Examples: Repeat all Euler Examples using RK2.

⑥ least Square Error Problems:

The method of least squares is a standard approach to approximate solution of systems. least squares proplems fall into two categories: linear or ordinary least squares and non-linear least squares. The least squares problem occur in statistical regression analysis. {Regression or curve fitting is used when an approximate data is available}.

* Linear least-squares Regression:

Here is the simplest fitting [straight line to a set of data]. The straight line Eq. $y = a_0 + a_1 x + e$ where e is the error between model and observations. $\therefore e = y - a_0 - a_1 x$. A stratrgy approach is to minimize the sum of the squares of the residual (error)

$$S_r = \sum_{i=1}^{n} (e_i)^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

with $a_1 = \dfrac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2}$, $a_0 = \bar{y} - a_1 \bar{x}$ where

$\bar{y}$ and $\bar{x}$ are the means of y & x respectively.

Ex:- Given the dependent variable (y) is the force, and velocity is the independent variable (x). The observations (Readings) can be set up in tabular form and the necessary sums are computed also:

i	x_i	y_i	x_i^2	$x_i y_i$
1	10	25	100	250
2	20	70	400	1400
3	30	380	900	11400
4	40	550	1600	22000
5	50	610	2500	30500
6	60	1220	3600	73200
7	70	830	4900	58100
8	80	1450	6400	116000
Σ	360	5135	20400	312850

The means can be calculated as: $\bar{x} = \dfrac{360}{8} = 45$

$\bar{y} = \dfrac{5135}{8} = 641.875$

Then: $a_1 = \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \dfrac{8(312850) - 360*5135}{8(20400) - (360)^2} = 19.47024$

$a_0 = \bar{y} - a_1\bar{x} = 641.875 - 19.47024(45) = -234.2857$

$\therefore y = a_0 + a_1 x = -234.2857 - 19.47024\,x$

Ex:- Below are the grades of 5 students for the subtotal and final exam. It is required to find the relation using linear regression to minimize (SSE).

$SSE = \sum_{1}^{n} (y_i - a_0 - a_1 x_i)$

X subtotal	y final exam
10	20
25	27
33	30
40	43
42	30

Ans: $y = a_0 + a_1 x = 14.73451 + 0.50885\,x$ {check for $y(10)$, $y(25)$}.

* Non-linear Regression: "Exponential Model"

$y = Ce^{DX}$

$\ln y = \ln(Ce^{DX}) = \ln C + \boxed{\ln e^{DX}}$

$\underbrace{\ln y}_{Y} = \underbrace{\ln C}_{a_0} + \underbrace{DX}_{a_1 x}$

Then: $Y = \ln C + DX \Rightarrow D = a_1,\; C = e^{a_0} \Rightarrow y = Ce^{DX}$

Example: Given the following Data, find ~~the~~ its exponential fit $y = Ce^{Dx}$

	X	y	$Y = \ln y$	X^2	XY
	0	1.5	0.40547	0	0
	1	2.5	0.91629	1	0.91629
	2	3.5	1.25276	4	2.50552
	3	5	1.60944	9	4.82832
	4	7.5	2.01490	16	8.0596
Σ	10		6.19886	30	16.30973

for $Y = a_0 + a_1 x$ $\quad a_0 = 0.45737,\; a_1 = 0.391201$

$D = a_1 = 0.391201,\quad C = e^{a_0} = e^{0.45737} = 1.57991$

thus $f(x) = y = Ce^{DX} = 1.57991\,e^{0.391201}$

⑦ Computer algorithmic Design:

Algorithm design is a specific method to create a mathematical process in solving problems. An algorithm design is not a programming language, but it can be a hand written process like set of equations and relations, and one of the most important aspects of algorithm design is creating an algorithm that has an efficient run time. The steps of development algorithms are:

1. Problem Definition.
2. Development of a model.
3. Specification of Algorithm.
4. Designing an Algorithm.
5. Checking the Correctness of Algorithm.
6. Analysis of Algorithm.
7. Implementation of Algorithm.
8. Program testing.
9. Documentation.

" Numerical Solution of non-linear equations "

(1) Characteristics of Numerical Methods:

Numerical methods have most of the following c/cs:

1. The Solution procedure is iterative, with the accuracy of the estimated solution improving with each iteration.
2. The solution procedure provides only an approximation to the true, but unknown, solution.
3. An initial estimate of the solution may be required.
4. The solution procedure is Conceptually simple, with algorithms representing the solution procedure that can be easily programmed on a digital Computer.
5. The Solution procedure may occasionally diverge from rather than Converge to the true solution.

