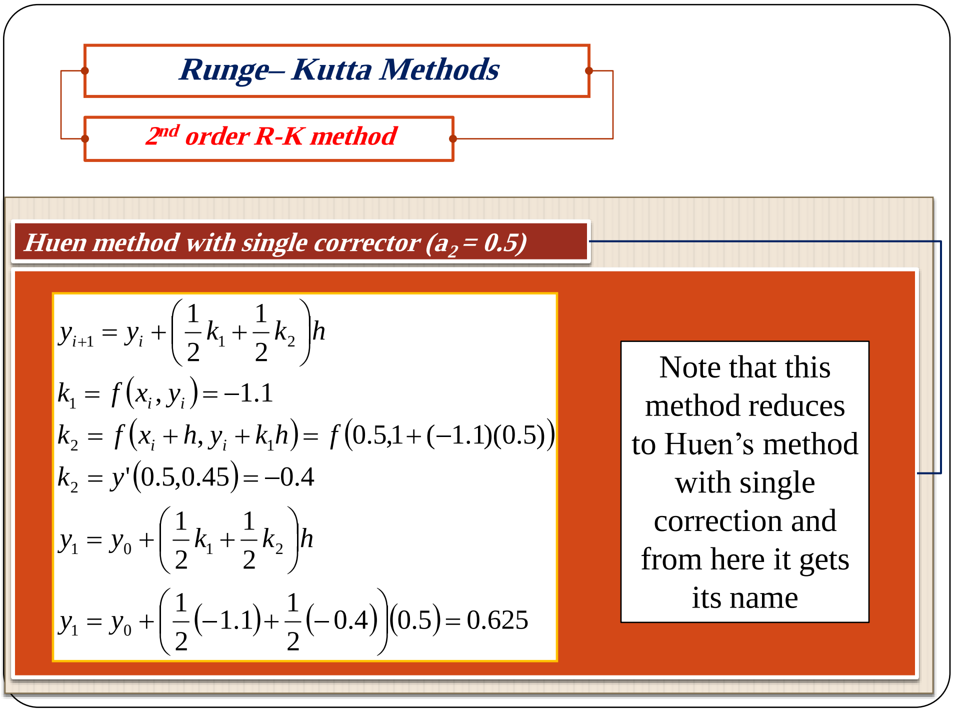
$$y_1 = y_0 + \frac{f(x_0, y_0) + f(x_1, y_1^0)}{2}h$$

$$f(x_1, y_1^0) = y'(0.5, 0.625)$$

$$= 0.625(0.5^2 - 1.1) = -0.53125$$

$$y_1 = 1 + \frac{-1.1 - 0.53125}{2}(0.5) = 0.5921875$$

$$\varepsilon_a = \left|\frac{0.6049 - 0.5921875}{0.6049}\right| * 100\% = 2.1\%$$



Runge– Kutta Methods

2nd order R-K method

Huen method with single corrector ($a_2 = 0.5$)

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_1 = f(x_i, y_i) = -1.1$$

$$k_2 = f(x_i + h, y_i + k_1 h) = f(0.5, 1 + (-1.1)(0.5))$$

$$k_2 = y'(0.5, 0.45) = -0.4$$

$$y_1 = y_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$y_1 = y_0 + \left(\frac{1}{2}(-1.1) + \frac{1}{2}(-0.4)\right)(0.5) = 0.625$$

Note that this method reduces to Huen's method with single correction and from here it gets its name

Runge– Kutta Methods

2^{nd} order R-K method

The midpoint method ($a_2 = 1$)

$$y_{i+1} = y_i + (k_2)h$$
$$k_1 = f(x_i, y_i) = -1.1$$
$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}0.5, 1 + \frac{1}{2}(-1.1)(0.5)\right)$$
$$k_2 = y'(0.25, 0.725) = -0.7521875$$
$$y_1 = y_1 + (k_2)h = 1 + (-0.7521875)0.5 = 0.6239$$
$$\varepsilon_a = \left|\frac{0.6049 - 0.6239}{0.6049}\right| * 100\% = 3.142\%$$

Runge– Kutta Methods

2^{nd} order R-K method

Ralston's method ($a_2 = 2/3$)

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$
$$k_1 = f(x_i, y_i) = -1.1$$
$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right) = f(0.375, 0.5875)$$
$$k_2 = y'(0.375, 0.5875) = -0.5636328$$
$$y_1 = y_0 + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h = 0.6287$$
$$\varepsilon_a = \left|\frac{0.6049 - 0.6287}{0.6049}\right| * 100\% = 3.949\%$$

Runge– Kutta Methods

3rd order R-K method

3rd order R-K method

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h$$

$$k_1 = f(x_i, y_i) = -1.1$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) = -0.7521875$$

