"Complex Variable Theory"

A) Introduction:

_ The complex number system:

A complex number z is an ordered pair (x,y) of real numbers x and y written $z = x + yi$ where $x = \operatorname{Re} z$, $y = \operatorname{Im} z$ with $i^2 = -1$, $i = \sqrt{-1}$

Addition: for $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ then $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2) i$

Multiplication: for $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ then $z_1 * z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$

subtraction: for $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ then $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2) i$

Division: for $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ then $\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} * \frac{(x_2 - y_2 i)}{(x_2 - y_2 i)}$

then $z_1/z_2 = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} i$

Complex Conjugate no.: for $z = x + iy$ the $\bar{z}$ = complex conjugate = $x - iy$

Ex:

1. prove that Multiplication by i is a counter clockwise rotation by $(90°)$ [take $z = 4 - 3i$ as an example.
2. Show that $z = x + iy$ is pure imaginary iff (if and only if) $[\bar{z} = -z]$
3. let $z_1 = 2 + 3i$, $z_2 = 4 - 5i$ find $(5z_1 + 3z_2)^2$ & $\bar{z}_1 + \bar{z}_2$ & z_2/z_1.
4. prove that $z_1 + z_2 = z_2 + z_1$, $z_1 z_2 = z_2 z_1$ (Commutative law).

_ Polar Form of Complex Numbers:-

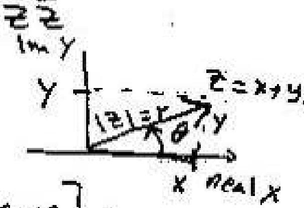
Besides the xy_ coordinates, there is polar coordinates as below:-

for a Complex no. $z = x + iy$ with $x = r\cos\theta$, $y = r\sin\theta$

then $z = r(\cos\theta + i\sin\theta)$

r is called the absolute value or modulus of z $[r = |z|]$ where $|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

& θ is called the argument of z $[\arg z]$ thus



$$\theta = \arg z = \arctan \frac{y}{x}$$

Note:- [all angles are measured in radians and + in the CCW sense].

– Multiplication and Division in Polar Form:-

Given $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then; $z_1 * z_2 = r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$. "prove it".

and $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$. "prove it".

i.e., $|z_1 z_2| = |z_1||z_2|$, $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $\arg \frac{z_1}{z_2} = \arg z_1 - \arg z_2$.

Ex:- let $z_1 = -2+2i$ and $z_2 = 3i$ then $z_1 * z_2 = -6-6i$, $z_1/z_2 = 2/3 + 2/3\, i$.

$|z_1 z_2| = 6\sqrt{2}$, $|z_1/z_2| = 2\sqrt{2/3}$,

and for the arguments $\arg z_1 = 3\pi/4$, $\arg z_2 = \pi/2$

$\arg(z_1 z_2) = -3\pi/4$, $\arg(z_1/z_2) = \pi/4$

– Integer Power of a Complex no. $z = x+iy = r(\cos\theta + i\sin\theta)$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

– Roots of a Complex no. $z = r(\cos\theta + i\sin\theta)$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n},\ i\sin\frac{\theta+2k\pi}{n}\right)$$

where $k = 0, 1, \ldots, n-1$.

Ex:- Represent the following in polar form:

1. $z = 3-3i$, 2. $z = 0.5 + 0.25\pi i$, 3. $z = \frac{1+i}{1-i}$ [Ans. 1. $3\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$. 3. $\cos(\pi/2) + i\sin(\pi/2)$]

Ex: Represent in $x+iy$ form the following:

1. $z = 3(\cos 0.2 + i\sin 0.2)$, 2. $z = \cos(-1) + i\sin(-1)$. [Ans. 1. $(2.94 + 0.596i)$ 2. $(0.54 - 0.841i)$]

Ex: Determine the argument of:

1. $z = -1-i$, 2. $z = (9+9i)^3$, $z = -\pi^2$. [Ans. 1. $-3\pi/4$ 2. $3\pi/4$]

Ex: Find all roots of $\sqrt[4]{-1}$ [Ans. $\pm(1 \pm i)/\sqrt{2}$].

- Elementary Complex functions:

* Exponential Function:-

for $z = x + iy$, $\Rightarrow$ $e^{z} = e^{x}(\cos y + i\sin y)$

for $z = r(\cos\theta + i\sin\theta) \Rightarrow$ $z = re^{i\theta}$

Note:- $e^{\pi i/2} = i$, $e^{\pi i} = -1$, $e^{-\pi i/2} = -i$, $e^{-\pi i} = -1$

Ex:- for the following complex numbers (z) compute e^{z}.

1. $1+2i$, 2. $7\pi i/2$ {Ans: 1. $(-1.1312 + 2.4717i)$, 2. $-i$, 1 }

Ex:- find Real and Im. part of e^{-2z}, e^{z^2} {Ans: $e^{-2x}\cos 2y$, $-e^{-2x}\sin 2y$ }

{ $\exp(x^2-y^2)\cos 2xy$, $\exp(x^2-y^2)\sin 2xy$ }.

Ex:- Write in polar form (-9) {Ans: $9e^{\pi i}$ }.

* Trigonometric Functions:- for complex no. $z = x+iy$,

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right), \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

with $\tan z = \dfrac{\sin z}{\cos z}$, $\cot z = \dfrac{\cos z}{\sin z}$, $\sec z = \dfrac{1}{\cos z}$, $\csc z = \dfrac{1}{\sin z}$

$(\cos z)' = -\sin z$, $(\sin z)' = \cos z$, $(\tan z)' = \sec^2 z$.

and $e^{iz} = \cos z + i\sin z$

Ex:- Compute $\cos(1+i)$ {Ans: $0.8337 - 0.9889i$}

* Logarithm: for complex no. $z = x+yi$, $z = r(\cos\theta + i\sin\theta)$

$$\ln z = \ln r + i\theta$$

Ex:- find $\ln z$ for

1. $z = -10$ {Ans: $\ln 10 + \pi i$}
2. $z = 2-2i$ {Ans: $\frac{1}{2}\ln 8 - \frac{\pi}{4}i$ }.

Taylor Series :-

The Taylor series of a function $f(z)$ is:

$$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n \quad \text{where, } a_n = \frac{1}{n!} f^{(n)}(z_0)$$

A Maclaurin Series is a Taylor Series with center $z_0 = 0$.

Important Special Tylor Series:

* Geometric Series: $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots = \frac{1}{1-z}$

* Exponential Function: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$

* Trigonometric Functions:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

* logarithm: $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$