(2) Numerical Error Analysis:

In general Errors Can be classified into:

- Non-Numerical Errors, which includes:
  - Modeling Errors
  - Mistakes Errors
  - Uncertainity in information and data
- Numerical Errors, which includes:
  - Round-off Error
  - trancation Error
  - propagation Error
  - approximation Error

Kinds of Numerical Error Calculations:

Error - Absolute = | True value - Approximated value | if True value exist.

Error - Absolute = | New Approx. value - Old approx. value | if True value does not exist.

$$E_{abs} = | X_{i+1} - X_i |$$

$$\text{Relative Error \%} = \frac{|\text{True value} - \text{Approx. value}|}{|\text{True value}|} * 100 \quad \text{if True value exists.}$$

$$E_{relative}\,\% = \frac{|X_{i+1} - X_i|}{|X_{i+1}|} * 100 \quad \text{if True value does not exist.}$$

③ Newton_Raphson method:

To numerically solve the non-linear algebric equations there are many available numerical methods such as Direct Method, Bi-section, secant, and Newton-Raphson methods.

The Newton_Raphson method was named after Isaac Newton and Joseph Raphson, is a method for finding successively better approximation to the solution (roots) of a real-valued function $\{f(x) = 0\}$ provided that an initial guess of solution is available.

$$X_{new} = X_{old} - \frac{f(X_0)}{f'(X_0)}$$

oR:

$$X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}$$

that $f(x)$ should be differentiable

Ex: find the $\sqrt{612}$ using Newton-Raphson method [True value = 24.738]

$X = \sqrt{612} \Rightarrow x^2 = 612 \Rightarrow f(x) = x^2 - 612 = 0 \Rightarrow f'(x) = 2x$

given an initial guess ($X_0 = 10$), then the sequence given by N-R method is:

$X_1 = X_0 - \frac{f(X_0)}{f'(X_0)} = 10 - \frac{100 - 612}{20} = 35.6 \qquad E_{rel}\% = \frac{|24.738 - 35.6|}{|24.738|} * 100 = 43.9\%$

$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)} = 26.395 \qquad E_{rel}\% = \frac{|24.738 - 26.395|}{|24.738|} * 100 = 8.8\%$

$X_3 = X_2 - \frac{f(X_2)}{f'(X_2)} = 24.79 \qquad E_{rel}\% = \frac{|24.738 - 24.79|}{|24.738|} * 100 = 0.2\%$

$X_4 = X_3 - \frac{f(X_3)}{f'(X_3)} = 24.738 \qquad E_{rel} = \frac{|24.738 - 24.738|}{|24.738|} * 100 = 0\%$

$X_5 = X_4 - \frac{f(X_4)}{f'(X_4)} = 24.738 \qquad E_{rel} = \frac{|24.738 - 24.738|}{|24.738|} * 100 = 0\%$

EX2: Solve $\cos(x) = x^3$ with $x_0 = 0.5$ using 5-iteration of N-R Method.

$f(x) = \cos(x) - x^3 = 0$, $f'(x) = -\sin(x) - 3x^2$

then; $X_1 = X_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{\cos(0.5) - (0.5)^3}{-\sin(0.5) - 3(0.5)^3} = 1.112$

Note: Triagonemetric function is in radian foll all numerical solutions.

Since True value doesn't exist then,

$$\epsilon_{rel,1}\% = \frac{|X_1 - X_0|}{|X_1|} * 100 = \frac{|1.112 - 0.5|}{|1.112|} * 100 = 55\%$$

Similarly one obtain:

$X_2 = 0.9096$, $X_3 = 0.8672$, $X_4 = 0.8654$, $X_5 = 0.8654$

with: $\epsilon_{re2}\% = \frac{|X_2 - X_1|}{|X_2|} * 100$ ---- $\epsilon_{rel5}\% = \frac{|X_5 - X_4|}{|X_5|} * 100$

$$\epsilon_{rel5}\% = \frac{|0.8654 - 0.8654|}{|0.8654|} * 100 = 0\%$$

EX3: Solve X^{100} with $X_0 = 0.1$ using 3-iteration of N-R Method:

$f(x) = x^{100} \Rightarrow f'(x) = 100\, X^{99}$

then $X_1 = X_0 - \frac{f(x_0)}{f'(x_0)} = 0.1 - \frac{(0.1)^{100}}{100(0.1)^{99}} \Rightarrow$ which is wrong approximation thus in such cases one should simplify N-R formula such as:

$$X_1 = X_0 - \frac{X_0^{100}}{100\, X_0^{99}} = X_0 - 0.01 X_0 = 0.99 X_0$$

thus $X_1 = 0.99 X_0$, $X_2 = 0.99 X_1$, $X_3 = 0.99 X_2$ --- & so on.

