Note: Cayley-Hamilton theorem can be applied also for multiple eigen values by finding $g'(\lambda)$ as in the following example.

Ex:- for $A=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ Compute A^{100} using C-H theorem.

solution: eigen values $= |\lambda I - A| = 0 \Rightarrow \left|\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}-\begin{pmatrix}1&2\\0&1\end{pmatrix}\right| = \begin{vmatrix}\lambda-1&-2\\0&\lambda-1\end{vmatrix}=0$

then $\lambda_1=\lambda_2=1$

$f(\lambda_1)=g(\lambda_1)=\alpha_0+\alpha_1\lambda_1 \Rightarrow (1)^{100}=\alpha_0+\alpha_1 \quad \therefore 1=\alpha_0+\alpha_1$ ---(1)

$f'(\lambda_1)=g'(\lambda_1)=\alpha_1 \Rightarrow 100=\alpha_1$ ---(2) then $\alpha_0=-99$

Hence $f(A)=A^{100}=\alpha_0 I+\alpha_1 A = -99\begin{pmatrix}1&0\\0&1\end{pmatrix}+100\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{bmatrix}1&200\\0&1\end{bmatrix}$.

Ex:- for $A=\begin{bmatrix}0&1\\-2&-3\end{bmatrix}$ find $f(A)=e^{At}$ using C-H theorem.

$|\lambda I - A|=0$ gives $\lambda_1=-1$, $\lambda_2=-2$

then $f(\lambda_1)=g(\lambda_1)=\alpha_0+\alpha_1\lambda_1 \Rightarrow e^{-t}=\alpha_0-\alpha_1$ ---(1)

$f(\lambda_2)=g(\lambda_2)=\alpha_0+\alpha_1\lambda_2 \Rightarrow e^{-2t}=\alpha_0-2\alpha_1$ ---(2) } solve for α_0, α_1

$\therefore \alpha_0=2e^{-t}-e^{-2t}$, $\alpha_1=e^{-t}-e^{-2t}$

then: $f(A)=e^{At}=\alpha_0 I+\alpha_1 A=\begin{pmatrix}2e^{-t}-e^{-2t}&0\\0&2e^{-t}-e^{-2t}\end{pmatrix}+\begin{pmatrix}0&e^{-t}-e^{-2t}\\-2e^{-t}+2e^{-2t}&-3e^{-t}+3e^{-2t}\end{pmatrix}$

$\therefore$

$$e^{At}=\begin{pmatrix}2e^{-t}-e^{-2t}&e^{-t}-e^{-2t}\\-2e^{-t}+2e^{-2t}&-e^{-t}+2e^{-2t}\end{pmatrix}.$$

* Sylvester's Expansion Theorem for finding e^{At}:- "Applied only for distinct eigen values"!

Algorithmic Steps:-

1. Determine n-distinct eigen values of the matrix A.
2. Find n F_i's as $F_i=\prod_{\substack{j=1\\ \neq i}}^{n}\dfrac{A-\lambda_j I}{\lambda_i-\lambda_j}$
3. Determine e^{At} as $\Phi(t)=e^{At}=\sum_{i=1}^{n} e^{\lambda_i t}F_i$

Ex:- for $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, find e^{At} using Sylvester's criterion.

Solution: $|\lambda I - A| = 0$ gives $\lambda_1 = -1$, $\lambda_2 = -2$

then $F_1 = \dfrac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \dfrac{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - (-2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{(-1) - (-2)} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$

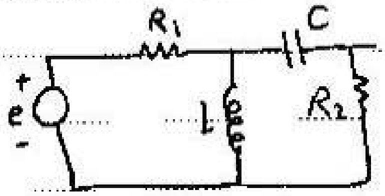
$F_2 = \dfrac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \dfrac{\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{(-2) - (-1)} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$

then $\Phi(t) = e^{At} = e^{\lambda_1 t} F_1 + e^{\lambda_2 t} F_2 = e^{-t}\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} + e^{-2t}\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}$

$\therefore \; e^{At} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$

"Problems & Solved Examples"

Ex: For the following RLC electrical circuit derive the state-space Equations.



Note: You've choose loops to involve all variables (e, R_1, L, C, R_2).

Note: Since two dynamic elements $(L, \& C)$ in the circuit then it is 2nd order sys.