$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h) = -0.67840625$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)h = 0.6011$$

$$\varepsilon_a = 0.6331\%$$

Runge– Kutta Methods

4th order R-K method

4th order R-K method

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$k_1 = f(x_i, y_i) = -1.1$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right) = -0.7521875$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right) = -0.8424$$

$$k_4 = f(x_i + h, y_i + k_3 h) = -0.49198$$

$$y_1 = 0.6016$$

$$\varepsilon_a = 0.55\%$$

Runge– Kutta Methods

System of higher order ODEs

First order system

$$
\begin{aligned}
y'_1 &= f_1(x, y_1, y_2, \dots, y_n) \\
y'_2 &= f_2(x, y_1, y_2, \dots, y_n) \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
y'_n &= f_n(x, y_1, y_2, \dots, y_n)
\end{aligned}
$$

Solving such system needs *n* of initial conditions and it done by applying the R-K method for each ODE individually with taking in consideration the involvement of unknowns (y's) in each ODE.

Runge– Kutta Methods

System of higher order ODEs

Example: Euler

Find the solutions y_1 and y_2 for the interval 0:0.5:2 using Euler's method if you know that $y_1(0) = 4$ and $y_2(0)=6$

$$
\begin{aligned}
y'_1 &= -0.5y_1 \\
y'_2 &= 4 - 0.3y_2 - 0.1y_1
\end{aligned}
$$

Solution:

$$
\begin{aligned}
y_{i+1} &= y_i + f(x_i, y_i)h \\
y_1(0.5) &= y_1(0) + y'_1\, h \\
y_1(0.5) &= 4 + [(-0.5)(4)](0.5) = 3 \\
y_2(0.5) &= y_2(0) + y'_2\, h \\
y_2(0.5) &= 6 + [4 - (0.3)(6) - (0.1)(4)](0.5) = 6.9
\end{aligned}
$$

Runge– Kutta Methods

System of higher order ODEs

Example ^{cont}

Continuo the previous procedure to find the rest of points which are presented in the following table:

x	y_1	y_2
0.0	4	6
0.5	3	6.9
1.0	2.25	7.715
1.5	1.6875	8.44525
2.0	1.265625	9.091087

Runge– Kutta Methods

System of higher order ODEs

Higher order ODE system

$$\begin{gathered} a_n y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \dots + a_1 y_1' + a_0 y_1 + c = 0 \\ a_n y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \dots + a_1 y_2' + a_0 y_2 + c = 0 \\ . \\ . \\ . \\ a_n y_n^{(n)} + a_{n-1} y_n^{(n-1)} + \dots + a_1 y_n' + a_0 y_n + c = 0 \end{gathered}$$

To solve such system, the higher order ODE must be reduced until it become a system of 1st order ODEs. Such systems must have initial conditions equal to order. Eg. 3rd order ODE must has 3 initial conditions ($y(x_0)$, $y'(x_0)$ and $y''(x_0)$)

Runge– Kutta Methods

System of higher order ODEs

Example

Solve the following system of ODEs

$$\left.\begin{array}{r} y''+3y'z+4y^2=0 \\ z''+z^2yy'=e^x \end{array}\right\} \begin{array}{l} y(0)=0, y'(0)=1 \\ z(0)=1, z'(0)=2 \end{array}$$

Solution:

1st step is to reduce each equation to system of 1st order ODEs. Note that two initial conditions are required for each ODE. Also, each equation is 2nd order in one of the variables (y and z).

$$y''+3y'z+4y^2=0 \left\} \begin{array}{l} y_1=y \\ y_2=y' \end{array} \right. \quad \Rightarrow \quad \left.\begin{array}{r} y'_1=y_2 \\ y'_2=-3y_2z_1-4y_1^2 \end{array}\right\} \begin{array}{l} y_1(0)=0 \\ y_2(0)=1 \end{array}$$

$$z''+z^2yy'=e^x=0 \left\} \begin{array}{l} z_1=z \\ z_2=z' \end{array} \right. \quad \Rightarrow \quad \left.\begin{array}{r} z'_1=z_2 \\ z'_2=e^x-z_1^2y_1y_2 \end{array}\right\} \begin{array}{l} z_1(0)=1 \\ z_2(0)=2 \end{array}$$

Runge– Kutta Methods

System of higher order ODEs

Example

This si a system of 1st order ODEs:

$$\left.\begin{array}{r} y'_1=y_2 \\ y'_2=-3y_2z_1-4y_1^2 \end{array}\right\} \begin{array}{l} y_1(0)=0 \\ y_2(0)=1 \end{array}$$

$$\left.\begin{array}{r} z'_1=z_2 \\ z'_2=e^x-z_1^2y_1y_2 \end{array}\right\} \begin{array}{l} z_1(0)=1 \\ z_2(0)=2 \end{array}$$

This system can now be solved by using R-K methods. Note that the desired solution is to find two solutions for the 1st ODE and other two solutions for the 2nd ODE. These solutions are : y_1 and y_2, and z_1 and z_2 respectively.