Examples: Apply 3-iteration of N-R method with 4SD accuracy to the following Problems:

* $\sin x = \cot x$, $x_0 = 1$
* $X = \cos x$, $x_0 = 1$
* $x^3 - 5X + 3 = 0$, $x_0 = 2$
* $X + \ln X = 2$, $x_0 = 2$

④ Lagrange Interpolating Polynomial:

Interpolation means finding (approximate) values of a function ($f(x)$) for an x between different x-value ($x_0, x_1, \ldots, x_n$) which their values $f(x)$ are given.
There are many Interpolation methods such as Lagrange and Newton interpolating methods.

Lagrange Interpolating Polynomial:

* Linear Interpolation: given x_0, $f(x_0)$, x_1 and $f(x_1)$

then; $L_0(x) = \dfrac{x - x_1}{x_0 - x_1}$, $L_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

and polynomial $(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$

Ex:- Compute 4.S.D value of $\ln 9.2$ from $\ln 9 = 2.1972$, $\ln 9.5 = 2.2513$ by linear lagrange interpolation then determine the $E_{rel}\%$ using True value of $\ln 9.2 = 2.2192$ (4.D)

$x_0 = 9$ $f(x_0) = 2.1972$

$x_1 = 9.5$ $f(x_1) = 2.2513$

then $L_0(x) = \dfrac{x - 9.5}{-0.5} = -2(x - 9.5)$, $L_0(9.2) = 0.6$

$L_1(x) = \dfrac{x - 9.0}{0.5} = 2(x - 9)$, $L_1(9.2) = 0.4$

then $P(x) = L_0(x) f(x_0) + L_1(x) f(x_1)$

Hence $P(9.2) = 2.2188$ $\quad E_{rel}\% = \dfrac{|2.2192 - 2.188|}{|2.2192|} * 100 = 0.95\%$

thus linear Interpolation is not sufficient here.

* Quadratic Interpolation: given $x_0, f(x_0), x_1, f(x_1)$ and $x_2, f(x_2)$

then $P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2)$

where; $L_0(x) = \dfrac{(x - x_1)}{(x_0 - x_1)} * \dfrac{(x - x_2)}{(x_0 - x_2)}$

$L_1(x) = \dfrac{(x - x_0)}{(x_1 - x_0)} * \dfrac{(x - x_2)}{(x_1 - x_2)}$

$L_2(x) = \dfrac{(x - x_0)}{(x_2 - x_0)} * \dfrac{(x - x_1)}{(x_2 - x_1)}$

Ex: Repeat previous example with $x_0 = 9$, $f(x_0) = 2.1972$

$x_1 = 9.5$, $f(x_1) = 2.2513$

$x_2 = 11$, $f(x_2) = 2.3979$

then, $L_0(x) = L_0(9.2) = 0.54$, $L_1(9.2) = 0.48$, $L_2(9.2) = -0.02$

Hence, $p(9.2) = L_0(9.2) * f(x_0) + L_1(9.2) * f(x_1) + L_2(9.2) * f(x_2)$

$= 2.2192$ Recall True value $= 2.2192$

so that Quadratic Lagrange interpolating polynomial gives exact 4D accuracy.

* Qubic Interpolation: it needs 4-point $x_0, f(x_0)$, $x_1, f(x_1)$, $x_2, f(x_2)$, $x_3, f(x_3)$

then $P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) + L_2(x) f(x_2) + L_3 f(x_3)$

where; $L_0(x) = \frac{(x-x_1)}{(x_0-x_1)} * \frac{(x-x_2)}{(x_0-x_2)} * \frac{(x-x_3)}{(x_0-x_3)}$

$$L_1(x) = \frac{(x-x_0)}{(x_1-x_0)} * \frac{(x-x_2)}{(x_1-x_2)} * \frac{(x-x_3)}{(x_1-x_3)}$$

$$L_2(x) = \frac{(x-x_0)}{(x_2-x_0)} * \frac{(x-x_1)}{(x_2-x_1)} * \frac{(x-x_3)}{(x_2-x_3)}$$

and

$$L_3(x) = \frac{(x-x_0)}{(x_3-x_0)} * \frac{(x-x_1)}{(x_3-x_1)} * \frac{(x-x_2)}{(x_3-x_2)}$$

The General Lagrange Interpolating formula is:

$$f(x) \simeq P_n(x) = \sum_{k=0}^{n} L_k(x) f_k = \sum_{k=0}^{n} \frac{l_k(x)}{l_k(x_k)} f_k$$

Examples:

* linear and Quadratic Lagrange Interpolation: Find $e^{-0.25}$ by linear interpolation with $x_0 = 0$, $x_1 = 0.5$. Then $P_2(x)$ to interpolate e^{-x} with $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$.