Note: Since 2nd order then there are two states x_1 = voltage at Capacitor
x_2 = coil current

Note: then "$C\dot{x}_1$" will be capacitor current.

and "$L\dot{x}_2$" will be voltage on the coil.

Hence; $e = R_1(x_2 + C\dot{x}_1) + x_1 + R_2 C\dot{x}_1$ ....(1) $\quad L\dot{x}_2 = x_1 + R_2 C\dot{x}_1$ ....(2)

Rearrange Eq.(1) to obtain $\dot{x}_1 = -\dfrac{1}{C(R_1+R_2)}x_1 - \dfrac{R_1}{C(R_1+R_2)}x_2 + \dfrac{e}{C(R_1+R_2)}$

Solve Eq.(2) to get: $\dot{x}_2 = \dfrac{R_1}{L(R_1+R_2)}x_1 - \dfrac{R_1 R_2}{L(R_1+R_2)}x_2 + \dfrac{R_2}{L(R_1+R_2)}e$

Hence: $\dot{\underline{x}} = A\underline{x} + Bu$ state-Equation will be:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{C(R_1+R_2)} & -\frac{R_1}{C(R_1+R_2)} \\ \frac{R_1}{L(R_1+R_2)} & -\frac{R_1 R_2}{L(R_1+R_2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{C(R_1+R_2)} \\ \frac{R_2}{L(R_1+R_2)} \end{bmatrix} e$$

"is of Random form"

Ex:- Solve the following s.s. Representation systems.

1. Harmonic Oscillator: $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{x}$, Ans. $x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \underline{x}(0)$.

2. Double Integrator: $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}$, Ans. $x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \underline{x}(0)$.

3. Determine the matrix exponential, and the homogenous response to the initial conditions $x_1(0) = 2$, $x_2(0) = 3$ of the system with state equations

$$\dot{x}_1 = -2x_1 + u, \quad \dot{x}_2 = x_1 - x_2$$

Ans. $e^{At} = \Phi(t) = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$, $x_1(t) = 2e^{-2t}$, $x_2(t) = 5e^{-t} - 2e^{-2t}$

4. Find the Solution of the System $\dot{x}_1 = -2x_1 + u$, $\dot{x}_2 = x_1 - x_2$ to a constant input $u(t) = 5$ for $t > 0$, if $\underline{x}(0) = 0$

Ans. $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2.5 - 2.5\, e^{-2t} \\ 2.5 - 5e^{-t} + 2.5\, e^{-2t} \end{bmatrix}$

5. As for Ex4: above find output solution if $y = [2 \;\; 1]\, \underline{x}$

Ans. $y(t) = 7.5 - 2.5\, e^{-2t} - 5 e^{-t}$.

6. Determine the eigen values of the System: $\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -9 & -4 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$

Ans. $\lambda_1 = -2$, $\lambda_2 = -1 + j2$, $\lambda_3 = -1 - j2$.

7. for $A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$ find Eigen values and Eigen Vectors.

Ans. Eigen values $\lambda_1 = -1$, $\lambda_2 = -4$, Note: Eigenvector can be found as $[\lambda_i I - A]\, m_i = 0$

then $m_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow \{m_{21} = m_{22}\}$, $m_2 = \begin{bmatrix} \alpha_2 \\ -2\alpha_2 \end{bmatrix}$ for $\alpha_2 \neq 0$, then $m_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} 15 \\ -30 \end{bmatrix}$ --- etc.

8. Given $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ find e^{At} using Cayley-Hamilton theorem

Knowing that $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2$. Ans. $e^{At} = \begin{bmatrix} -3e^{t}+4e^{2t} & 6e^{t}-6e^{2t} & 6e^{t}-6e^{2t} \\ e^{t}-e^{2t} & -2e^{t}+3e^{2t} & -2e^{t}+2e^{2t} \\ -3e^{t}+3e^{2t} & 6e^{t}-6e^{2t} & 6e^{t}-5e^{2t} \end{bmatrix}$

9. A wide variety of wave propagation problems in a stratified medium reduce the equation $\dot{\underline{x}} = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} \underline{x}(0)$, use Sylvester criterion to find e^{At} for a&b are +ve, and for a&b are -ve. then find $\underline{x}(t)$ for each case.

10. Extract state-space Equations Canonical form for the following Differential Equations:

(a) $y''' + 2y'' + 3y' + 5y = x''' + x'' + x' + x$ "Check Your Answer"

(b) $y''' + 4y' + 2y = x' + 3x$ "Check Your Answer"

(C) $y'' + 2y' + y = x$ "Check Your Answer".

11. Find the general solution for scalar state-space Equation:

$$\dot{x} = ax + bu \quad , \quad y = cx + du$$

12. For $\dfrac{Y(s)}{U(s)} = \dfrac{b_0 s^n + b_1 s^{n-1} + \cdots b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$, use Partial Fraction to get: $\dfrac{Y(s)}{U(s)} = b_0 + \dfrac{c_1}{s+p_1} + \dfrac{c_2}{s+p_2} + \cdots$

then $\dot{\underline{x}} = \begin{bmatrix} -p_1 & & 0 \\ & -p_2 & \\ 0 & & \ddots -p_n \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} u$, $y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \underline{x} + b_0 u$

is called Diagonal Canonical form. for the Sys. $\dfrac{Y(s)}{U(s)} = \dfrac{s+3}{s^2+3s+2} = \dfrac{s+3}{(s+1)(s+2)}$ write down the Diagonal & Controllable Cannonical forms, then check Your Answe, Finally find e^{At} for both forms using $\mathcal{L}^{-1}(sI-A)^{-1}$. what Conclusion Can You make?

"Solved Examples"

EX1: A state-space representation of a system in the Controllable canonical form is given by: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -0.4 & -1.3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$, $y=\begin{bmatrix}0.8 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

The same system can be represented by the following observable cannonical form: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & -0.4\\ 1 & -1.3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0.8\\ 1\end{bmatrix}u$, $y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

Show that both representations belong to the same system.

Solution: $T.F=\frac{Y(s)}{U(s)}=C(SI-A)^{-1}B+D \Rightarrow \begin{bmatrix}0.8 & 1\end{bmatrix}\left[\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix}-\begin{pmatrix}0 & 1\\ -0.4 & -1.3\end{pmatrix}\right]^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}$

$$\begin{bmatrix}0.8 & 1\end{bmatrix}\begin{bmatrix}s & -1\\ 0.4 & s+1.3\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix} \Rightarrow \frac{\begin{bmatrix}0.8 & 1\end{bmatrix}\begin{bmatrix}s+1.3 & 1\\ -0.4 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}}{s^2+1.3s+0.4} \Rightarrow \frac{\begin{bmatrix}0.8 & 1\end{bmatrix}\begin{bmatrix}1\\ s\end{bmatrix}}{s^2+1.3s+0.4}=\frac{s+0.8}{s^2+1.3s+0.4}$$

"Try to find a way to check it directly."

for observable form: $\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s & +0.4\\ -1 & s+1.3\end{bmatrix}^{-1}\begin{bmatrix}0.8\\ 1\end{bmatrix} \Rightarrow \frac{\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}s+1.3 & -0.4\\ 1 & s\end{bmatrix}\begin{bmatrix}0.8\\ 1\end{bmatrix}}{s^2+1.3s+0.4}$

$$\frac{\begin{bmatrix}1 & s\end{bmatrix}\begin{bmatrix}0.8\\ 1\end{bmatrix}}{s^2+1.3s+0.4}=\frac{s+0.8}{s^2+1.3s+0.4},$$

What Conclusion can you make?

EX2: Consider $\frac{Y(s)}{U(s)}=\frac{s+6}{s^2+5s+6}$, obtain the state-space representation a- Controllable Cannonical form, b- observable Cannonical form.

C.C.F: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$, $y=\begin{bmatrix}6 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \Rightarrow$ $\dot{x}_1=x_2$, $\dot{x}_2=-6x_1-5x_2+u$, $y=6x_1+x_2$

O.C.F: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & -6\\ 1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}6\\ 1\end{bmatrix}u$, $y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \Rightarrow$ $\dot{x}_1=-6x_2+6u$, $\dot{x}_2=x_1-5x_2+u$, $y=x_2$

"Conclude the relationship."