* Find $P_3(x)$ for $x = 0.5$ for the data $(0,1)$, $(1, 0.7652)$, $(2, 0.2239)$, $(3, -0.26)$.

(5) Numerical (Integration) Solution of non-linear dynamic Equations:

There are many methods to solve dynamic equations numerically such as Euler's and Modified Euler's methods, 2nd and 4th order Runge-Kutta methods. Note(1) "Sometimes are called initial value problems" "Note(2) Numerical Integration methods for algebric solution equations are for ex: Trapezoidal Rule, Simpson ($\frac{1}{3}$rd and $\frac{1}{8}$ rules)."

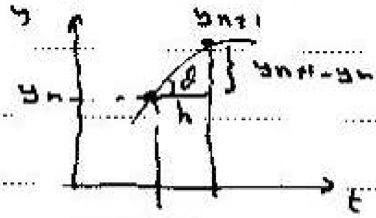
* Euler's Method:

Initial value problems for 1st order systems of ODEs are of the form $y' = f(x, y)$, with $y(x_0) = y_0$

Then $y_{n+1} = y_n + h\, f(t_n, y_n)$ wher h = step size

$f(t_n, y_n) = y'_{(n)}$

which can be derived easily from the following figure:



then $\tan\theta$ = slope = $y'(n) = \frac{y_{n+1} - y_n}{h}$ then $y_{n+1} = y_n + hy' = y_n + hf(t_n, y_n)$

Note:- "for h = stepsize is very small, the accuracy will be high on the expense of more calculations, and vice-versa."

Example:- Given the initial value problem $y' = y$, $y(0) = 1$, we would like to use Euler method to approximate $y(4)$ using step size $h = 1$. [knowing that the analytical exact solution is $y = e^t$].

$y_1 = y_0 + hf(y_0) = 1 + 1 \times 1 = 2$

$y_2 = y_1 + hf(y_1) = 2 + 1 * 2 = 4$

$y_3 = y_2 + hf(y_2) = 4 + 1 * 4 = 8$

$y_4 = y_3 + hf(y_3) = 8 + 1 * 8 = 16$

Here, $y(4) = e^4 = 54.598$ so, the approximation here is not very good.

using different step size we repeat the previous calculations to obtain

h	$y(4)$
1	16
0.25	35.53
0.1	45.26
0.05	49.56
0.025	51.98
0.0125	53.26

check for $h = 0.001$.

Examples: Solve using Euler Method.

* for $\frac{Y(s)}{R(s)} = \frac{1}{s+2}$ with $R(s)$= Unit impulse solve for $y(t)$ with step size of $h = 0.1$
* $y_1' = -3y_1 + y_2$, $y_2' = y_1 - 3y_2$, $y_1(0) = 2$, $y_2(0) = 0$, $h = 0.1$ (5 steps).
* $y'' - y = x$, $y(0) = 1$, $y'(0) = 2$, $h = 0.1$ (5 steps).

* 2nd order Runge-Kutta Method (Heun's method)

Give the Initial value problems for 1st order systems of ODEs are of the form $y' = f(x, y)$, with $y(x_0) = y_0$. Then its numerical integration (solution) using 2nd order RK method (Heun's Method) can be obtain by:

$$y_{n+1} = y_n + h(0.5K_1 + 0.5K_2)$$

where; $K_1 = f(x_n, y_n)$

$$K_2 = f(x_n + h, y_n + K_1 h)$$

Example: Given $y' = 4e^{0.8t} - 0.5y$ ($t = 0$ to $t = 4$) $y(0) = 2$, use $h = 1$ apply 2nd order Runge Kutta method to find $y(4)$.

$y(1) = y(0) + h(0.5K_1 + 0.5K_2) = 2 + 1(0.5K_1 + 0.5K_2)$

$K_1 = f(x_0, y_0) = f(0, 2) = 4e^{0.8*0} - 0.5*2 = 3$, $K_2 = f(x_0+1, y_0 + 3*1) = f(1, 5) = 6.4$

$y(1) = 2 + 1(0.5*3 + 0.5*6.4) = 6.7$

Repeat for $y(2)$, $y(3)$, $y(4)$.