EX3: Consider the Random form S.S-Representation with $A=\begin{bmatrix}1 & 2\\ -4 & -3\end{bmatrix}$, $B=\begin{bmatrix}1\\ 2\end{bmatrix}$, $C=\begin{bmatrix}1 & 1\end{bmatrix}$ it's required to obtain Controllable Cannonical form:

Sol: $T.F=C(SI-A)^{-1}B+D \Rightarrow \begin{bmatrix}1 & 1\end{bmatrix}\left[\begin{pmatrix}s & 0\\ 0 & s\end{pmatrix}-\begin{pmatrix}1 & 2\\ -4 & -3\end{pmatrix}\right]^{-1}\begin{bmatrix}1\\ 2\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}s-1 & -2\\ 4 & s+3\end{bmatrix}^{-1}\begin{bmatrix}1\\ 2\end{bmatrix}$

$$\Rightarrow \frac{\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}s+3 & 2\\ -4 & s-1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}}{s^2+2s-3+8} \Rightarrow \frac{\begin{bmatrix}s-1 & s+1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}}{s^2+2s+5}=\frac{s-1+2s+2}{s^2+2s+5}=\frac{3s+1}{s^2+2s+5}$$

Then: C.C.F: $A=\begin{bmatrix}0 & 1\\ -5 & -2\end{bmatrix}$, $B=\begin{bmatrix}0\\ 1\end{bmatrix}$, $C=\begin{bmatrix}1 & 3\end{bmatrix}$, $D=0$

Ex4: Consider the following system: $\dddot{y}+6\ddot{y}+11\dot{y}+6y=6u \Rightarrow \frac{Y(s)}{U(s)}=\frac{6}{s^3+6s^2+11s+6}$

obtain the s.s. representation of this system in: (a) C.C.F, (b) O.C.F, (c) diagonal C.F.

Solution: (a) C.C.F $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u, \quad y=\begin{bmatrix}6 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

then: $\dot{x}_1=x_2, \quad \dot{x}_2=x_3, \quad \dot{x}_3=-6x_1-11x_2-6x_3+u, \quad y=6x_1$

(b) O.C.F: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 0 & -6\\ 1 & 0 & -11\\ 0 & 1 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}6\\ 0\\ 0\end{bmatrix}u, \quad y=\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

then: $\dot{x}_1=-6x_3+6u, \quad \dot{x}_2=x_1-11x_3, \quad \dot{x}_3=x_2-6x_3, \quad y=x_3$

(c) diagonal C.F: $\frac{Y(s)}{U(s)}=\frac{6}{s^3+6s^2+11s+6}=\frac{6}{(s+1)(s+2)(s+3)}=\frac{A}{(s+1)}+\frac{B}{(s+2)}+\frac{C}{(s+3)}$

using Partial Fraction to obtain $\frac{Y(s)}{U(s)}=\frac{3}{(s+1)}-\frac{6}{(s+2)}+\frac{3}{(s+3)}$

then: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u, \quad y=\begin{bmatrix}3 & -6 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

then: $\dot{x}_1=-x_1+u, \quad \dot{x}_2=-2x_2+u, \quad \dot{x}_3=-3x_3+u, \quad y=3x_1-6x_2+3x_3$

Ex5:- Given $\frac{Y(s)}{U(s)}=\frac{10.4s^2+47s+160}{s^3+14s^2+56s+160}$ obtain s.s. Representation using (a) MatlaB,

b) Analytical method:

Solution: C.C.F is: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -160 & -56 & -14\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u, \quad y=\begin{bmatrix}160 & 47 & 10.4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

Note: "student should review to find MATLAB Code that transfer T.F ⟹ S.S. and S.S. ⟹ T.F"

Ex6: Consider the System with $A=\begin{bmatrix}-1 & 0 & 1\\ 1 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}, B=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}, C=\begin{bmatrix}1 & 1 & 0\end{bmatrix}$, obtain Transfer Function $Y(s)/U(s)$: $T.F=C(sI-A)^{-1}B+D=\begin{bmatrix}1 & 1 & 0\end{bmatrix}\left\{\begin{pmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{pmatrix}-\begin{pmatrix}-1 & 0 & 1\\ 1 & -2 & 0\\ 0 & 0 & -3\end{pmatrix}\right\}^{-1}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}$

$$\Rightarrow \frac{\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}(s+2)(s+3) & 0 & (s+2)\\ (s+3) & (s+1)(s+3) & 1\\ 0 & 0 & (s+1)(s+2)\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}{s^3+6s^2+11s+6} \Rightarrow \frac{s+3}{s^3+6s^2+11s+6}$$